Math 676, Homework 5: due Oct 7

(1) If K is a number field, show that $\mathcal{O}_K$ is a principal ideal domain if and only if every ideal I of $\mathcal{O}_K$ contains an element α with $|N(\alpha\mathcal{O}_K)| = N(I)$.

(2) Find an integral basis for $\mathcal{O}_{\mathbb{Q}(\alpha)}$ where α is a root of either $x^3 - 2x + 3$ or $x^3 - x - 4$.

(3) Show that every ideal in a Dedekind domain can be generated by two elements.

(4) If K is a degree-n number field, and $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$, then show that $\Delta_{K/\mathbb{Q}}(\alpha_1, \ldots, \alpha_n)$ is congruent to 0 or 1 mod 4.
Hint: consider odd and even permutations separately.