Math 156 Applied Honors Calculus II Review Sheet for Final Exam Fall 2009

The final exam is on Thursday, December 17, 8-10am, in 140 Lorch Hall.

You may use the integrals $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$, $\int \frac{dx}{x} = \ln x$, $\int e^x dx = e^x$, $\int \sin x\, dx = -\cos x$, $\int \cos x\, dx = \sin x$, $\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta)$, $\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))$, but all other integrals should be derived.

1. **True or false?** Justify your answer with a reason or a counterexample.

a) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+(i/n))\cdot\frac{1}{n} = \frac{3}{2}$ b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+(i/n)}\cdot\frac{1}{n} = \ln 2$

c) If $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, then $\lim_{n\to\infty}\sum_{i=1}^{n} f'(x_i)\Delta x = f(b) - f(a)$.

d) If $f(0) = f(1) = g(0) = g(1) = 0$, then $\int_0^1 f(x)g''(x)dx = \int_0^1 f''(x)g(x)dx$.

e) $\int_0^\infty \frac{dx}{x^2}$ is a convergent improper integral.

f) A spring has natural length 10 cm. If 2 Joule of work are needed to stretch it from length 10 cm to 15 cm, then 4 Joules of work are needed to stretch it from length 10 cm to 20 cm.

g) The center of mass of the region $\{(x,y) : -1 \le x \le 1, 0 \le y \le \cosh x\}$ is $(\overline{x}, \overline{y}) = (0, \frac{1}{2})$.

h) If $f(x)$ is the pdf of a random variable with mean μ, then $f(x)$ attains its maximum value at $x = \mu$.

i) If a radioactive material has a half-life of 100 years and a given sample has mass 1 kg, then after 400 years there will be 0.25 kg remaining.

j) If \$2000 is invested at 5% interest compounded continuously, then after 2 years the investment is worth more than \$2210.

k) Suppose that a differential equation $y' = f(y)$ has a constant solution $y_1(t) = c$. If $y_2(t)$ is another solution with initial condition $y_2(0)$ sufficiently close to c, then $\lim_{t\to\infty} y_2(t) = c$.

l) $1 + \frac{2008}{2009} + (\frac{2008}{2009})^2 + (\frac{2008}{2009})^3 + \cdots = 2009$

m) If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty} a_n b_n = 0$.

n) If $0 \le a_n \le b_n$ and $\sum_{n=0}^\infty a_n$ converges, then $\sum_{n=0}^\infty b_n$ also converges.

o) If $a_n \to 0$ as $n \to \infty$, then $\sum_{n=0}^\infty a_n$ converges.

p) If $\sum_{n=0}^\infty a_n$ converges, then $\sum_{n=0}^\infty |a_n|$ also converges.

q) The ratio test can be used to show that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges.

r) If the power series $\sum_{n=0}^\infty c_n x^n$ converges for $x = 2$, then it also converges for $x = 1$.

s) If $f(x) = e^{-x^2}$, then $f^{(3)}(0) = 0$ and $f^{(6)}(0) = -6$. t) $\frac{1}{(1+x)^2} = \sum_{n=1}^\infty (-1)^{n+1} n x^{n-1}$

u) $2 < e < 3$ v) $\int_0^1 e^{-x^2} dx > \frac{2}{3}$ w) $1 = \frac{\pi}{2} - \frac{1}{3!}(\frac{\pi}{2})^3 + \frac{1}{5!}(\frac{\pi}{2})^5 - \frac{1}{7!}(\frac{\pi}{2})^7 + \cdots$

x) $\lim_{x\to 0}\frac{\sin x}{x} = 0$ y) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$ z) $\lim_{n\to\infty}(1 - \frac{x}{n})^n = -e^x$

aa) $\cosh^2 x - \sinh^2 x = 1$ bb) $\int \tanh x\, dx = \operatorname{sech}^2 x$

cc) $y(t) = \sinh t$ is a solution of the differential equation $y' = \sqrt{1+y^2}$.

dd) $\sqrt{1+x^2} = 1 + x^2 + \cdots$ ee) $\cosh ix = \cos x$ ff) $\log(-1) = \pi i$

gg) $\binom{6}{3} = 2$ hh) $\binom{10}{2} = \binom{10}{8}$ ii) $\sum_{n=0}^{k}\binom{k}{n}(-1)^n = 0$

integration

2. Evaluate the limit (by any means).

a) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}$ b) $\lim\limits_{h\to 0}\frac{(x+h)^4-x^4}{h}$ c) $\lim\limits_{h\to 0}\frac{1}{h}\int_0^h f(x)\,dx$ d) $\lim\limits_{h\to 0}\frac{1}{h^2}\int_0^h xf(x)\,dx$

3. Find the antiderivative.

a) $\int\frac{dx}{4x^2}$ b) $\int\frac{x}{4+x^2}\,dx$ c) $\int\frac{dx}{4+x^2}$ d) $\int\frac{dx}{\sqrt{4+x^2}}$ e) $\int\frac{dx}{4-x^2}$ f) $\int\frac{dx}{4x-x^2}$

g) $\int x\sin x\,dx$ h) $\int e^{-x}\sin x\,dx$ i) $\int\sin^2 x\,dx$ j) $\int\sin^3 x\,dx$ k) $\int\sin^4 x\,dx$

4. Show that $\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx=\frac{\pi}{4}$. (hint: substitute $u=\frac{\pi}{2}-x$, add the resulting integral to the original integral.)

5. Determine whether the integral converges or diverges. If it converges, find the value.

a) $\int_1^\infty\frac{dx}{x^2}$ b) $\int_1^\infty\frac{dx}{x}$ c) $\int_1^\infty\frac{dx}{x-1}$ d) $\int_0^1\frac{dx}{x^2}$ e) $\int_0^1\frac{dx}{\sqrt{x}}$ f) $\int_{-1}^1\frac{dx}{x}$

6. Consider a metal sphere of radius a carrying electric charge $q>0$. Let r be the distance from the center of the sphere to a point in space. It is known from electromagnetic theory that the induced electric potential is $V(r)=\frac{q}{2a}\int_{-a}^{a}\frac{dx}{(r^2-2rx+a^2)^{1/2}}$.

a) Evaluate $V(r)$. Consider the two cases $0\le r\le a$ and $r>a$ separately.

b) Sketch the graph of $V(r)$ for $r\ge 0$.

7. An aquarium full of water is 2 m long, 1 m high, and 0.5 m wide. How much work is done in pumping the water out the top of the aquarium? If the width of the aquarium is doubled, is the work also doubled? If the height is doubled, is the work also doubled?

8. Two identical ions repel each other with force $F=-\frac{q^2}{r^2}$, where q is the ion charge and r is the distance between them.

a) An ion is held fixed at $x=0$. Find the work done in moving another ion from $x=3$ to $x=2$.

b) An ion is held fixed at $x=1$. Find the work done in moving another ion from $x=3$ to $x=2$.

c) Two ions are held fixed at $x=0$ and $x=1$. Find the work done in moving a third ion from $x=3$ to $x=2$.

d) A metal rod of uniform charge density is held fixed on the interval $0\le x\le 1$. The total charge on the rod is q. Find the work done in moving an ion from $x=3$ to $x=2$.

9. A cable hanging between two poles has the shape $y=\cosh x$, $-1\le x\le 1$.

a) Find the arclength of the cable.

b) Find the surface area obtained by rotating the cable about the x-axis.

10. Sketch the region in the xy-plane and find the center of mass.

a) $\{(x,y):0\le y\le x^2\ ,\ 0\le x\le 2\}$ b) $\{(x,y):x^2\le y\le 4\ ,\ 0\le x\le 2\}$

c) $\{(x,y):-1\le x\le 1,\ 0\le y\le 1-x^2\}$ d) $\{(x,y):0\le y\le\frac{1}{1+x^2}\ ,\ 0\le x<\infty\}$

11. The lifetime of a light bulb is described by an exponential distribution with mean 1000 hours. Find the probability that the lightbulb (a) fails within the first 200 hours, (b) lasts more than 800 hours.

12. Let $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ for $0 < x < 1$ and zero otherwise. Show that $f(x)$ is a valid pdf.

differential equations

13. Find the solution of the differential equation with initial condition $y(0) = y_0$. Sketch the solution for $t \geq 0$. Find $\lim_{t\to\infty} y(t)$.

a) $y' = -2y\,,\ y_0 = 1$ b) $y' = 1 - 2y\,,\ y_0 = 0$ c) $y' = 1 - y^2\,,\ y_0 = 0$ d) $y' = -ty\,,\ y_0 = 1$

14. Consider the differential equation $y'' = y$.

a) Show that $y(t) = c_1e^t + c_2e^{-t}$ is a solution for any constants c_1, c_2.

b) Find the solution $y(t)$ subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

c) Repeat part (b) for initial conditions $y(0) = 0$, $y'(0) = 1$.

15. Polonium-214 has a half-life of 1.4×10^{-4} s. If a sample has initial mass 40 mg, how long will it take for the mass to decay to 30 mg?

16. A tiger consumes 2500 calories per day and expends 20 calories per kg of its mass per day in daily activity. Assume that 10,000 calories is equivalent to 1 kg of the tiger's mass. Formulate a differential equation for the mass of the tiger as a function of time. Solve the equation and sketch the graph for $t \geq 0$. What value does the tiger's mass approach as time increases?

17. A thermometer at room temperature 70°F is placed in a patient's mouth. After one minute the thermometer reads 95°F and after two minutes it reads 100°F. Find the patient's temperature.

18. A common model for an epidemic assumes that the rate of spread of infection is proportional to the product of the number of people infected and the number of people not infected. In a town with 4000 inhabitants, if 10 people are infected at the beginning of the week and 20 people are infected at the end of the week, how long does it take for half the population to be infected?

series

19. Determine whether the series converges or diverges. Justify your answer.

a) $\sum_{n=1}^{\infty} \frac{1}{2n}$ b) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ c) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ e) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

20. Express the repeating decimal as a rational number (i.e. a ratio of two integers).

a) $0.11111111\ldots$ b) $0.1212121212\ldots$ c) $0.4999999999\ldots$

21. Find the sum of the series.

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ b) $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$ c) $\sum_{n=1}^{\infty} \frac{1}{3^n}$ d) $\sum_{n=1}^{\infty} \frac{n}{3^n}$ e) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

22. It is known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this to evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

23. For each of the following series, find a bound for $|s - s_{10}|$. a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

24. Two students walk towards each other at 2 mi/hr starting from a separation of 20 miles. At the same time, a dog starts running back and forth between the students at 10 mi/hr. Let D be the total distance traveled by the dog when the students finally meet. Express D as an infinite series and find the sum of the series.

25. Winning a ping-pong game requires a lead of two points, i.e. if the final score is tied, you must score two consecutive points in order to win the game. Suppose your probability of scoring a point is p, where $0 \le p \le 1$. If the final score is tied, what is the probability that you will eventually win the game? Evaluate for $p = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$. Interpret.

26. Start with the closed interval $[0, 1]$. Remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Remove the middle third of each of those. That leaves four intervals. Remove the middle third of each of those. Continue the process indefinitely. The Cantor set is the set of all points remaining after all the intervals have been removed.

a) Show that the total length of all the intervals removed is 1.

b) Show that, nonetheless, the Cantor set contains infinitely many numbers.

power series, Taylor series

27. Find the radius of convergence, interval of convergence, and sum of the power series.

a) $\sum_{n=0}^{\infty} x^n$ b) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ c) $\sum_{n=0}^{\infty} (x-1)^n$ d) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ e) $\sum_{n=1}^{\infty} nx^n$

28. Find the power series representation for $f(x) = \frac{1}{1-x}$ about $a = -1$. Find the interval of convergence. Set $x = \frac{1}{4}$ in the power series to obtain a formula for $\frac{4}{3}$. Write out the first four terms in the series. (Note: in class we considered the cases $a = 0$ and $a = \frac{1}{2}$.)

29. Expand $f(x)$ in a power series about $x = a$ and find the interval of convergence.

a) $\frac{1}{1+x}$, $a = 0$ b) $\frac{1}{1+x}$, $a = 1$

30. Find the Taylor series for $\sinh x$ and $\cosh x$ about $x = 0$.

31. Compute the first three nonzero terms in the power series for $\sin^2 x + \cos^2 x$ about $x = 0$ by squaring the power series for $\sin x$ and $\cos x$ and adding the results.

32. Let $T_1(x)$ and $T_2(x)$ be the Taylor polynomials of degree 1 and 2 for $f(x) = e^{-x^2}$ about $x = 0$. Sketch $f(x)$, $T_1(x)$, $T_2(x)$ on the same graph in a neighborhood of $x = 0$.

33. Let $f(x) = \begin{cases} 0 & \text{for } x \le 0, \\ e^{-1/x} & \text{for } x > 0. \end{cases}$

Sketch the graph of $f(x)$, but first evaluate the following limits.

a) $\lim_{x\to\infty} f(x)$ b) $\lim_{x\to 0^+} f(x)$ c) $\lim_{x\to 0^+} f'(x)$ d) $\lim_{x\to 0^+} f''(x)$ e) $\lim_{x\to 0^+} f^{(n)}(x)$ for $n \ge 3$

Note: $f(x)$ is a function whose Taylor series about $x = 0$ converges for all x, but such that the limit of the Taylor series is not equal to $f(x)$ for any $x > 0$.

34. Find an approximate value for $\sqrt{10}$ which is accurate to within 0.005.

35. Use the Taylor series for $f(x) = \ln(1+x)$ about $x = 0$ to evaluate $\ln\frac{3}{2}$ to within 10^{-3}.

36. Find the first two nonzero terms in the Taylor series for $f(x)$ about $x = 0$.

a) $\tan x$ b) $e^{-x}\sin x$ c) $(1-\cos x)/x$

37. The Bernoulli numbers B_n are defined by $\dfrac{x}{e^x - 1} = \displaystyle\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$. Find B_0, B_1, B_2.

38. Show that the following functions satisfy $f(0) = 0$ and $f'(0) = 1$. Find $f''(0)$ in each case. If the functions are graphed in a neighborhood of $x = 0$, in what order do they appear (from top to bottom)? a) x b) $\sin x$ c) $\ln(1+x)$ d) $e^x - 1$

39. Recall the power series for the Bessel function of order zero, $J_0(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

a) Evaluate $\int_0^1 J_0(x)\,dx$ using 2 terms in the series. Find an upper bound for the error.

b) Show that $J_0(x)$ satisfies the differential equation $xy'' + y' + xy = 0$.

40. Let $f(t) = \displaystyle\sum_{n=0}^{\infty} t^n$. We will derive the sum of the series using differential equations.

a) Show that $f(t)$ satisfies the differential equation $y' = y^2$ with initial condition $y(0) = 1$.

b) Solve the differential equation for $f(t)$ by separation of variables.

41. Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ converges, but $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges. (hint: sketch the graph of each integrand, express the first integral as an alternating series)

42. In class we used the 1st order Taylor approximation for $\sin x$ about $x = 0$ to show that $|\sin\frac{\pi}{5} - \frac{\pi}{5}| \le \frac{1}{2}(\frac{\pi}{5})^2$. Derive a more accurate result using the 3rd degree Taylor approximation.

43. a) expand $\frac{a}{a+b}$ in powers of $\frac{a}{b}$ b) expand $\sqrt{R^2 - r^2}$ in powers of $\frac{r}{R}$
Find the first three nonzero terms in each case.

44. Show that $f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots$

45. The equation $\frac{x^2}{(1+\epsilon)^2} + y^2 = 1$ defines an ellipse in the xy-plane (assume $0 \le \epsilon < 1$).

a) Find the intercepts on the x-axis and y-axis. Sketch the ellipse.

b) Let $A(\epsilon)$ be the area of the ellipse. Express $A(\epsilon)$ as a definite integral.

c) Find the first 2 nonzero terms in the power series expansion of $A(\epsilon)$ about $\epsilon = 0$.

46. The gravitational potential energy function due to a pair of point masses m_1, m_2 located at x_1, x_2 is $V(x) = \frac{Gm_1}{|x-x_1|} + \frac{Gm_2}{|x-x_2|}$, where G is the gravitational constant. For $x \to \infty$, the potential can be approximated by $V(x) \approx \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \cdots$, where the coefficients $a, b, c, \ldots$ are constants that depend on m_1, m_2, x_1, x_2. Find the values of a, b, c and give a physical interpretation of the results. (hint: set $y = 1/x$ and expand the potential in powers of y.)

47. The Lennard-Jones potential energy function, $V(x) = V_0((\frac{x_0}{x})^{12} - 2(\frac{x_0}{x})^6)$, describes the interaction between two molecules, where V_0 and x_0 are positive constants and x is the distance between the molecules. Sketch the graph of $V(x)$ for $x > 0$. Find $T_2(x)$, the quadratic Taylor approximation for $V(x)$ at $x = x_0$ and sketch its graph.

48. Use the 2nd degree Taylor approximation of $\sqrt{1+x^2}$ at $x = 0$ to approximate $\int_0^1 \sqrt{1+x^2}\,dx$. Find an upper bound for the error.

binomial series

49. Recall the binomial expansion, $(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}b^n$, where $k \geq 1$ is an integer.

a) Show that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$.

b) Explain the connection between the formula in (a) and Pascal's triangle below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

c) Fill in the next two rows of the triangle. Use this to expand $(a+b)^6$.

complex numbers

50. Express the complex numbers below in Cartesian form $x + iy$ and polar form $re^{i\theta}$. Plot each number in the complex plane.

a) $1+i$ b) $(1+i)^2$ c) $(1+i)^3$ d) $\frac{1}{1+i}$ e) $\sqrt{1+i}$

51. Compute $(1+i)^6$ two ways, using: (a) binomial formula , (b) polar form.

52. Find the roots of the equation. Plot the roots in the complex plane.

a) $z^2 + 2z - 2 = 0$ b) $z^2 + 2z + 2 = 0$ c) $z^2 = 1$ d) $z^3 = 1$ e) $z^4 = 1$ f) $e^z = 1$

53. Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

54. a) Use integration by parts to find the antiderivatives $\int e^{ax}\cos bx\,dx$, $\int e^{ax}\sin bx\,dx$.

b) Show that $e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$ and $\int e^{(a+ib)x}dx = \frac{a-ib}{a^2+b^2}e^{(a+ib)x}$.

c) Take the real and imaginary parts in (b) to rederive the formulas you obtained in (a).

55. Derive the following results using Euler's formula, $e^{ix} = \cos x + i\sin x$.

a) $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ b) $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Derive the following formulas using the results in (a) and (b).

c) $\frac{d}{dx}\cos x = -\sin x$ d) $\frac{d}{dx}\sin x = \cos x$

e) $\sin^2 x + \cos^2 x = 1$ f) $\sin 2x = 2\sin x\cos x$ g) $\cos 2x = \cos^2 x - \sin^2 x$