Math 116 – Team Homework #4, Winter 2023

Some guidelines for your assignment

- You must *read* and *attempt* the problems *before* meeting with your team. Even if you aren't able to obtain all the answers, being prepared during the team meetings helps your group work more efficiently during the meeting.
- Don't be discouraged if you cannot solve most of the problems on your own — this is perfectly normal. This is part of why you are being assigned to work on these assignments as a group; make sure to discuss your questions and ideas with your teammates.
- If your team is having trouble with a particular problem, try utilizing the Math Lab (our math tutoring center - see more details here: `https://lsa.umich.edu/math/undergraduates/course-resources/math-lab.html`) with your teammates to get help.
- Make sure *everyone* is involved and no-one feels excluded during the meetings. If you notice someone is shy, actively encourage them to contribute to the group!
- Ask your teammates to explain their reasoning behind their answers if you don't understand it. Remember that all members of the team are responsible for this assignment, and *everyone* should be on board with what the team turns in.
- Write up your final solutions neatly, and make sure your explanations are clear and complete.
- Consult pages 12-14 of the Student Guide on the course website for more details regarding best practices and team homework roles.

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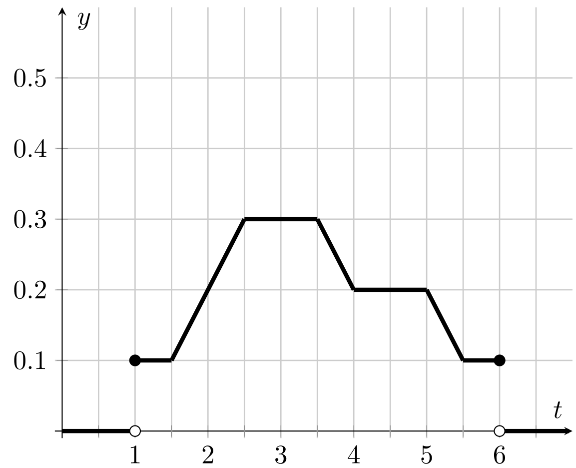
1. *Endure and Survive* is a reality competition that involves survival games. One of the games involves contestants completing a timed run of an obstacle course. All contestants failing to complete the course within 3.5 minutes will be eliminated. Alice, a contestant, has trained for several months on a replica of the course. From records of her training, Alice has obtained a probability density function (shown in the graph below) for the amount of time t it takes her to finish the obstacle course, where t is given in minutes.



Hint: Use areas of larger geometric shapes, rather than counting boxes, to calculate areas in this problem.

(a) What was Alice's median finishing time during her training?

> **Solution:** We want to find the median M where $\int_{-\infty}^{M} p(t)\,dt = 0.5$.
> Note: We use the variable M instead of the text's T since T is used in later parts.
> Notice $\int_{-\infty}^{1} p(t)\,dt = 0$, $\int_{1}^{1.5} p(t)\,dt = 0.05$, $\int_{1.5}^{2.5} p(t)\,dt = 0.2$, and $\int_{2.5}^{3.5} p(t)\,dt = 0.3$.
> These calculations imply the median M lies somewhere between 2.5 and 3.5
> (since 0.25 and 0.55 units of area accumulate under the pdf at $t = 2.5$ and $t = 3.5$, respectively).
> To find the exact value of M, we use the fact that $0.25 + \int_{2.5}^{M} p(t)\,dt = 0.5$.
> Since $p(t) = 0.3$ when $2.5 < t < 3.5$, this means $\int_{2.5}^{M} 0.3\,dt = 0.25$.
> This implies $0.3 \cdot M - 0.3 \cdot 2.5 = 0.25$, or equivalently $M = 3\frac{1}{3}$ minutes.

(b) Assuming Alice's performance in the competition is consistent with her training, what is the probability that she will not be eliminated in this game?

> **Solution:**
> By the calculations in the third line of the part (a) solution and the fact that $\int_{2.5}^{3.5} p(t)\,dt = 0.3$, we know $\int_{-\infty}^{3.5} p(t)\,dt = 0.05 + 0.2 + 0.3 = 0.55$.
> To put it plainly, Alice has a 55% chance of not being eliminated.

Alice clears the obstacle course game and moves on to the next one. In the next game, each remaining contestant competes head-to-head against another contestant in speed climbing. The loser of each head-to-head contest is eliminated. The probability density function, $r(T)$, for the amount of time T (measured in seconds) it takes for Alice to climb the wall is given by

$$r(T) = \begin{cases} T - 7 & \text{for } 7 < T \leq 8, \\ a(T-12)^2 & \text{for } 12 \leq T \leq 15, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Find the value of a so that $r(T)$ is a probability density function.

Solution: For $r(T)$ to be a pdf, we need $\int_{-\infty}^{\infty} r(T)\, dT = 1$.
In particular, we need $\int_{7}^{8} (T-7)\, dT + \int_{12}^{15} a(T-12)^2\, dT = 1$.
Notice $\frac{(T-7)^2}{2}$ and $\frac{a(T-12)^3}{3}$ are antiderivatives of the integrands.
Using the antiderivatives and FTC, we have $\frac{(8-7)^2}{2} + \frac{a(15-12)^3}{3} = 1$.
This implies $0.5 + 9 \cdot a = 1$ or that $a = 1/18$.

(d) The maximum of $r(T)$ occurs at $T = 8$, where the value $r(8) = 1$. Provide an interpretation, in context, for the fact that $r(8) = 1$.

Solution: The fact that r(8)=1 says that it will take it will take Alice between 7.9 and 8 seconds to climb the wall in approximately 10 percent of her attempts.
Notes: While interpreting, we must keep in mind to not extend a full unit away from $T = 8$ in our interpretation. For example, it is inaccurate to claim Alice will complete the climb in 7-8 seconds about 100 percent of the time. We must also be careful to not extend our interpretation to times above $T = 8$ (since $r(T) = 0$ when $8 \le T < 12$). For example, it is inaccurate to claim Alice will complete the climb in 8-8.5 seconds about 50 percent of the time.

(e) What is the average amount of time it takes Alice to speed climb the wall?

Solution: Recall the average (also known as the mean) equals $\int_{-\infty}^{\infty} T \cdot r(T)\, dT$.
Using part (c), the average equals $\int_{7}^{8} T(T-7)\, dT + \int_{12}^{15} \frac{T}{18}(T-12)^2\, dT$.
Antiderivatives of the integrands are $T^3/3 - 7 \cdot T^2/2$ and $T^4/72 - 4 \cdot T^3/9 + 4 \cdot T^2$, respectively.
Using the FTC, the average is $(8^3/3 - 7 \cdot 8^2/2) - (7^3/3 - 7 \cdot 7^2/2) + (15^4/72 - 4 \cdot 15^3/9 + 4 \cdot 15^2) - (12^4/72 - 4 \cdot 12^3/9 + 4 \cdot 12^2)$.
Therefore her average is $10\frac{23}{24}$ seconds.

(f) Find the cumulative distribution function, $R(T)$, associated with $r(T)$.

Solution:
We can use the antiderivatives from part (c) to help find our cdf.
The cdf corresponding to the first nontrivial piece of the pdf is $R_1(T) = \frac{(T-7)^2}{2} + C_1$.
The cdf corresponding to the second nontrivial piece of the pdf is $R_2(t) = \frac{(T-12)^3}{54} + C_2$.
To ensure continuity of our cdf, we need $R_1(7) = \frac{(7-7)^2}{2} + C_1 = 0$ and $R_2(12) = \frac{(12-12)^3}{54} + C_2 = 1/2$. Thus, $C_1 = 0$ and $C_2 = 1/2$.
Using this work we see the cdf takes the form

$$R(T) = \left\{ \begin{array}{cc} 0 & T \le 7 \\ \frac{(T-7)^2}{2} & 7 \le T \le 8 \\ 1/2 & 8 \le T \le 12 \\ \frac{(T-12)^3}{54} + 1/2 & 12 \le T \le 15 \\ 1 & 15 \le T \end{array} \right\}$$

(g) Given Alice's head-to-head opponent climbs the wall in 13.5 seconds, what is the probability that Alice does not get eliminated in this round?

Solution: We can evaluate the cdf at the opponent's time of 13.5 seconds to help solve this part.
Notice $R(13.5) = \frac{(13.5-12)^3}{54} + 1/2 = 9/16 = 0.5625$.
Therefore, Alice has a 56.25% chance of not getting eliminated.

2. One of the games on *Endure and Survive* involves contestants completing a stepping stones course. The course consists of several circular stones, placed in a straight line. The gap between any two consecutive stones on the course is 1.5 meters. The starting platform is a stone with diameter 2 meters. Each subsequent stone has diameter two-thirds that of the preceding stone.

(a) Write a closed form formula for the diameter d_n of the n-th stepping stone on the course. You may treat the starting platform as the first stepping stone, i.e. $d_1 = 2$ meters.

Solution: Since $d_n = \frac{2}{3}d_{n-1}$ and $d_1 = 2$, we compute

$$d_n = 2 \cdot \left(\frac{2}{3}\right)^{n-1} \text{ meters.}$$

(b) Given the finishing platform is the first stone on the course with diameter less than 15 centimeters, how many stones form the course (including the starting and the finishing platforms)?

Solution: The number of stones in the course is the smallest positive integer n for which $d_n < 0.15$. This says that

$$2 \cdot \left(\frac{2}{3}\right)^{n-1} < 0.15,$$

or equivalently

$$\frac{2}{0.15} < \left(\frac{3}{2}\right)^{n-1}.$$

Take logs to rewrite this as

$$\ln\left(\frac{2}{0.15}\right) < (n-1)\ln(1.5),$$

or equivalently

$$\frac{\ln\left(\frac{2}{0.15}\right)}{\ln(1.5)} < n - 1.$$

Since the left side is approximately 6.388, the smallest positive integer n satisfying the last inequality is $n = 8$, so the course has 8 stones.

(c) Write an expression involving series (i.e. use summation notation $\sum$, do not use the symbol "d_n", but instead explicitly write out the expression for d_n) for the total length of the course, measured from the back of the starting platform to the front of the finishing platform.

Solution: The length of the course (including the diameters of both the starting and finishing platforms) is

$$7(1.5) + \sum_{n=1}^{8} d_n = \boxed{7(1.5) + \sum_{n=1}^{8} 2 \cdot \left(\frac{2}{3}\right)^{n-1} \text{ meters,}}$$

since there are eight stones and seven 1.5-meter gaps between consecutive stones.

(d) The organizers of *Endure and Survive* want to call the starting platform the "0-th stepping stone" instead. Re-index your series in part (c) to reflect this; i.e. Rewrite all series in your expression so that they now have starting index 0.

Solution: Writing $m := n - 1$, the boxed expression in (c) becomes

$$7(1.5) + \sum_{m=0}^{7} 2 \cdot \left(\frac{2}{3}\right)^{m} \text{ meters.}$$

(e) Find a closed-form expression (i.e. no summation notation $\sum$, no ellipsis (...), no recursion, no d_n) for the total length of the course. *Do not simply write out the terms from your series in part (c) (and/or part (d)) explicitly.*

Solution: By using the formula for the sum of a finite geometric series, we may rewrite the expression in (d) as

$$7(1.5) + 2\frac{1 - \left(\frac{2}{3}\right)^8}{1 - \frac{2}{3}} \text{ meters.}$$