Math 676, Homework 9: due before class Nov 4

(1) Let K be a number field, and let $\alpha \in K$ be a root of a monic polynomial $f(x) \in \mathbb{Z}[x]$. Show that if $r \in \mathbb{Z}$ satisfies $f(r) = \pm 1$ then $\alpha - r$ is a unit in $\mathcal{O}_K$. For extra credit, combine this with the lemma stated in class on Wednesday in order to describe all units in $\mathcal{O}_{\mathbb{Q}(\sqrt[3]{7})}$.

(2) Show that $1 - \zeta_m$ is a unit in $\mathbb{Z}[\zeta_m]$ if and only if m is not a prime power.

(3) Let p be an odd prime and put $K := \mathbb{Q}(\zeta_p)$ and $L := \mathbb{Q}(\zeta_p + \zeta_p^{-1})$.
  (a) Show that $\mathcal{O}_L = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$.
  (b) Show that $\mathcal{O}_K^\times = \langle \zeta_p \rangle \times \mathcal{O}_L^\times$.