Math 116 — Practice for Exam 2

Generated November 1, 2017

Name: **SOLUTIONS** ______________________

Instructor: ______________________ Section Number: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2012	3	3		12	
Fall 2013	3	2		11	
Fall 2016	3	3		8	
Winter 2013	3	8		8	
Total				39	

Recommended time (based on points): 47 minutes

3. [12 points]

a. [6 points] State whether each of the following series converges or diverges. Indicate which test you use to decide. Show all of your work to receive full credit.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: The function $\frac{1}{n\sqrt{\ln n}}$ is decreasing and positive for $n \geq 2$, then the Integral test says that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ behaves as $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}} dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}} du = \lim_{b\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{\ln b} = \infty.$$

Hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$

Solution: Since $0 \leq \dfrac{\cos^2(n)}{\sqrt{n^3}} \leq \dfrac{1}{n^{\frac{3}{2}}}$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by p-series test ($p = \frac{3}{2} > 1$), then comparison test yields the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$.

b. [6 points] Decide whether each of the following series converges absolutely, converges conditionally or diverges. Circle your answer. No justification required.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$

Converges absolutely **Converges conditionally** Diverges

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{3n}}{5^n}$

Converges absolutely Converges conditionally **Diverges**

Solution: $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right| = \sum_{n=0}^{\infty}\frac{\sqrt{n^2+1}}{n^2+n+8}$ behaves as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ since

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n^2+1}}{n^2+n+8}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n\sqrt{n^2+1}}{n^2+n+8} = 1 > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series test $p=1$), then by limit comparison test $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right|$ diverges.

The convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$ follows from alternating series test since for $a_n = \frac{\sqrt{n^2+1}}{n^2+n+8}$:

- $\lim_{n\to\infty} a_n = 0$.
- a_n is decreasing

$$\frac{d}{dn}\left(\frac{\sqrt{n^2+1}}{n^2+n+8}\right) = \frac{-1+6n-n^3}{\sqrt{1+n^2}(n^2+n+8)^2} < 0$$

for n large.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^{3n}}{5^n} = \sum_{n=0}^{\infty}\left(-\frac{8}{5}\right)^n$ is a geometric series with ratio $r = -\frac{8}{5} < -1$, hence it diverges.

2. [11 points] Determine the convergence or divergence of the following series. In parts (a) and (b), support your answers by stating and properly justifying any test(s), facts or computations you use to prove convergence or divergence. Circle your final answer. Show all your work.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ CONVERGES **DIVERGES**

Solution:

$$\lim_{n\to\infty} \frac{9n}{e^{-n}+n} = \lim_{n\to\infty} \frac{9n}{n} = 9 \neq 0.$$

Since $\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ diverges.

b. [4 points] $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$. **CONVERGES** DIVERGES

Solution: The function $f(n) = \frac{4}{n(\ln n)^2}$ is positive and decreasing for $n > 2$, then by Integral Test the convergence or divergence of $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$ can be determined with the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{4}{x(\ln x)^2} dx$

$$\int \frac{4}{x(\ln x)^2} dx = \int \frac{4}{u^2} du \qquad \text{where } u = \ln x.$$
$$= -\frac{4}{u} + C = -\frac{4}{\ln x} + C$$

Hence

$$\int_2^{\infty} \frac{4}{x(\ln x)^2} dx = \lim_{b\to\infty} -\frac{4}{\ln x}\Big|_2^b = -\frac{4}{\ln 2} \qquad \text{converges.}$$

or

$$\int_2^{\infty} \frac{4}{x(\ln x)^2} dx = 4\int_{\ln 2}^{\infty} \frac{1}{u^2} du \qquad \text{converges by } p\text{-test with } p = 2 > 1.$$

c. [4 points] Let r be a **real** number. For which values of r is the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+4}$ absolutely convergent? Conditionally convergent? No justification is required.

Solution:

Absolutely convergent if : $r > 3$

Conditionally convergent if : $2 < r \leq 3$

Justification (not required):

- Absolute convergence:
  The series $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n^2}{n^r+4}\right| = \sum_{n=1}^{\infty}\frac{n^2}{n^r+4}$ behaves like $\sum_{n=1}^{\infty}\frac{n^2}{n^r} = \sum_{n=1}^{\infty}\frac{1}{n^{r-2}}$. The last series is a p-series with $p = r - 2$ which converges if $r - 2 > 1$. Hence the series converges absolutely if $r > 3$.

- Conditionally convergence:
  The function $\frac{n^2}{n^r+4}$ is positive and decreasing (for large values of n) when $r > 2$. Hence by the Alternating series test $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+4}$ converges in this case.

3. [8 points] For $n = 1, 2, 3, \ldots$ consider the sequence a_n given by

$$a_n = \frac{-1}{2^{(n+1)/2}} \text{ if } n \text{ is odd,} \qquad a_n = \frac{1}{3^{n/2}} \text{ if } n \text{ is even.}$$

a. [2 points] Write out the first 5 terms of the sequence a_n.

Solution: The first five terms are

$$-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{8}.$$

b. [2 points] The series $\sum_{n=1}^{\infty} a_n$ is alternating. In a sentence or two, explain why the Alternating Series Test **cannot** be used to determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Solution: The condition $|a_{n+1}| < |a_n|$ does not hold for all n. (It does not even hold eventually.)

c. [4 points] The series $\sum_{n=1}^{\infty} a_n$ converges. Show that it converges, either by using theorems about series, or by computing its exact value.

Solution: One possible answer is that the series is equal to the difference of two convergent geometric series:

$$\sum_{k=1}^{\infty} \frac{1}{3^k} - \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} - \frac{\frac{1}{2}}{1 - \frac{1}{2}} = -\frac{1}{2}.$$

Another answer uses the Comparison Test; for $n = 1, 2, \ldots$, let $b_n = \frac{1}{n^2}$, and notice that $|a_n| \leq b_n$ eventually. Since $\sum_{n=1}^{\infty} b_n$ converges by the p-Test ($p = 2$), $\sum_{n=1}^{\infty} |a_n|$ converges by comparison. Hence the original series converges.

8. [8 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n}} (x-5)^n.$$

In the following questions, you need to support your answers by stating and properly justifying the use of the test(s) or facts you used to prove the convergence or divergence of the series. Show all your work.

a. [2 points] Does the series converge or diverge at $x = 3$?

Solution: At $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the alternating series test, since $1/\sqrt{n}$ is decreasing and converges to 0.

b. [2 points] What does your answer from part (**a**) imply about the radius of convergence of the series?

Solution: Because it converges at $x = 3$, we know that the radius of convergence $R \geq 2$.

c. [4 points] Find the interval of convergence of the power series.

Solution: Using the ratio test, we have

$$\lim_{n \to \infty} \frac{\frac{1}{2^{n+1}\sqrt{n+1}} |x-5|^{n+1}}{\frac{1}{2^n \sqrt{n}} |x-5|^n} = \frac{1}{2}|x-5| = L,$$

so the radius of convergence is 2. Now we have to check the endpoints. We know from part (a) that it converges at $x = 3$. For $x = 7$, we get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges. Thus, the interval of convergence is $3 \leq x < 7$.