Math 494, Homework 4: due Feb 18

(1) Define a *Euclidean function* on an integral domain $R$ to be a function $\phi\colon R \to \mathbf{N}_0$ such that $\forall a, b \in R$ with $b \neq 0$, $\exists q, r \in R$ with $a = bq + r$ and $\phi(r) < \phi(b)$. Although there might be several different Euclidean functions on a given ring $R$, it turns out that there is a distinguished Euclidean function, namely the unique "minimal" Euclidean function. More precisely, let $S$ be the set of all Euclidean functions $\phi$ on a given Euclidean domain $R$, and define a function $\psi\colon R \to \mathbf{N}_0$ by $\psi(r) := \min(\{\phi(r)\colon \phi \in S\})$. Show that $\psi$ is a Euclidean function on $R$. For any Euclidean domain $R$, what are all $r \in R$ for which $\psi(r) = 0$? What are all $r \in R$ for which $\psi(r) = 1$? If $\psi(r) = 2$ then must $r$ be irreducible? Describe $\psi$ precisely in case $R = \mathbf{Z}$.

(2) Show that $\mathbf{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a principal ideal domain which is not a Euclidean domain. *(Hints available upon request. Note that in order to show a ring $R$ is not Euclidean, it does not suffice to show that some specific function $\phi\colon R \to \mathbf{N}_0$ such as $\phi(r) := |r|^2$ is not a Euclidean function for $R$, instead one must show that* every *function $\phi\colon R \to \mathbf{N}_0$ is not a Euclidean function for $R$.)*

(3) Problem 9.7 from chapter 11 of Artin and problems 2.1–2.3, 2.6–2.8, and 2.10 from chapter 12 of Artin.