Math 676, Homework 8: due by midnight Oct 29 (Thursday!)

(1) Find the class group of the ring of integers of $\mathbb{Q}(\alpha)$, where $\alpha^3+\alpha^2+5\alpha-16=0$.

(2) Let K be a number field with real embeddings $\sigma_1,\dots,\sigma_{r_1}$ and with $\tau_1,\dots,\tau_{r_2}$ being representatives for the pairs of complex-conjugate non-real complex embeddings of K. Assuming Dirichlet's theorem, pick $\alpha_1,\dots,\alpha_{r_1+r_2-1}$ such that $\mathcal{O}_K^*$ is generated by the α_i's and the roots of unity in $\mathcal{O}_K^*$. Let M be the $(r_1+r_2-1)\times(r_1+r_2)$ matrix whose j-th row is

$$(\log|\sigma_1(\alpha_j)|,\dots,\log|\sigma_{r_1}(\alpha_j)|,2\log|\tau_1(\alpha_j)|,\dots,2\log|\tau_{r_2}(\alpha_j)|).$$

Let M' be the matrix obtained by deleting any column of M. Show that $|\det(M')|$ is a positive real number which depends only on K, and not on the choice of α_i's or the choice of column. (This number is called the *regulator* of K.)

(3) Let p be a prime number.
  (a) For any prescribed integer a coprime to p, show that $\Lambda := \{(x,y)\in\mathbb{Z}^2 : y\equiv ax \pmod{p}\}$ is a lattice in $\mathbb{R}^2$ with covolume p.
  (b) For any prescribed integers a,b with $a^2+b^2\equiv -1 \pmod{p}$, show that $\Lambda := \{(x,y,z,w)\in\mathbb{Z}^4 : z\equiv ax+by \pmod{p}$ and $w\equiv bx-ay \pmod{p}\}$ is a lattice in $\mathbb{R}^4$ with covolume p^2.

(4) Let Λ be a rank-n lattice in $\mathbb{R}^n$, and let $S\subset\mathbb{R}^n$ be compact, convex, and centrally symmetric. Show that if $\operatorname{vol}(S)\ge 2^n\operatorname{vol}(\mathbb{R}^n/\Lambda)$ then S contains a nonzero element of Λ.
(In class we showed that if "compact" is replaced by "bounded" then this is true if the hypothesized inequality is strict; the problem here is to deduce that if S is compact and the inequality is an equality then the conclusion still holds.)