MATH 116 — PRACTICE FOR EXAM 3

Generated November 19, 2017

NAME: SOLUTIONS

INSTRUCTOR: ______________________ SECTION NUMBER: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2012	3	9		10	
Fall 2013	3	8		12	
Winter 2014	3	3		10	
Winter 2015	3	4	Bessel	9	
Total				41	

Recommended time (based on points): 49 minutes

9. [10 points] A second way to approximate the function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

is by using its Taylor polynomials.

a. [2 points] Find the first three nonzero terms in the Taylor series for the function $\sqrt{1+u}$ about $u = 0$.

Solution:

$$\sqrt{1+u} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2$$

(using binomial theorem –
$(1+u)^p = 1 + pu + \frac{p(p-1)}{2}u^2 + \dots$ for $|u| < 1$,
and substituting $p = \frac{1}{2}$)

b. [2 points] Find the first three nonzero terms in the Taylor series for $\sqrt{1+9t^4}$ about $t = 0$.

Solution: Let $u = 9t^4$ then

$$\sqrt{1+9t^4} \approx 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8$$

c. [2 points] Find the first three nonzero terms in the Taylor series for $F(x)$ about $x = 0$.

Solution:

$$F(x) \approx \int_0^x 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8 dt = x + \frac{9}{10}x^5 - \frac{9}{8}x^9$$

d. [2 points] For which values of x do you expect the Taylor series for $F(x)$ about $x = 0$ to converge? Justify your answer.

Solution: We substituted $u = 9t^4$ into the Binomial series. The interval of convergence for the Binomial series is $-1 < u < 1$. Then we expect the series to converge for $0 \leq 9x^4 < 1$. Hence the Taylor series for $F(x)$ about $x = 0$ converges if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

e. [2 points] Use the fifth degree Taylor polynomial for $F(x)$ about $x = 0$ to approximate the value of $F(\frac{1}{2})$.

Solution: $P_5(x) = x + \frac{9}{10}x^5$, then $F\left(\frac{1}{2}\right) \approx P_5\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{9}{10}\left(\frac{1}{2}\right)^5 = 0.528$

8. [12 points]

a. [4 points] Let a be a positive constant. Determine the first three nonzero terms of the Taylor series for

$$f(x) = \frac{1}{(1+ax^2)^4}$$

centered at $x = 0$. Show all your work. Your answer may contain a.

Solution: Using the binomial series $(1+u)^p$ with $u = ax^2$ and $p = -4$

$$f(x) = \frac{1}{(1+ax^2)^4} \approx 1 + pu + \frac{p(p-1)}{2}u^2 = 1 - 4ax^2 + 10a^2x^4$$

b. [2 points] What is the radius of convergence of the Taylor series for $f(x)$? Your answer may contain a.

Solution: Since the interval of convergence of the binomial series is $-1 < u < 1$, then the interval of convergence of the series for $f(x)$ is $-1 < ax^2 < 1$, or $-\sqrt{\frac{1}{a}} < x < \sqrt{\frac{1}{a}}$. Hence the radius of convergence is $R = \sqrt{\frac{1}{a}}$.

c. [3 points] Determine the first three nonzero terms of the Taylor series for

$$g(t) = \int_0^t \frac{1}{(1+ax^2)^4}dx,$$

centered at $t = 0$. Show all your work. Your answer may contain a.

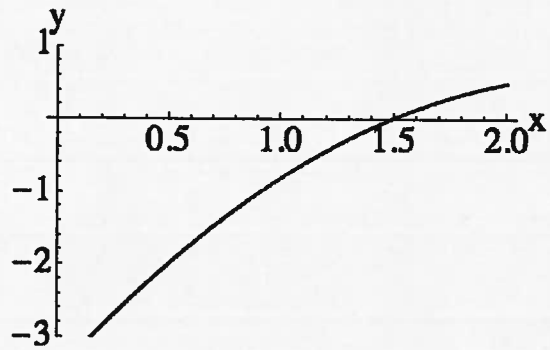
Solution:

$$g(t) = \int_0^t \frac{1}{(1+ax^2)^4}dx \approx \int_0^t 1 - 4ax^2 + 10a^2x^4 dx = x - \frac{4}{3}ax^3 + 2a^2x^5|_0^t = t - \frac{4}{3}at^3 + 2a^2t^5$$

d. [3 points] The degree-2 Taylor polynomial of the function $h(x)$, centered at $x = 1$, is

$$P_2(x) = a + b(x-1) + c(x-1)^2.$$

The following is a graph of $h(x)$:



What can you say about the values of a, b, c? You may assume a, b, c are nonzero. Circle your answers. No justification is needed.

Solution:

a is:	Positive	**NEGATIVE**	Not enough information
b is:	**POSITIVE**	Negative	Not enough information
c is:	Positive	**NEGATIVE**	Not enough information

Note: $a = h(1)$ which by the graph is ≈ -1.

$b = h'(1)$ which by the graph is positive.

$c = \frac{h''(1)}{2}$ which is negative since the graph is concave down at $x=1$.

Here we're using that the coeff. of $(x-1)^n$ in the Taylor series of $h(x)$ about $x=1$ is $\frac{h^{(n)}(1)}{n!}$.

3. [10 points] For each of the following questions circle the correct answer.

a. [2 points] What is the value of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!}$?

(This is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ at $x = -2^2 = -4$.)

this is e^x

$\cos(2)$ $\quad$ e^{-2} $\quad$ $\cos(4)$ $\quad$ $\boxed{e^{-4}}$

b. [2 points] What is the value of the series $\sum_{n=1}^{\infty} \frac{2^{2n}(-1)^n}{(2n+1)!}$?

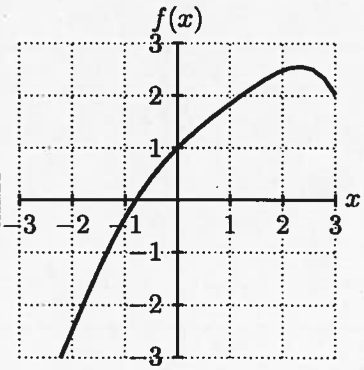
(This is $\sum_{n=1}^{\infty} \frac{x^{2n}(-1)^n}{(2n+1)!}$ at $x=2$

this is $\frac{1}{x}\sum_{n=1}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1)!}$ $\|$ $\frac{1}{x}(\sin x - x)$)

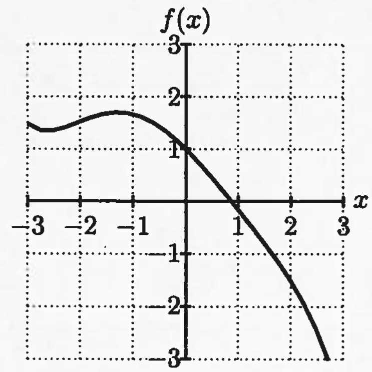
$\frac{1}{2}\sin(2)$ $\quad$ $\sin(2)-2$ $\quad$ $\sin(2)$ $\quad$ $\boxed{\frac{1}{2}(\sin(2)-2)}$

c. [2 points] Suppose that $1 + x - \frac{1}{4}x^2 + \frac{1}{10}x^3$ is the 3rd degree Taylor polynomial for a function $f(x)$. Which of the following pictures could be a graph of $f(x)$?

$\boxed{\text{I}}$

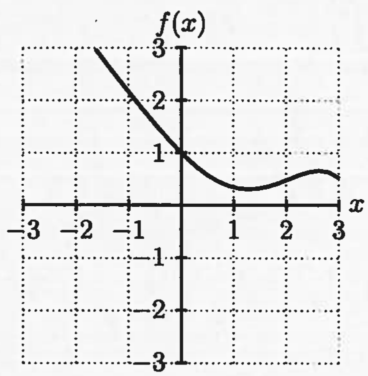


II

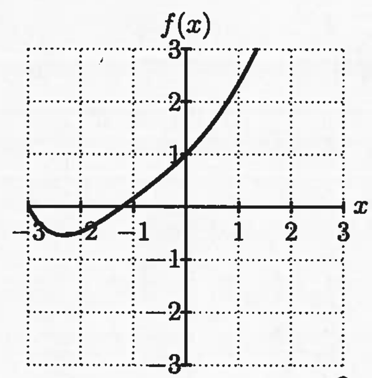


Know:
$f'(0) = 1$
(ruling out II and III)
$f''(0) = -\frac{1}{2}$
(ruling out IV)

III



IV



d. [2 points] What is the Taylor series of $2xe^{x^2}$ centered at $x = 0$?

[$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, substitute x^2 for x to get
$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$
and then multiply by $2x$]

$\boxed{\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}}$ $\quad$ $\sum_{n=1}^{\infty} \frac{2x^{2n-1}}{n!}$ $\quad$ $\sum_{n=1}^{\infty} \frac{2x^{2n+1}}{(n-1)!}$ $\quad$ $\sum_{n=0}^{\infty} \frac{2x^{2n-1}}{n!}$

e. [2 points] The radius of convergence of the Taylor series $\sum_{n=1}^{\infty} \frac{(x+5)^n 5^{-n}}{n+5}$ is $R = 5$. What is the interval of convergence of the series?

$\boxed{[-10, 0)}$ $\quad$ $(-10, 0)$ $\quad$ $(0, 10]$ $\quad$ $[-10, 0]$ $\quad$ $[0, 10)$

Power series centered at -5, so interval of convergence goes from -10 to 0. At -10 it's $\sum_{n=1}^{\infty} \frac{(-5)^n 5^{-n}}{n+5} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$, converges by AST since $\lim_{n\to\infty} \frac{1}{n+5} = 0$ and $\frac{1}{n+6} < \frac{1}{n+5}$; at 0 it's $\sum_{n=1}^{\infty} \frac{1}{n+5} = \sum_{n=6}^{\infty} \frac{1}{n}$, diverges by p-test ($p=1$).

4. [9 points] We can define the Bessel function of order one by its Taylor series about $x = 0$,

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}.$$

a. [3 points] Compute $J_1^{(2015)}(0)$. Write your answer in exact form and do not try evaluate using a calculator.

Solution: The 2015th derivative of J_1 at $x = 0$ corresponds to 2015! times the 1007th coefficient in the Taylor series above. This gives us that $J_1^{(2015)}(0) = \dfrac{-(2015)!}{(1007)!(1008)!2^{2015}}$.

since $n=1007$ $\Downarrow$ $2n+1=2015$

$$J_1^{(2015)}(0) = \underline{\frac{--(2015)!}{(1007)!(1008)!2^{2015}}}$$

b. [4 points] Find $P_5(x)$, the Taylor polynomial of degree 5 that approximates $J_1(x)$ near $x = 0$.

Solution: $P_5(x) = \dfrac{x}{2} - \dfrac{x^3}{16} + \dfrac{x^5}{384}$

these are just the terms of degree ≤ 5 in the power series defining $J_1(x)$.

$$P_5(x) = \underline{\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}}$$

c. [2 points] Use the Taylor polynomial from the previous part to compute $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3}$.

Solution: $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \lim_{x\to 0} \frac{-\frac{x^3}{16} + \frac{x^5}{384}}{x^3} = -\frac{1}{16}$

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \underline{-\frac{1}{16}}$$