Math 676, Homework 1: due Sep 9

(1) Solve $x^3 - y^2 = 2$ in integers x, y.
(You may assume that $\mathbb{Z}[\sqrt{-2}]$ is a unique factorization domain, which follows from the fact that this ring is Euclidean.)

(2) Show that $\mathbb{Z}[\sqrt{-13}]$ and $\mathbb{Z}[\sqrt{10}]$ are not unique factorization domains.

(3) Let $d \neq 1$ be an integer which is not divisible by the square of any prime. Show that the ring of all algebraic integers in $\mathbb{Q}(\sqrt{d})$ is

$$\begin{aligned} &\mathbb{Z}[\sqrt{d}] && \text{if } d \equiv 2, 3 \pmod 4 \\ &\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] && \text{if } d \equiv 1 \pmod 4. \end{aligned}$$

Determine the ring of all algebraic integers in $\mathbb{Q}(\sqrt[3]{2})$.

(4) Let $R = \mathbb{Z}[\sqrt{-5}]$, and write $\alpha := 1 + \sqrt{-5}$ and $\overline{\alpha} := 1 - \sqrt{-5}$. Show that $6 = 2 \cdot 3 = \alpha \cdot \overline{\alpha}$ are two inequivalent factorizations into irreducible elements of R. Then write the ideals (2), (3), (α), $(\overline{\alpha})$ of R as products of prime ideals, and show that the two resulting prime factorizations of the ideal (6) induced from the factorizations $6 = 2 \cdot 3 = \alpha \cdot \overline{\alpha}$ are identical.
(Hint: one way to show that an ideal I of R is prime is by showing that R/I is an integral domain.)