Math 116 — Practice for Exam 3

Generated April 13, 2023

Name: ____

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2021	3	3		7	
Fall 2021	3	4		6	
Fall 2021	3	6		13	
Fall 2021	3	7	fan	17	
Fall 2021	3	8	bumper cars	16	
Total				59	

Recommended time (based on points): 71 minutes

2. [5 points] Suppose that the power series $\sum_{n=1}^{\infty} C_n(x+2)^n$ converges at $x = 4$ and diverges at $x = -10$. What can you say about the behavior of the power series at the following values of x?

a. [1 point] At $x = 0$, the power series...

CONVERGES DIVERGES CANNOT DETERMINE

b. [1 point] At $x = -8$, the power series...

CONVERGES DIVERGES CANNOT DETERMINE

c. [1 point] At $x = 8$, the power series...

CONVERGES DIVERGES CANNOT DETERMINE

d. [1 point] At $x = -4$, the power series...

CONVERGES DIVERGES CANNOT DETERMINE

e. [1 point] At $x = 6$, the power series...

CONVERGES DIVERGES CANNOT DETERMINE

3. [7 points] A function $F(x)$ has Taylor series given by

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^n(n^2+1)}(x-1)^{4n+1}$$

Answer the following questions regarding the Taylor series:

a. [3 points] Circle the appropriate answer. No work is needed, but partial credit may be given for correct work.

At $x = 1$, F is... INCREASING DECREASING CANNOT DETERMINE

b. [4 points] What is $F^{(2021)}(1)$? Give your answer in exact form and do not try to simplify. Show your work.

$F^{(2021)}(1) =$ ________________________.

4. [6 points] Find the **radius of convergence** of the following power series:

$$\sum_{n=0}^{\infty} \frac{8^n (n!)^3}{(3n)!}(x-5)^{3n}.$$

Show your work including full justifications of any tests you use.

The radius of convergence is ________________________________.

6. [13 points] Values of a function $g(x)$ and some of its derivatives at $x = 2$ are given in the table below. Use this information for some of the problems below.

$g(2)$	$g'(2)$	$g''(2)$	$g'''(2)$	$g^{(4)}(2)$
1	2	-4	0	4

a. [4 points] Find the first 4 nonzero terms of the Taylor series of $g(x)$ about $x = 2$. Write your final answer as a polynomial $P(x)$ in the blank below.

$P(x) =$ ______________________________

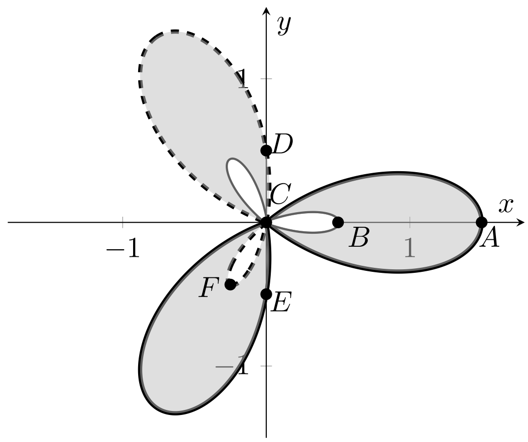
b. [4 points] Using known Taylor series, find the first 3 nonzero terms of the Taylor series of $f(x) = (x-2)\ln\left(\frac{x}{2}\right)$ about $x = 2$. Write your final answer as a polynomial $Q(x)$ in the blank below. (*Hint:* $f(x) = (x-2)\ln\left(1 + \frac{(x-2)}{2}\right)$)

$Q(x) =$ ______________________________

c. [5 points] Let $H(x) = 1 + \int_2^x f(t) + g(t)dt$. Find the first 4 nonzero terms of the Taylor series of H about $x = 2$. Write your final answer as a polynomial $R(x)$ in the blank below. Partial credit may be given for finding the appropriate terms of $\int_2^x f(t)dt$ or $\int_2^x g(t)dt$.

$R(x) =$ ______________________________

7. [17 points] John is holding a Fan Fair to celebrate the success of his burgeoning fan business. At the fair, John is debuting his new fan, which has blades given by the shaded region of the graph of the polar equation $r = \cos(3\theta) + \frac{1}{2}$ shown below. Note that the graph of $r = \cos(3\theta) + \frac{1}{2}$ is comprised of both the inner and outer loops of the fan blades. One of the activities at the Fan Fair is to guess the perimeter and area of the blades, which can actually be computed explicitly.



a. [4 points] For the values of θ listed below, write on the line the letter of the point corresponding to it.

$\theta = 0$: __________ $\theta = \frac{\pi}{3}$: __________

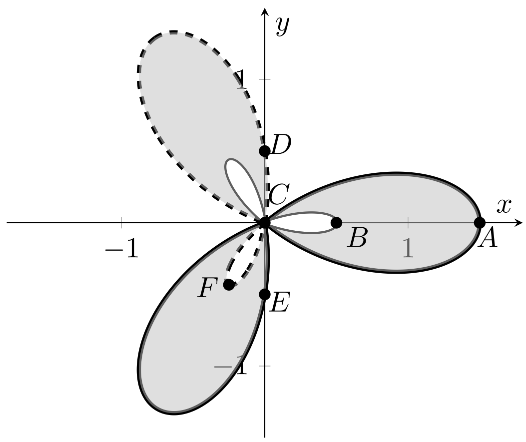
$\theta = \frac{\pi}{2}$: __________ $\theta = \pi$: __________

b. [5 points] Find the 3 values of θ which correspond to the point C (the origin) for $0 \le \theta \le \pi$. Then, determine the interval within $[0, 2\pi]$ for which θ traces out the **dashed** loops in the graph above. (*Hint:* $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$)

$\theta =$ __________, __________, __________

Interval giving θ-values that trace out the dashed loops: __________

7. (continued) Here is a reproduction of the graph from the previous page of the polar equation $r = \cos(3\theta) + \frac{1}{2}$:



c. [4 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total perimeter of the fan blades, including both the inner and outer edges of the fan blades.

Total Perimeter = ____

d. [4 points] Write, but do not evaluate, an expression giving the total area of all 3 fan blades (the shaded region of the graph). (*Hint*: Your answer from (b) may be handy, but is not strictly necessary)

Total Area = ____

8. [16 points] Molly and Erin are two bumper car enthusiasts who hate bumping into things. So, they get on the bumper cars, ride until they bump into each other, and then stop riding the bumper cars. At time t minutes after they start driving their bumper cars, Molly's position is given by

$$M(t) = \left(4 - 6\cos t, 3t + \pi^2\right)$$

and Erin's position is given by

$$E(t) = \left(2\cos t, (t - 2\pi)^2 + 3t + \frac{8\pi^2}{9}\right),$$

where all distances are in meters.

a. [4 points] How long do they ride the bumper cars? Make sure to include units.

b. [4 points] Find an explicit expression for Erin's speed t minutes after she starts driving her bumper cars, before the collision (your expression should not contain any integrals nor the letters M, E). Make sure to include units.

c. [4 points] Write, but do not evaluate, an expression involving integrals that gives the total distance that Erin travelled before the collision. Make sure to include units.

d. [4 points] Suppose the positive y-direction in the xy-plane is North. At $t = 0$, Molly is facing directly North. Find all other times $t > 0$ (if any) after they start riding their bumper cars, but before their collision, when Molly is facing directly North. Make sure to include units.

2. [15 points] The parts of this problem are unrelated to each other. Be sure to show work for all parts, and circle your final answer.

a. [5 points] A leaking bag of sugar is lifted vertically from the ground to a height of 10 feet above the ground. The **weight** of the bag of sugar is $6 - \sqrt{x}$ lbs when it has been lifted x feet above the ground. Find the work done lifting the bag, including units. Fully evaluate any integrals, but you do not need to simplify your answer.

Answer: ______________________

b. [5 points] Write an expression involving one or more integrals that gives the volume of the solid obtained by rotating the region in the xy-plane bounded between the x-axis, the parabola $y = x^2 + 1$, the line $x = -1$ and the line $x = 1$, about the line $x = -2$. Do not evaluate your integral(s).

Answer: ______________________

c. [5 points] The function $f(x) = x^4 + 5$ can be rewritten in the form $f(x) = (x+1)^4 + A(x+1)^3 + B(x+1)^2 + C(x+1) + D$, where A, B, C, D are constants. Find the values of A, B, C, D using Taylor series. Other methods used to find the constants will not be given credit.

A= __________

B= __________

C= __________

D= __________

3. [13 points] A function $g(x)$ has Taylor series centered at $x = 5$ given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^{n+1}}{(n+1) \cdot 4^n}.$$

a. [2 points] Is $g(x)$ increasing or decreasing near $x = 5$? Briefly justify your answer.

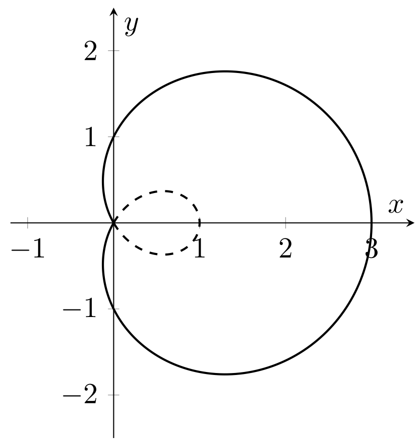
b. [3 points] Find $g^{(1001)}(5)$.

$g^{(1001)}(5)=$ ____________

c. [8 points] Given that the radius of convergence of this Taylor series is 4 (do NOT show this), find the **interval** of convergence of this Taylor series. Show all your work, including full justification for series behavior.

Interval of convergence: ____________

5. [11 points] The parts of this question relate to the following polar graph, defined by the polar curve $r(\theta) = -1 + 2\cos(\theta)$, on the domain $[0, 2\pi]$. Both the solid and dashed curves are part of the graph of $r(\theta)$.



a. [2 points] What are all the angles θ, with $0 \leq \theta \leq 2\pi$, for which the graph passes through the origin?

Answer(s): ______________________

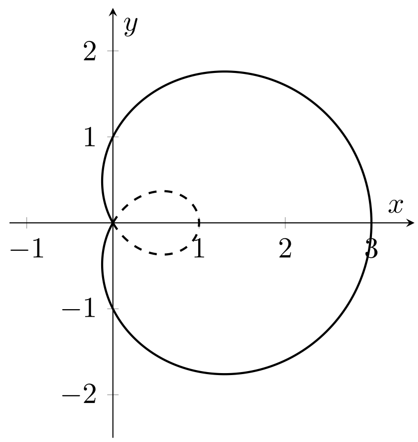
b. [2 points] Determine the interval(s) within $[0, 2\pi]$ for which θ traces out the **dashed** portion of the graph.

Answer(s): ______________________

c. [3 points] Write, but do not evaluate, an expression involving one or more integrals which gives the **area** enclosed by the **dashed** portion of the graph.

The area is ______________________

5. (continued) For your convenience, the polar graph referenced by this problem is reproduced here:



d. [4 points] Write, but do not evaluate, an expression involving one or more integrals which gives the **arc length** of the **solid** portion of the graph.

The arc length is ______________________________

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$