# Math 116 — Practice for Exam 3

Generated April 13, 2023

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

| Semester | Exam | Problem | Name | Points | Score |
|---|---|---|---|---|---|
| Fall 2021 | 3 | 3 | | 7 | |
| Fall 2021 | 3 | 4 | | 6 | |
| Fall 2021 | 3 | 6 | | 13 | |
| Fall 2021 | 3 | 7 | fan | 17 | |
| Fall 2021 | 3 | 8 | bumper cars | 16 | |
| Total | | | | 59 | |

**Recommended time (based on points): 71 minutes**

**2**. [5 points] Suppose that the power series $\sum_{n=1}^{\infty} C_n(x+2)^n$ converges at $x = 4$ and diverges at $x = -10$. What can you say about the behavior of the power series at the following values of $x$?

**a**. [1 point] At $x = 0$, the power series...

[CONVERGES] DIVERGES CANNOT DETERMINE

**b**. [1 point] At $x = -8$, the power series...

CONVERGES DIVERGES [CANNOT DETERMINE]

**c**. [1 point] At $x = 8$, the power series...

CONVERGES [DIVERGES] CANNOT DETERMINE

**d**. [1 point] At $x = -4$, the power series...

[CONVERGES] DIVERGES CANNOT DETERMINE

**e**. [1 point] At $x = 6$, the power series...

CONVERGES DIVERGES [CANNOT DETERMINE]

**3**. [7 points] A function $F(x)$ has Taylor series given by

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^n(n^2+1)}(x-1)^{4n+1}$$

Answer the following questions regarding the Taylor series:

**a**. [3 points] Circle the appropriate answer. No work is needed, but partial credit may be given for correct work.

At $x = 1$, $F$ is... [INCREASING] DECREASING CANNOT DETERMINE

*Solution:* Using the Taylor series,

$$F'(1) = \frac{(-1)^0(1)}{2^0(1)} = 1 > 0,$$

so $F$ is increasing at $x = 1$.

**b**. [4 points] What is $F^{(2021)}(1)$? Give your answer in exact form and do not try to simplify. Show your work.

*Solution:* Using the Taylor series, the term $(x-1)^{2021}$ appears when $n = 505$, so

$$\frac{F^{(2021)}(1)}{2021!} = \frac{(-1)^{505}(505+1)}{2^{505}(505^2+1)}$$

$F^{(2021)}(1) =$ $-\frac{(506)(2021)!}{2^{505}(505^2+1)}$.

**4**. [6 points] Find the **radius of convergence** of the following power series:

$$\sum_{n=0}^{\infty} \frac{8^n (n!)^3}{(3n)!}(x-5)^{3n}.$$

Show your work including full justifications of any tests you use.

*Solution:* Setting $a_n = \frac{8^n(n!)^3}{(3n)!}(x-5)^{3n}$, compute

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{8^{n+1}\left((n+1)!\right)^3}{(3n+3)!}\frac{(3n)!}{8^n(n!)^3}|x-5|^3 \\ &= \lim_{n\to\infty} 8\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}|x-5|^3 \\ &= \frac{8}{27}|x-5|^3.\end{aligned}$$

By the ratio test, the power series converges for

$$\frac{8}{27}|x-5|^3 < 1 \iff |x-5| < \left(\frac{27}{8}\right)^{1/3} = \frac{3}{2}.$$

The radius of convergence is $\frac{3}{2}$.

**6**. [13 points] Values of a function $g(x)$ and some of its derivatives at $x = 2$ are given in the table below. Use this information for some of the problems below.

| $g(2)$ | $g'(2)$ | $g''(2)$ | $g'''(2)$ | $g^{(4)}(2)$ |
|---|---|---|---|---|
| 1 | 2 | -4 | 0 | 4 |

**a**. [4 points] Find the first 4 nonzero terms of the Taylor series of $g(x)$ about $x = 2$. Write your final answer as a polynomial $P(x)$ in the blank below.

$P(x) =$ $1 + 2(x-2) - 2(x-2)^2 + \frac{1}{6}(x-2)^4$

**b**. [4 points] Using known Taylor series, find the first 3 nonzero terms of the Taylor series of $f(x) = (x-2)\ln\left(\frac{x}{2}\right)$ about $x = 2$. Write your final answer as a polynomial $Q(x)$ in the blank below. (*Hint:* $f(x) = (x-2)\ln\left(1 + \frac{(x-2)}{2}\right)$)

*Solution:* For $-1 < x \leq 1$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots,$$

about $x = 0$. So,

$$(x-2)\ln\left(1 + \frac{(x-2)}{2}\right) = (x-2)\left(\frac{x-2}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 + \ldots\right)$$

$Q(x) =$ $\frac{(x-2)^2}{2} - \frac{(x-2)^3}{8} + \frac{(x-2)^4}{24}$

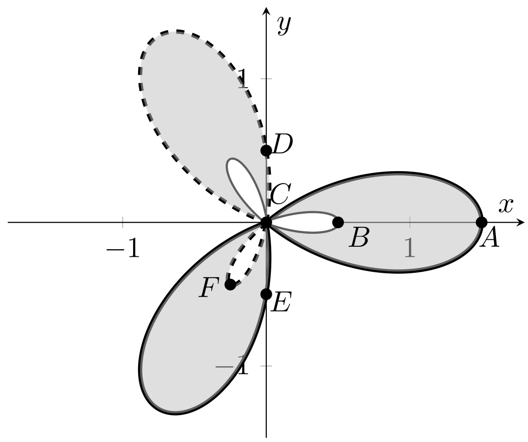
**c**. [5 points] Let $H(x) = 1 + \int_2^x f(t) + g(t)dt$. Find the first 4 nonzero terms of the Taylor series of $H$ about $x = 2$. Write your final answer as a polynomial $R(x)$ in the blank below. Partial credit may be given for finding the appropriate terms of $\int_2^x f(t)dt$ or $\int_2^x g(t)dt$.

*Solution:* Putting together (a) and (b), $R$ is the first 4 nonzero terms of $1 + \int_2^x P(t) + Q(t)dt$. Note that we only need the first 3 terms of $P$ and the first term of $Q$:

$$\begin{aligned} 1 + \int_2^x P(t) + Q(t)dt &= 1 + \left[t + (t-2)^2 - \frac{2}{3}(t-2)^3\right]\Big|_2^x + \left[\frac{(t-2)^3}{6}\right]\Big|_2^x \\ &= 1 + (x-2) + (x-2)^2 - \frac{2}{3}(x-2)^3 + \frac{(x-2)^3}{6}. \end{aligned}$$

$R(x) =$ $1 + (x-2) + (x-2)^2 - \frac{(x-2)^3}{2}$

**7.** [17 points] John is holding a Fan Fair to celebrate the success of his burgeoning fan business. At the fair, John is debuting his new fan, which has blades given by the shaded region of the graph of the polar equation $r = \cos(3\theta) + \frac{1}{2}$ shown below. Note that the graph of $r = \cos(3\theta) + \frac{1}{2}$ is comprised of both the inner and outer loops of the fan blades. One of the activities at the Fan Fair is to guess the perimeter and area of the blades, which can actually be computed explicitly.




**a**. [4 points] For the values of $\theta$ listed below, write on the line the letter of the point corresponding to it.

$\theta = 0$ : *A* $\quad$ $\theta = \frac{\pi}{3}$ : *F*

$\theta = \frac{\pi}{2}$ : *D* $\quad$ $\theta = \pi$ : *B*

**b**. [5 points] Find the 3 values of $\theta$ which correspond to the point $C$ (the origin) for $0 \leq \theta \leq \pi$. Then, determine the interval within $[0, 2\pi]$ for which $\theta$ traces out the **dashed** loops in the graph above. (*Hint:* $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$)

*Solution:* Note that the solutions to $\cos(3\theta) = -\frac{1}{2}$ are, for $k$ an integer

$$3\theta = \frac{2\pi}{3} + 2\pi k$$
$$3\theta = \frac{4\pi}{3} + 2\pi k.$$

So, the first three values are

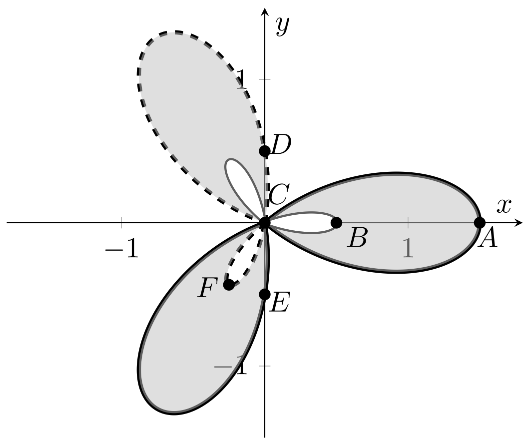
$$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}.$$

Using (a), we know that the dashed loops lie between the first value and the third value when $\theta$ corresponds to the point $C$.

$\theta =$ $\frac{2\pi}{9}$ , $\frac{4\pi}{9}$ , $\frac{8\pi}{9}$

Interval giving $\theta$-values that trace out the dashed loops: $[\frac{2\pi}{9}, \frac{8\pi}{9}]$

**7. (continued)** Here is a reproduction of the graph from the previous page of the polar equation $r = \cos(3\theta) + \frac{1}{2}$:




**c.** [4 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total perimeter of the fan blades, including both the inner and outer edges of the fan blades.

*Solution:* Note that $\frac{dr}{d\theta} = -3\sin(3\theta)$ and the perimeter is given by

$$\int_0^{2\pi} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Total Perimeter = $\int_0^{2\pi} \sqrt{(\cos(3\theta) + \frac{1}{2})^2 + 9\sin^2(3\theta)}\, d\theta$

**d.** [4 points] Write, but do not evaluate, an expression giving the total area of all 3 fan blades (the shaded region of the graph). (*Hint*: Your answer from (b) may be handy, but is not strictly necessary)

*Solution:* Using (b), the small dashed loop is traced out for $\frac{2\pi}{9} \le \theta \le \frac{4\pi}{9}$ and the large dashed loop is traced out for $\frac{4\pi}{9} \le \theta \le \frac{8\pi}{9}$, so the area of one small loop is

$$\int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta$$

and the area of one large loop is

$$\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta.$$

Exploiting the symmetry of the fan, we get the total area below.

Total Area = $3\left[\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta - \int_{\frac{2\pi}{9}}^{\frac{4\pi}{9}} \frac{(\cos(3\theta) + \frac{1}{2})^2}{2} d\theta\right]$

**8**. [16 points] Molly and Erin are two bumper car enthusiasts who hate bumping into things. So, they get on the bumper cars, ride until they bump into each other, and then stop riding the bumper cars. At time $t$ minutes after they start driving their bumper cars, Molly's position is given by

$$M(t) = \left(4 - 6\cos t, 3t + \pi^2\right)$$

and Erin's position is given by

$$E(t) = \left(2\cos t, (t - 2\pi)^2 + 3t + \frac{8\pi^2}{9}\right),$$

where all distances are in meters.

**a**. [4 points] How long do they ride the bumper cars? Make sure to include units.

*Solution:* We set

$$4 - 6\cos t = 2\cos t \quad \text{and} \quad 3t + \pi^2 = (t - 2\pi)^2 + 3t + \frac{8\pi^2}{9},$$

we get

$$\cos t = \frac{1}{2} \quad \text{and} \quad (t - 2\pi)^2 = \frac{\pi^2}{9},$$

so the $x$-coordinates are equal when $t = \frac{\pi}{3}, \frac{5\pi}{3} + 2\pi k$ and the $y$-coordinates are equal when $t = \frac{5\pi}{3}, \frac{7\pi}{3}$. So, they ride the bumper cars for $\frac{5\pi}{3}$ minutes.

**b**. [4 points] Find an explicit expression for Erin's speed $t$ minutes after she starts driving her bumper cars, before the collision (your expression should not contain any integrals nor the letters $M$, $E$). Make sure to include units.

*Solution:* For Erin, $\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2(t - 2\pi) + 3$. So, her speed is

$$\sqrt{4\sin^2 t + (2(t - 2\pi) + 3)^2} \text{ meters/minute}$$

**c**. [4 points] Write, but do not evaluate, an expression involving integrals that gives the total distance that Erin travelled before the collision. Make sure to include units.

*Solution:* Using the answer to (a),

$$\int_0^{\frac{5\pi}{3}} \sqrt{4\sin^2 t + (2(t - 2\pi) + 3)^2} dt \text{ meters}$$

**d**. [4 points] Suppose the positive $y$-direction in the $xy$-plane is North. At $t = 0$, Molly is facing directly North. Find all other times $t > 0$ (if any) after they start riding their bumper cars, but before their collision, when Molly is facing directly North. Make sure to include units.

*Solution:* For, Molly $\frac{dx}{dt} = 6\sin t$ and $\frac{dy}{dt} = 3$. So, $\frac{dx}{dt} = 0$ when $t = k\pi$. The only $t > 0$ before the collision where $\frac{dx}{dt} = 0$ is $\pi$ minutes.

**2**. [15 points] The parts of this problem are unrelated to each other. Be sure to show work for all parts, and circle your final answer.

**a**. [5 points] A leaking bag of sugar is lifted vertically from the ground to a height of 10 feet above the ground. The **weight** of the bag of sugar is $6 - \sqrt{x}$ lbs when it has been lifted $x$ feet above the ground. Find the work done lifting the bag, including units. Fully evaluate any integrals, but you do not need to simplify your answer.

*Solution:* The work is obtained by integrating the force over the distance the bag is lifted. The force on the bag is equal to its weight, so we have:

$$\int_0^{10} (6 - \sqrt{x})\,dx = 60 - \frac{2}{3}x^{3/2}\Big|_0^{10}$$
$$= 60 - \frac{2}{3}10^{3/2}.$$

Answer: $60 - \frac{2}{3}10^{3/2}$ **lbs·ft**

**b**. [5 points] Write an expression involving one or more integrals that gives the volume of the solid obtained by rotating the region in the $xy$-plane bounded between the $x$-axis, the parabola $y = x^2 + 1$, the line $x = -1$ and the line $x = 1$, about the line $x = -2$. Do not evaluate your integral(s).

*Solution:* Using the shell method, the volume is

$$\int_{-1}^{1} 2\pi(x+2)(x^2+1)\,dx.$$

Answer: $\int_{-1}^{1} 2\pi(x+2)(x^2+1)\,dx$

**c**. [5 points] The function $f(x) = x^4 + 5$ can be rewritten in the form $f(x) = (x+1)^4 + A(x+1)^3 + B(x+1)^2 + C(x+1) + D$, where $A, B, C, D$ are constants. Find the values of $A, B, C, D$ using Taylor series. Other methods used to find the constants will not be given credit.

$A=$ $-4$

$B=$ $6$

$C=$ $-4$

$D=$ $6$

**3**. [13 points] A function $g(x)$ has Taylor series centered at $x = 5$ given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^{n+1}}{(n+1) \cdot 4^n}.$$

**a**. [2 points] Is $g(x)$ increasing or decreasing near $x = 5$? Briefly justify your answer.

*Solution:* The coefficient of $(x-5)$ in the Taylor series is equal to $g'(5)$. Therefore $g'(5) = \frac{(-1)^0}{(0+1) \cdot 4^0} = 1 > 0$, so $g(x)$ is increasing near $x = 5$.

**b**. [3 points] Find $g^{(1001)}(5)$.

*Solution:* The coefficient of $(x-5)^n$ in the Taylor series for $g(x)$ is $\frac{g^{(n)}(5)}{n!}$. We see that the exponent on $(x-5)$ is 1001 when $n = 1000$. Therefore $g^{(1001)}(5)$ is equal to 1001! times $\frac{(-1)^{1000}}{(1000+1) \cdot 4^{1000}}$.

$g^{(1001)}(5) =$ $\frac{1001!}{1001 \cdot 4^{1000}}$

**c.** [8 points] Given that the radius of convergence of this Taylor series is 4 (do NOT show this), find the **interval** of convergence of this Taylor series. Show all your work, including full justification for series behavior.

*Solution:* Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $5 \pm 4 = 1, 9$. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-4)^{n+1}}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{4(-1)^n(-1)^{n+1}}{n+1} = -4\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

To determine the behavior of this, we use the limit comparison test with comparison series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have (with $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since 1 is a positive number, and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the $p$-series test with $p = 1$, the limit comparison tells us that the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. Therefore $-4\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. So $x = 1$ is not included in the interval of convergence.

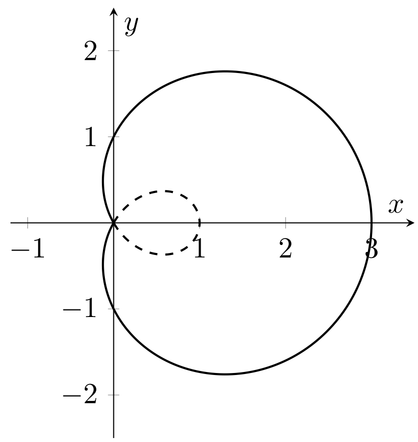
At $x = 9$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1}}{(n+1)4^n} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

Since $a_n = \frac{1}{n+1} \to 0$ as $n \to \infty$, and $a_n$ is decreasing, by the alternating series test this series converges.

Interval of convergence: $(1, 9]$

**5**. [11 points] The parts of this question relate to the following polar graph, defined by the polar curve $r(\theta) = -1 + 2\cos(\theta)$, on the domain $[0, 2\pi]$. Both the solid and dashed curves are part of the graph of $r(\theta)$.




**a**. [2 points] What are all the angles $\theta$, with $0 \leq \theta \leq 2\pi$, for which the graph passes through the origin?

*Solution:* The curve passes through the origin when $r(\theta) = 0$, so we need to solve $\cos(\theta) = 1/2$. This occurs for $\theta$ in $[0, 2\pi]$ at the values $\theta = \pi/3, 5\pi/3$.

Answer(s): $\theta = \pi/3, 5\pi/3$

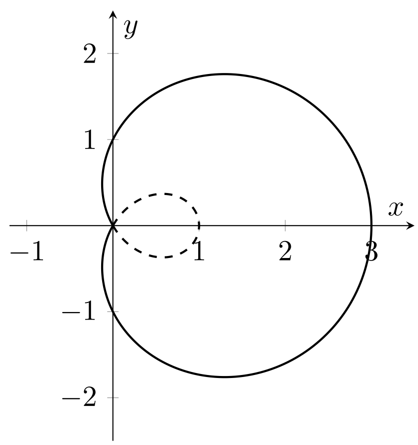
**b**. [2 points] Determine the interval(s) within $[0, 2\pi]$ for which $\theta$ traces out the **dashed** portion of the graph.

Answer(s): $[0, \pi/3]$ **and** $[5\pi/3, 2\pi]$

**c**. [3 points] Write, but do not evaluate, an expression involving one or more integrals which gives the **area** enclosed by the **dashed** portion of the graph.

The area is $\frac{1}{2}\int_0^{\pi/3}(-1+2\cos(\theta))^2\,d\theta + \frac{1}{2}\int_{5\pi/3}^{2\pi}(-1+2\cos(\theta))^2\,d\theta$

**5. (continued)** For your convenience, the polar graph referenced by this problem is reproduced here:




**d**. [4 points] Write, but do not evaluate, an expression involving one or more integrals which gives the **arc length** of the **solid** portion of the graph.

*Solution:* The arclength of the graph is given by

$$\int_{\pi/3}^{5\pi/3}\sqrt{(r(\theta))^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

We have $dr/d\theta = -2\sin(\theta)$. This gives the answer below.

The arc length is $\int_{\pi/3}^{5\pi/3}\sqrt{(-1+2\cos(\theta))^2+(-2\sin(\theta))^2}\,d\theta$