Math 116 — Practice for Exam 3

Generated December 7, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: ________

1. This exam has 11 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	3	6		7	
Fall 2016	3	7	Legendre	7	
Fall 2015	3	4		8	
Fall 2014	3	2		6	
Winter 2012	2	9	wire	14	
Winter 2014	2	5	telemarketing	7	
Fall 2015	3	3	rumor	13	
Winter 2016	2	8	hiking	13	
Winter 2014	2	3		11	
Fall 2013	3	6		6	
Winter 2013	2	2		13	
Total				105	

Recommended time (based on points): 110 minutes

6. [7 points] Consider the function F defined by $F(x) = \int_{-\infty}^{x} e^{-t^2}\,dt$.

For each value of x, the right hand side is an improper integral that converges. The function $F(x)$ is an antiderivative of e^{-x^2}. (You do not need to verify this.)

a. [4 points] It can be shown that $F(0) = \frac{\sqrt{\pi}}{2}$. Using this fact, write the first four nonzero terms of the Taylor series for the function $F(x)$ centered at $x = 0$.

$$F(x) \qquad \boxed{\frac{\sqrt{\pi}}{2} + x - \frac{1}{3}x^3 + \frac{1}{10}x^5} + \dots$$

$$\frac{d}{dx}\downarrow \qquad \uparrow \int dx$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2} + \dots$$

b. [3 points] Use your answer from part **a.** to approximate the integral $\int_0^1 e^{-t^2}\,dt$.

$$\int_0^1 e^{-t^2}\,dt = \int_{-\infty}^{1} e^{-t^2}\,dt - \int_{-\infty}^{0} e^{-t^2}\,dt$$

$$= \left[\frac{\sqrt{\pi}}{2} + (1) - \frac{1}{3}(1)^3 + \frac{1}{10}(1)^5\right] - \left[\frac{\sqrt{\pi}}{2}\right]$$

$$= 1 - \frac{1}{3} + \frac{1}{10} = \frac{30}{30} - \frac{10}{30} + \frac{3}{30} = \boxed{\frac{23}{30}}$$

7. [6 points]

a. [3 points] Find the interval of convergence of the power series $\sum_{k=3}^{\infty} \frac{x^k}{(k^2+1)3^k}$.

You do not need to show your work.

Ratio test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{((k+1)^2+1)3^{k+1}} \cdot \frac{(k^2+1)3^k}{x^k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\right| \frac{(k^2+1)}{(k^2+2k+2)} \cdot \frac{3^k}{3^{k+1}}$

$$= \frac{|x|}{3} < 1 \iff |x| < 3 \iff -3 < x < 3$$

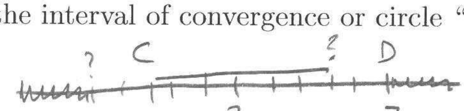
$x = -3$: $\sum \frac{(-3)^k}{(k^2+1)3^k} = \sum \frac{(-1)^k}{k^2+1}$ converges by AST

$x = 3$: $\sum \frac{3^k}{(k^2+1)3^k} = \sum \frac{1}{k^2+1}$ converges by comparison, p-test ($p=2$).

Answer: Interval of Convergence = $[-3, 3]$

b. [3 points] Consider the power series $\sum_{j=0}^{\infty} C_j (x-2)^j$.

This power series converges when $x = -1$ and diverges when $x = 7$.
Which, if any, of the following intervals could be exactly equal to the interval of convergence for this power series? Circle all the intervals below that could be exactly equal to the interval of convergence or circle "NONE OF THESE".



$[-2, 6]$ $\qquad$ ~~$[-2, 7)$~~ $\qquad$ $[-3, 7)$ $\qquad$ ~~$(-3, 7]$~~ $\qquad$ ~~$[-1, 4]$~~ $\qquad$ ~~$[-4, 7)$~~ $\qquad$ NONE OF THESE

7. [7 points] The *Legendre equation* is a differential equation that arises in the quantum mechanical study of the hydrogen atom. In one of its forms, the Legendre equation is

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

For this problem, let y be a solution to the Legendre equation satisfying $y(\frac{1}{2}) = 2$ and $y'(\frac{1}{2}) = 3$. Assume that the Taylor series for $y(x)$ about $x = \frac{1}{2}$ converges to $y(x)$ for all $-\frac{1}{2} < x < \frac{3}{2}$.

a. [4 points] In the blank below, write down $P_2(x)$, the degree 2 Taylor polynomial of $y(x)$ near $x = \frac{1}{2}$. Your answer should not contain the function $y(x)$ or any of its derivatives.

$$P_2(x) = 2 + 3\left(x - \frac{1}{2}\right) - 14\left(x - \frac{1}{2}\right)^2$$

b. [3 points] Compute the limit

$$\lim_{x \to 1/2} \frac{y(x) - \frac{1}{2} - 3x}{(x - \frac{1}{2})^2}.$$

Solution: The limit is -14.

4. [8 points] Let $f(x) = \sqrt[3]{1+2x^2}$.

a. [5 points] Find the first 3 nonzero terms of the Taylor series for f centered at $x = 0$.

Solution: Using the Taylor series for $(1+y)^{1/3}$ centered at $y = 0$,

$$\sqrt[3]{1+y} = 1 + \frac{1}{3}y + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}y^2 + \dots$$
$$= 1 + \frac{y}{3} - \frac{y^2}{9} + \dots$$

Substituting $y = 2x^2$,

$$\sqrt[3]{1+2x^2} = 1 + \frac{2x^2}{3} - \frac{4x^4}{9} + \dots$$

b. [3 points] For what values of x does the Taylor series converge?

Solution: The binomial series for $\sqrt[3]{1+y}$ converges when $-1 < y < 1$. Substituting $y = 2x^2$, this converges when $1 < 2x^2 < 1$, or $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

5. [3 points] Determine the **exact** value of the infinite series

$$-1 + \frac{1}{3!} - \frac{1}{5!} + \dots + \frac{(-1)^{n+1}}{(2n+1)!} + \dots$$

No decimal approximations are allowed. You **do not** need to show your work.

Solution:

$$-1 + \frac{1}{3!} - \frac{1}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{(2n+1)!} = \sin(-1).$$

2. [6 points] Let $f(x) = xe^{-x^2}$.

a. [4 points] Find the Taylor series of $f(x)$ centered at $x = 0$. Be sure to include the first 3 nonzero terms and the general term.

Solution: We can use the Taylor series of e^y to find the Taylor series for e^{-x^2} by substituting $y = -x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$

Therefore the Taylor series of xe^{-x^2} is

$$xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} = x - x^3 + \frac{x^5}{2!} + \cdots + \frac{(-1)^n x^{2n+1}}{n!} + \cdots$$

b. [2 points] Find $f^{(15)}(0)$.

Solution: We know that $\frac{f^{(15)}(0)}{15!}$ will appear as the coefficient of the degree 15 term of the Taylor series. Using part (**a**), we see that the degree 15 term has coefficient $\frac{-1}{7!}$. Therefore

$$f^{(15)}(0) = \frac{-15!}{7!} = -259,459,200$$

3. [3 points] Determine the exact value of the infinite series

$$1 - \frac{2}{1!} + \frac{4}{2!} - \frac{8}{3!} + \cdots + \frac{(-1)^n 2^n}{n!} + \cdots$$

Solution: Notice that this is the Taylor series for e^y applied to $y = -2$. Therefore, the series has exact value e^{-2}.

9. [14 points] A machine produces copper wire, and occasionally there is a flaw at some point along the wire. The length x of wire produced between two consecutive flaws is a continuous variable with probability density function

$$f(x) = \begin{cases} c(1+x)^{-3} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work in order to receive full credit.

a. [5 points] Find the value of c.

Solution: Since $f(x)$ is a density function $\int_{-\infty}^{\infty} f(x)dx = 1$. Then

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} c(1+x)^{-3}dx = \lim_{b\to\infty} \int_0^b c(1+x)^{-3}dx$$
$$= \lim_{b\to\infty} \frac{-c}{2(1+x)^2}\Big|_0^b = \frac{c}{2} = 1$$

Hence $c = 2$.

b. [3 points] Find the cumulative distribution function $P(x)$ of the density function $f(x)$. Be sure to indicate the value of $P(x)$ for **all** values of x.

Solution:

$$P(x) = \int_{-\infty}^{x} f(t)dt = \int_0^x c(1+t)^{-3}dt = \frac{-c}{2(1+t)^2}\Big|_0^x = \frac{c}{2} - \frac{c}{2(1+x)^2} = 1 - \frac{1}{(1+x)^2}.$$

$$P(x) = \begin{cases} 1 - \frac{1}{(1+x)^2} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

c. [5 points] Find the mean length of wire between two consecutive flaws.

Solution:

$$\text{mean} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} c\frac{x}{(1+x)^3}dx = c\lim_{b\to\infty}\int_0^b \frac{x}{(1+x)^3}dx$$

$$u = x \qquad v' = (1+x)^{-3}$$

$$u' = 1 \qquad v = \frac{-1}{2(1+x)^2}$$

$$= c\lim_{b\to\infty} \frac{-x}{2(1+x)^2}\Big|_0^b + \int_0^b \frac{1}{2(1+x)^2}dx$$

$$= c\lim_{b\to\infty} \frac{-x}{2(1+x)^2} - \frac{1}{2(1+x)}\Big|_0^b = \frac{c}{2} = 1.$$

or

$$\text{mean} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} c\frac{x}{(1+x)^3}dx = c\lim_{b\to\infty}\int_0^b \frac{x}{(1+x)^3}dx$$

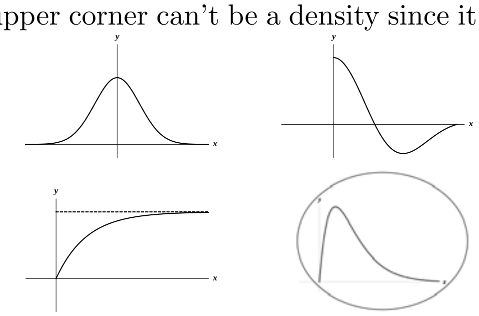
$$u = 1+x$$

$$= c\lim_{b\to\infty}\int_1^{b+1} \frac{u-1}{u^3}dx = c\lim_{b\to\infty}\int_1^{b+1} u^{-2} - u^{-3}dy$$

$$= c\lim_{b\to\infty} -u^{-1} + \frac{u^{-2}}{2}\Big|_1^{b+1} = \frac{c}{2} = 1.$$

d. [1 point] A second machine produces the same type of wire, but with a different probability density function (pdf). Which of the following graphs could be the graph of the pdf for the second machine? Circle all your answers.

Solution: The graph on the left upper corner can't be the density since$x$ is the distance between flaws, hence the probability density function can't be positive for $x < 0$.
The graph on the left lower corner can't be the density since the area under the curve for $x \geq 0$ is infinite (it has a positive horizontal asymptote).
The graph on the right upper corner can't be a density since it is negative on an interval.



5. [7 points] The Terrible Telemarketing corporation has realized that people often hang up on their telemarketing calls. After collecting data they found that the probability that someone will hang up the phone at time t seconds after the call begins is given by the probability density function $p(t)$. The formula for $p(t)$ is given below.

$$p(t) = \begin{cases} 0 & t < 0 \\ te^{-ct^2} & t \geq 0 \end{cases}$$

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a. [5 points] Find the value of c so that $p(t)$ is a probability density function.

Solution:
We must have $\int_0^\infty te^{-ct^2}\,dt = 1$. Let $u = ct^2$ then $du = 2ctdt$. Thus we get the integral $\frac{1}{2c}\int_0^\infty e^{-u}du = \frac{1}{2c}\lim_{N\to\infty} -e^{-u}|_0^N = \frac{1}{2c}$. Thus we have $1 = \frac{1}{2c}$ so $c = \frac{1}{2}$.

b. [2 points] What is the probability that someone will stay on the phone with a telemarketer for more than 4 seconds?

Solution:
The probability is $\int_4^\infty te^{-\frac{1}{2}t^2}\,dt = e^{-8}$.

6. [6 points] Consider the probability density function $q(t)$ shown below.

$$q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2} & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

a. [4 points] What is the cumulative distribution function $Q(t)$ of the density given by $q(t)$? Write your final answer in the answer blanks provided.

Solution:
$Q(t) = \int_{-\infty}^t q(s)\,ds = \int_0^t q(s)\,ds$. If $t < 0$ then $Q(t) = 0$ and if $0 \leq t < 2$ then $Q(t) = \int_0^t q(s)\,ds = \int_0^t \frac{s}{2}\,ds = t^2/4$. Then it follows that $Q(t) = 1$ for $t \geq 2$.

If $t < 0$ then $Q(t) = 0$

If $0 \leq t < 2$ then $Q(t) = t^2/4$

If $t \geq 2$ then $Q(t) = 1$

b. [2 points] What is the median of the distribution?

Solution: The median is the number T such that $Q(T) = \frac{1}{2}$. Thus we want $T^2/4 = \frac{1}{2}$. Therefore $T = \sqrt{2}$.

3. [13 points]

a. [4 points] The number of people R that have heard a rumor increases at a rate proportional to the product of the number of people that have heard the rumor and the number of people that haven't yet heard the rumor. Write a differential equation for R which models the scenario described assuming that the total number of people is 1,000. Use $k > 0$ for the constant of proportionality.

Solution: The number of people that have heard the rumor is R, so the number of people that have not yet heard the rumor is $1000 - R$. Thus the differential equation is $\frac{dR}{dt} = kR(1000 - R)$.

$$\frac{dR}{dt} = \underline{\qquad kR(1000-R) \qquad}$$

b. [4 points] For what values of A, B is $y(t) = At\cos t + Bt$ a solution to the differential equation $ty' = y + t^2 \sin t$ satisfying the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$? Be sure to show your work.

Solution: Since $y'(t) = A\cos t - At\sin t + B$, $y(t)$ is a solution if

$$t(A\cos t - At\sin t + B) = At\cos t + Bt + t^2\sin t \qquad \Rightarrow \qquad -At^2\sin t = t^2 \sin t$$

Thus $A = -1$. Plugging in the initial condition $y\left(\frac{\pi}{2}\right) = 2\pi$,

$$-\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + B\left(\frac{\pi}{2}\right) = 2\pi.$$

Thus $B = 4$.

$A =$ **-1**

$B =$ **4**

c. [5 points] Find the solution to the differential equation

$$e^{-x} + y^2\frac{dy}{dx} = 0, \quad \text{with initial condition} \quad y(0) = 2.$$

Solution: Moving the e^{-x} to the right side of the equation and separating variables,

$$\int y^2 dy = \int -e^{-x} dx$$
$$\frac{1}{3}y^3 = e^{-x} + C$$
$$y = \sqrt[3]{3e^{-x} + C}$$

Plugging in the initial condition $y(0) = 2$, $2 = \sqrt[3]{3 + C}$. Therefore $C = 5$.

$$y = \underline{\qquad \sqrt[3]{3e^{-x} + 5} \qquad}$$

8. [13 points] Brianne is hiking, and the temperature of the air in °C after she's traveled x km is a solution to the differential equation

$$y' + y \sin x = 0$$

a. [7 points] Find the general solution of the differential equation.

Solution: Writing the equation as $\frac{dy}{dx} = -y \sin x$ and separating the variables we get

$$\int \frac{1}{y} dy = \int -\sin x dx$$

$$\ln |y| = \cos x + C$$

$$y = Ae^{\cos x}$$

b. [2 points] If the temperature was 10 °C at the beginning of the hike, find $T(x)$, the temperature of the air in °C after she's traveled x km. Show your work.

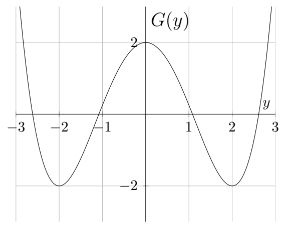
Solution: From (a), $T(x) = Ae^{\cos x}$. Since $T(0) = 10$ we get $A = \frac{10}{e}$. Thus,

$$T(x) = \frac{10}{e} e^{\cos x}$$

c. [4 points] Brianne traveled 7 km on the hike. Using the information given in (b), find the coldest air temperature she encountered on the hike. Give an **exact** answer (i.e. no decimal approximations).

Solution: We want to find the minimum of the function $T(x) = \frac{10}{e} e^{\cos x}$ over the interval $[0, 7]$. The critical points are $0, \pi, 2\pi$. Checking the outputs of T at those points and the endpoint 7 we find that the minimum is $T(\pi) = \frac{10}{e^2}$.

3. [11 points] The graph of $G(y)$ is shown below. Suppose that $G'(y) = g(y)$. Consider the differential equation $\dfrac{dy}{dt} = g(y)$.



Note again that $\frac{dy}{dt} = g(y)$ and the given graph depicts $G(y)$ **not** $g(y)$.

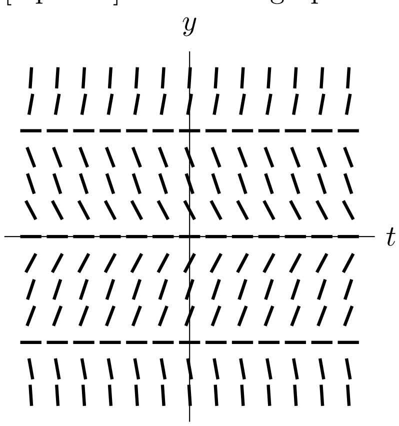
a. [6 points] The differential equation has 3 equilibrium solutions. Find the 3 solutions and indicate whether they are stable or unstable by circling the correct answer.

Equilibrium solution 1: -2 **Stable** $\boxed{\textbf{Unstable}}$

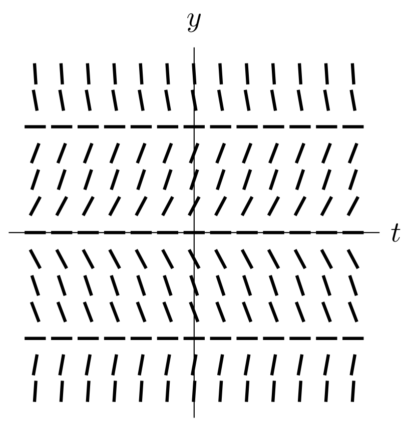
Equilibrium solution 2: 0 $\boxed{\textbf{Stable}}$ **Unstable**

Equilibrium solution 3: 2 **Stable** $\boxed{\textbf{Unstable}}$

b. [2 points] Circle the graph that could be the slope field of the above differential equation.



$\boxed{\text{I}}$



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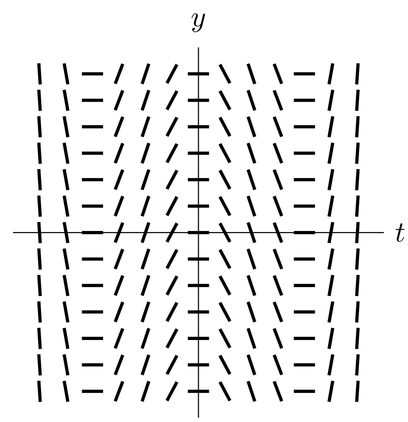


III

c. [3 points] Suppose $y_1(t), y_2(t)$ and $y_3(t)$ are all solutions of the differential equation with different initial conditions as indicated below:

- $y_1(t)$ solves the differential equation with initial condition $y(0) = -2$.
- $y_2(t)$ solves the differential equation with initial condition $y(0) = 1.5$.
- $y_3(t)$ solves the differential equation with initial condition $y(0) = -2.1$.

Compute the following limits:

$\lim\limits_{t\to\infty} y_1(t) = -2$ $\lim\limits_{t\to\infty} y_2(t) = 0$ $\lim\limits_{t\to\infty} y_3(t) = -\infty$ or DNE

6. [6 points] Consider the following differential equation

$$\frac{dy}{dx} = (x-y)(y-2)$$

a. [2 points] Find all the equilibrium solutions of the differential equation (if any). If the differential equation has no equilibrium solutions, write none.

Solution: $y = 2$

b. [4 points] Use inequalities to describe the regions in the slope field of the differential equation where the solution curves are increasing.

Solution: The regions in the slope field in which the solution curves are increasing can be determined by finding where

$$\frac{dy}{dx} = (x-y)(y-2) > 0.$$

Region 1: $x - y > 0$ and $y - 2 > 0$. In other words $x > y$ and $y > 2$.

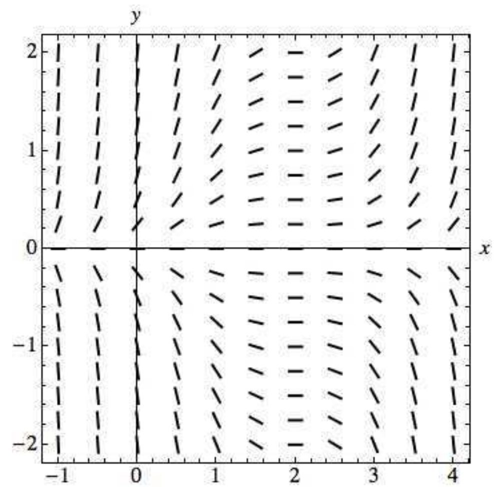
Region 2: $x - y < 0$ and $y - 2 < 0$, or $x < y$ and $y < 2$

2. [13 points]

a. [7 points] Consider the following differential equations:

A. $y' = y(x-2)^2$ B. $y' = y(x-2)$ C. $y' = -y(1-y)$ D. $y' = -y^2(1-y)$

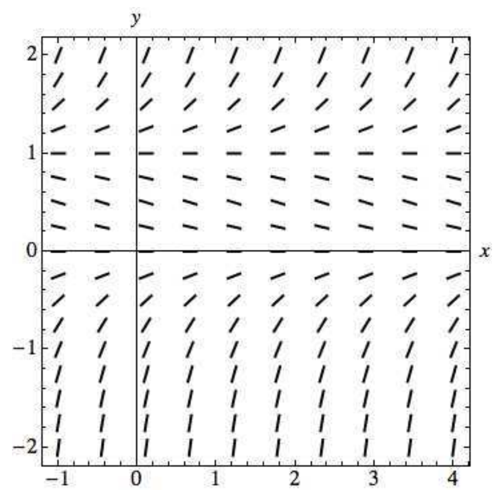
Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line. Find the equation of the equilibrium solutions and their stability. If a slope field has no equilibrium solutions, write none.



Differential equation: **A**

Equilibrium solutions and stability:

$y = 0$ **unstable**.



Differential equation: **C**

Equilibrium solutions and stability:

$y = 1$ **unstable**.

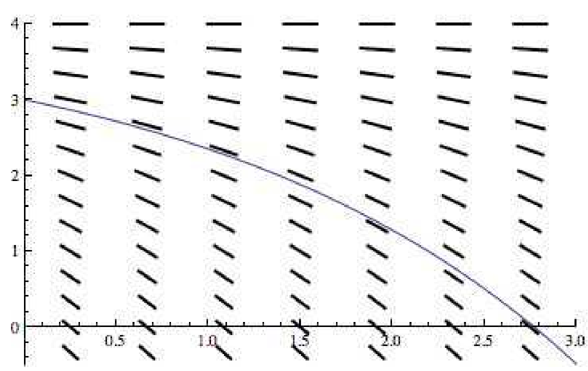
$y = 0$ **stable**.

b. [4 points] A bank account earns a p percent annual interest compounded continuously. Continuous payments are made out of the account at a rate of q thousands of dollars per year. Let $B(t)$ be the amount of money (**in thousands of dollars**) in the account t years after the account was opened. Write the differential equation satisfied by $B(t)$.

Solution:

$$\frac{dB}{dt} = \frac{p}{100}B - q.$$

c. [2 points] The slope field shown below corresponds to the differential equation satisfied by $B(t)$ (for certain values of p and q). Sketch on the slope field below the solution to the differential equation that corresponds to an account opened with an initial deposit of 3,000 dollars.



Solution: