Math 116 — Practice for Exam 1

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	1	1		12	
Fall 2016	1	7		6	
Fall 2016	1	8		6	
Fall 2016	1	1		13	
Fall 2019	1	2		14	
Fall 2018	1	1		12	
Winter 2022	1	8	Poseidon	12	
Total				75	

Recommended time (based on points): 68 minutes

1. [12 points] Suppose that f is a twice differentiable function with continuous second derivative. (That is, both f and f' are differentiable, and f'' is continuous.) The following table gives some values of f and f'.

x	0	1	2	3	4	5	6	e^3
$f(x)$	7	5	-1	0	11	-3	2	9
$f'(x)$	3	-4	-2	4	-5	0	-1	2

In parts (a) through (c) below, calculate the exact numerical value of the integral. Write "NOT ENOUGH INFO" if there is not enough information to find the exact value. Be sure to show your work clearly. No partial credit will be given for estimates.

a. [4 points] $\displaystyle\int_1^{e^3} \frac{f'(\ln x)}{x}\, dx$

Solution: The substitution $w = \ln(x)$ gives $dw = \frac{dx}{x}$ and

$$\int_1^{e^3} \frac{f'(\ln x)}{x}\, dx = \int_0^3 f'(w)\, dw = f(3) - f(0) = 0 - 7 = -7.$$

b. [4 points] $\displaystyle\int_0^4 x f''(x)\, dx$

Solution: Integration by parts with $u = x$ and $dv = f''(x)\, dx$ gives

$$\begin{aligned}\int_0^4 x f''(x)\, dx &= x f'(x)\Big|_0^4 - \int_0^4 f'(x)\, dx \\ &= \big(4 \cdot f'(4) - 0 \cdot f'(0)\big) - \big(f(4) - f(0)\big) \\ &= -20 - (11 - 7) = -24.\end{aligned}$$

c. [4 points] $\displaystyle\int_2^6 f'(x)\,[f(x)]^2\, dx$

Solution:
One Approach: substitution with $w = f(x)$ so $dw = f'(x)\, dx$

$$\int_2^6 f'(x)\,[f(x)]^2\, dx = \int_{f(2)}^{f(6)} w^2\, dw = \frac{w^3}{3}\Big|_{-1}^{2} = \frac{8}{3} - \frac{-1}{3} = 3.$$

Another Approach: integration by parts with $u = f(x)^2$ and $dv = f'(x)\, dx$

$$\int_2^6 f'(x)\,[f(x)]^2\, dx = [f(x)]^3\Big|_2^6 - 2\int_2^6 f'(x)[f(x)]^2\, dx.$$

Moving the last term to the left hand side and dividing both sides of the resulting equation by 3 gives

$$\int_2^6 f'(x)\,[f(x)]^2\, dx = [f(x)]^3\Big|_2^6 = \frac{8 - (-1)}{3} = 3.$$

7. [6 points] Suppose that g is a continuous function, and define another function G by

$$G(x) = \int_0^x g(t)\,dt.$$

Given that $\int_0^7 g(x)\,dx = 5$, compute

$$\int_0^7 g(x)(G(x))^2\,dx.$$

Show each step of your computation.

Solution: Substitution gives

$$\int_0^7 g(x)(G(x))^2\,dx = \int_{G(0)}^{G(7)} u^2\,du = \left.\frac{u^3}{3}\right|_0^5 = \frac{125}{3}.$$

Alternatively, integrate by parts to obtain

$$\int_0^7 g(x)(G(x))^2\,dx = (G(x))^3\Big|_0^7 - 2\int_0^7 g(x)(G(x))^2\,dx,$$

which after rearranging gives

$$\int_0^7 g(x)(G(x))^2\,dx = \frac{1}{3}\left((G(x))^3\Big|_0^7\right) = \frac{125}{3}.$$

8. [6 points] Suppose that f is a continuous, **odd** function, and define another function F by

$$F(x) = \int_{-12}^{x} f(3t - c)\, dt,$$

where c is some constant. You do not need to show your work for this problem.

a. [3 points] Find a value of c for which the graph of F goes through the origin.

Solution: The correct value is $c = -18$.

b. [3 points] Find a value of c for which the graph of F' goes through the origin.

Solution: The correct value is $c = 0$.

1. [13 points] Suppose that f is a twice-differentiable, function that satisfies

$$f(0) = 1 \qquad f(2) = 2 \qquad f(4) = 4 \qquad f'(2) = 3$$

$$\int_0^2 f(x)\,dx = 5 \qquad \int_2^4 f(x)\,dx = 7.$$

Evaluate the following integrals.

a. [4 points] $\int_0^2 x f'(x)\,dx$

Solution:

$$\int_0^2 x f'(x)\,dx = x f(x)\big|_0^2 - \int_0^2 f(x)\,dx = -1.$$

b. [4 points] $\int_{\sqrt{2}}^2 x f'(x^2)\,dx$

Solution:

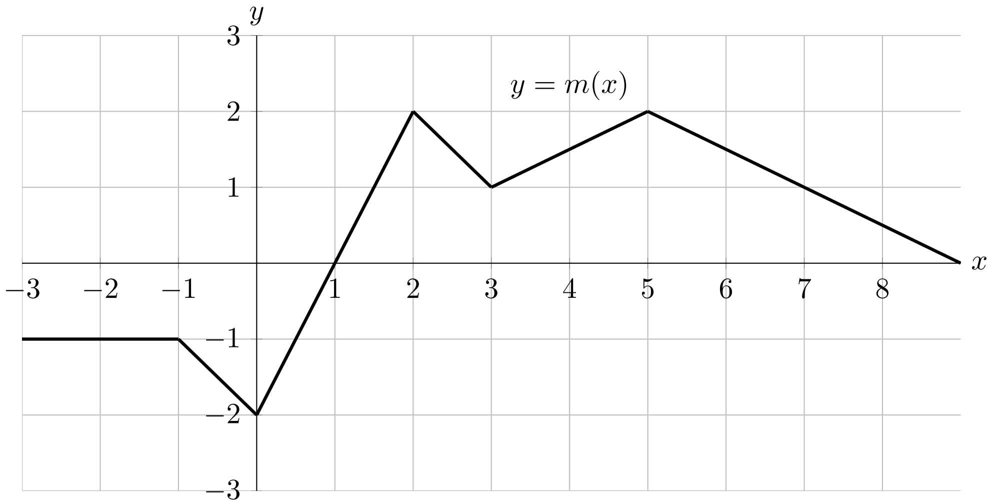
$$\int_{\sqrt{2}}^2 x f'(x^2)\,dx = \frac{1}{2}\int_2^4 f'(u)\,du = 1.$$

c. [5 points] $\int_0^2 x^3 f'(x^2)\,dx$

Solution:

$$\int_0^2 x^3 f'(x^2)\,dx = \frac{1}{2}\int_0^4 u f'(u)\,du = \frac{1}{2}\left(u f(u)\big|_0^4 - \int_0^4 f(u)\,du\right) = 2.$$

2. [14 points] Part of the graph of a continuous, piecewise-linear function $m(x)$ is given below. The domain of $m(x)$ is all real numbers.



Let:

- $F(x) = \int_1^x m(t)\,dt$
- $G(x) = \int_2^{x/2} m(t)\,dt$
- $H(x)$ is an antiderivative of $m(x)$ with $H(2) = 8$.

You do not need to show work for this problem.

a. [11 points] Find the following values. If it is not possible to do so based on the information provided, write "NI". If the value does not exist, write "DNE".

(i) $F(1) = $ 0

(ii) $F(3) = $ 2.5

(iii) $F(-2) = $ 3.5

(iv) $F'(4) = $ 1.5

(v) $G(2) = $ −1

(vi) $G(6) = $ 1.5

(vii) $G'(8) = $ 0.75

(viii) $H(3) = $ 9.5

(ix) $H(10) - F(10) = $ 7

b. [3 points] On which of the following intervals is $H(x)$ concave up on the entire given interval? Circle all correct answers.

$\boxed{(0,2)}$ $(1,3)$ $(2,5)$ $\boxed{(3,5)}$ NONE OF THESE

1. [12 points] The table below gives several values of a differentiable function f such that f' is also differentiable and f'' is continuous.

x	-3	-2	-1	0	1	2	3
$f(x)$	14	20	4	11	24	5	8
$f'(x)$	3	-4	-6	2	5	-3	4

For each of the following, calculate the exact numerical value of the integral. If it is not possible to do so based on the information provided, write "NOT POSSIBLE" and clearly indicate why it is not possible. Show your work.
Note that no variables or function names (such as f, f', or f'') should appear in your answers.

a. [3 points] $\int_0^1 f'(2x)\,dx$

let $w = 2x$
$dw = 2dx$
$x = 0 \Rightarrow w = 0$
$x = 1 \Rightarrow w = 2$

$= \int_0^2 f'(w) \cdot \frac{dw}{2}$

$= \frac{1}{2}\left[f(w)\right]_0^2 = \frac{1}{2}\left[f(2) - f(0)\right] = \frac{1}{2}\left[5 - 11\right]$

Answer: -3

b. [3 points] $\int_2^3 s f''(s)\,ds$ $= \int_2^3 uv' = uv\Big|_2^3 - \int_2^3 u'v = sf'(s)\Big|_2^3 - \int_2^3 (1)(f'(s))\,ds$

Parts:
Let $u = s$ $v' = f''(s)$
$u' = 1$ $v = f'(s)$

$= sf'(s) - f(s)\Big|_2^3 = \left[3f'(3) - f(3)\right] - \left[2f'(2) - f(2)\right]$

$= \left[3(4) - 8\right] - \left[2(-3) - 5\right] = \left[12 - 8\right] - \left[-6 - 5\right] = 4 + 11$

Answer: 15

c. [3 points] $\int_{-2}^{-1} q \cdot \left[\frac{d}{dq}\left(f'(q)e^{f(q)}\right)\right] dq$ $= \int_{-2}^{-1} uv' = \underbrace{uv\Big|_{-2}^{-1}}_{①} - \underbrace{\int_{-2}^{-1} u'v}_{②}$

Parts:
Let $u = q$ $v' = \frac{d}{dq}\left(f'(q)e^{f(q)}\right)$
$u' = 1$ $v = f'(q)e^{f(q)}$

① $uv\Big|_{-2}^{-1} = qf'(q)e^{f(q)}\Big|_{-2}^{-1} = (-1)f'(-1)e^{f(-1)} - (-2)f'(-2)e^{f(-2)}$
$= (-1)(-6)e^4 - (-2)(-4)e^{20} = 6e^4 - 8e^{20}$

② $\int_{-2}^{-1} u'v = \int_{-2}^{-1} (1)f'(q)e^{f(q)}\,dq$

let $w = f(q)$
$dw = f'(q)\,dq$
$q = -2 \Rightarrow w = f(-2) = 20$
$q = -1 \Rightarrow w = f(-1) = 4$

$= \int_{20}^{4} e^w\,dw = e^w\Big|_{20}^{4} = e^4 - e^{20}$

So we have ① − ② $= (6e^4 - 8e^{20}) - (e^4 - e^{20})$

Answer: $5e^4 - 7e^{20}$

d. [3 points] $\int_{-1}^{1} f'(y) \cdot f''(f(y))\,dy$

Let $w = f(y)$
$dw = f'(y)dy$
$y = -1 \Rightarrow w = f(-1) = 4$
$y = 1 \Rightarrow w = f(1) = 24$

$= \int_4^{24} f''(w)\,dw = f'(w)\Big|_4^{24} = f'(24) - f'(4).$

But we don't know $f'(24)$ and $f'(4)$, so we can't go any further.

Answer: NOT POSSIBLE

8. [12 points] After constructing their boat, Brad and Shawna departed the island, hoping to return home. However, their victory at Troy angered Poseidon, the god of the sea, who created large waves to further complicate their journey. The waves were so high Brad and Shawna could not see land, making navigation difficult. Hermes, the messenger of the gods, was sympathetic. He stole the formula Poseidon used to create the waves and gave it to Brad and Shawna. The formula is given by:

$$H(t) = 12 \int_0^{\sin^2(t)} e^{x^2} dx.$$

The function $H(t)$ gives the wave height in meters, t minutes after Hermes stole the formula.

a. [7 points] Brad and Shawna can only see land at the moment their ship is at the top of a wave. If they know exactly when that time is coming, they can be prepared to correct their course toward land. If Hermes brought them the formula 4 minutes after he stole it, when is the first time they can see land? (Brad and Shawna did not know when to look for land before obtaining the formula.)

Solution: We find the first maximum of $H(t)$ after $t = 4$. We need to find where $H'(t) = 0$. By applying chain rule and 2^{nd} Fundamental Theorem of calculus, we find

$$H'(t) = 24 \sin(t) \cos(t) e^{\sin^4(t)}$$

Since e^x is always positive, $e^{\sin^4(t)}$ can never be zero, so the critical points are where $\sin(t) = 0$ and $\cos(t) = 0$. These are $n\pi$ for $\sin(t)$ and $n\pi + \frac{\pi}{2}$ for $\cos(t)$. Now, we need to determine which are maximum and which are minimum. Since $\sin(n\pi) = 0$, if we plug these into $H(t)$, we see that $H(n\pi) = \displaystyle\int_0^0 e^{x^2} dx = 0$. If we check $n\pi + \frac{\pi}{2}$, $\sin(n\pi + \frac{\pi}{2}) = \pm 1$ so $\sin^2(n\pi + \frac{\pi}{2}) = 1$ and so $H(n\pi + \frac{\pi}{2}) = \displaystyle\int_0^1 e^{x^2} dx$, which is positive since e^{x^2} is always positive. Therefore, these are the maximum. If we start counting, we see that $\frac{\pi}{2} \approx 1.5$ and $\frac{3\pi}{2} \approx 4.5$ so the answer is $\frac{3\pi}{2}$.

b. [5 points] The same poet who recorded the tale of the Trojan war would like to record parts of Brad and Shawna's odyssey. Seeking a more appealing version of the expression above, he asks you for a different formula. Write a function equivalent to $H(t)$ with only t in the upper bound of the integral.

Solution: Using second fundamental theorem, we integrate $H'(t)$, which we solved for above. This gives

$$H(t) = \int_0^t 24 \sin(x) \cos(x) e^{\sin^4(x)} dx$$