Math 676, Homework 10: due Nov 11

(1) Let R be a Dedekind domain with field of fractions K, and let S be a finite set of nonzero prime ideals of R.
  (a) Show that $R^\times$ is the intersection of $R_\mathfrak{p}^\times$, taken over all maximal ideals $\mathfrak{p}$ of R.
  (b) Show that there is a canonical exact sequence of abelian groups
$$1 \to R^\times \to (R^S)^\times \to \oplus_{P\in S}(K^\times/R_P^\times) \to \mathrm{Cl}(R) \to \mathrm{Cl}(R^S) \to 1.$$
  (c) Show that $K^\times/R_P^\times \cong \mathbb{Z}$ for each $P \in S$.
  (d) Show that if K is a number field and $R = \mathcal{O}_K$ then $(R^S)^\times \cong W_K \times \mathbb{Z}^{r_1+r_2-1+|S|}$, where W_K is the group of roots of unity in K, r_1 is the number of real embeddings of K, and r_2 is the number of complex-conjugate pairs of non-real complex embeddings of K.

(2) Let a, b be squarefree integers congruent to 1 mod 3 such that $a, b, 1$ are pairwise distinct, and let $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Show that it is not possible to write $\mathcal{O}_K = \mathbb{Z}[\alpha]$ with $\alpha \in K$.
(You may assume that if L, M, N are number fields with $L \subseteq M \cap N$, and $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_L$ which splits completely in both $\mathcal{O}_M$ and $\mathcal{O}_N$, then $\mathfrak{p}$ splits completely in $\mathcal{O}_{LM}$. This fact will be proved in class next week.)