2. Let $g(x) = x^2$, and let

$$h(x) = \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational.} \end{cases}$$

(i) For which y is $h(y) \leq y$?
(ii) For which y is $h(y) \leq g(y)$?
(iii) What is $g(h(z)) - h(z)$?
(iv) For which w is $g(w) \leq w$?
(v) For which ε is $g(g(\varepsilon)) = g(\varepsilon)$?

3. Find the domain of the functions defined by the following formulas.

(i) $f(x) = \sqrt{1 - x^2}$.
(ii) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$.
(iii) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$.
(iv) $f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$.
(v) $f(x) = \sqrt{1 - x} + \sqrt{x - 2}$.

8. For which numbers a, b, c, and d will the function

$$f(x) = \frac{ax + b}{cx + d}$$

satisfy $f(f(x)) = x$ for all x (for which this equation makes sense)?

9. (a) If A is any set of real numbers, define a function C_A as follows:

$$C_A(x) = \begin{cases} 1, & x \text{ in } A \\ 0, & x \text{ not in } A. \end{cases}$$

Find expressions for $C_{A \cap B}$ and $C_{A \cup B}$ and $C_{\mathbf{R}-A}$, in terms of C_A and C_B. (The symbol $A \cap B$ was defined in this chapter, but the other two may be new to you. They can be defined as follows:

$$A \cup B = \{x : x \text{ is in } A \text{ or } x \text{ is in } B\},$$
$$\mathbf{R} - A = \{x : x \text{ is in } \mathbf{R} \text{ but } x \text{ is not in } A\}.)$$

(b) Suppose f is a function such that $f(x) = 0$ or 1 for each x. Prove that there is a set A such that $f = C_A$.
(c) Show that $f = f^2$ if and only if $f = C_A$ for some set A.

12. A function f is **even** if $f(x) = f(-x)$ and **odd** if $f(x) = -f(-x)$. For example, f is even if $f(x) = x^2$ or $f(x) = |x|$ or $f(x) = \cos x$, while f is odd if $f(x) = x$ or $f(x) = \sin x$.

(a) Determine whether $f + g$ is even, odd, or not necessarily either, in the four cases obtained by choosing f even or odd, and g even or odd. (Your answers can most conveniently be displayed in a 2×2 table.)
(b) Do the same for $f \cdot g$.
(c) Do the same for $f \circ g$.
(d) Prove that every even function f can be written $f(x) = g(|x|)$, for infinitely many functions g.

*13. (a) Prove that any function f with domain $\mathbf{R}$ can be written $f = E + O$, where E is even and O is odd.
(b) Prove that this way of writing f is unique. (If you try to do part (b) first, by "solving" for E and O you will probably find the solution to part (a).)

***17.** If $f(x) = 0$ for all x, then f satisfies $f(x+y) = f(x) + f(y)$ for all x and y, and also $f(x \cdot y) = f(x) \cdot f(y)$ for all x and y. Now suppose that f satisfies these two properties, but that $f(x)$ is not always 0. Prove that $f(x) = x$ for all x, as follows:

(a) Prove that $f(1) = 1$.
(b) Prove that $f(x) = x$ if x is rational.
(c) Prove that $f(x) > 0$ if $x > 0$. (This part is tricky, but if you have been paying attention to the philosophical remarks accompanying the problems in the last two chapters, you will know what to do.)
(d) Prove that $f(x) > f(y)$ if $x > y$.
(e) Prove that $f(x) = x$ for all x. Hint: Use the fact that between any two numbers there is a rational number.

21. Prove or give a counterexample for each of the following assertions:

(a) $f \circ (g+h) = f \circ g + f \circ h$.
(b) $(g+h) \circ f = g \circ f + h \circ f$.
(c) $\dfrac{1}{f \circ g} = \dfrac{1}{f} \circ g$.
(d) $\dfrac{1}{f \circ g} = f \circ \left(\dfrac{1}{g}\right)$.