Math 116 — Practice for Exam 2

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2015	3	7		6	
Fall 2016	3	1		4	
Fall 2016	3	12		9	
Winter 2016	3	12		8	
Winter 2015	3	5		10	
Fall 2017	2	3		12	
Total				49	

Recommended time (based on points): 56 minutes

6. [4 points] For each of the following questions, circle the answer which correctly completes the statement. You **do not** need to show your work.

a. [2 points] The integral $\displaystyle\int_1^\infty \frac{\ln x}{x^{3/2}}dx$

$\boxed{\textbf{converges}}$ **diverges**

Solution: $\ln x \le x^{1/4}$ for sufficiently large values of x, so

$$\frac{\ln x}{x^{3/2}} \le \frac{x^{1/4}}{x^{3/2}} = \frac{1}{x^{5/4}}$$

eventually. Since $\int_1^\infty \frac{1}{x^{5/4}}\,dx$ converges by the p-test ($p = \frac{5}{4} > 1$), $\int_1^\infty \frac{\ln x}{x^{3/2}}\,dx$ also converges by direct comparison.

b. [2 points] The integral $\displaystyle\int_0^1 \frac{x}{x^2 + x^{3/2}}dx$

$\boxed{\textbf{converges}}$ **diverges**

Solution:

$$\frac{x}{x^2 + x^{3/2}} = \frac{1}{x + \sqrt{x}} \le \frac{1}{\sqrt{x}}$$

for all positive values of x. Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges by the p-test ($p = \frac{1}{2} < 1$), $\int_0^1 \frac{x}{x^2+x^{3/2}}\,dx$ also converges by direct comparison.

7. [6 points] The power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{5^n n}$$

has a radius of convergence of 5. For each of the endpoints of the interval of convergence, fill in the first two blanks with the endpoint and the series at that endpoint (in sigma notation or by writing out the first 4 terms), and then indicate whether the series converges at that endpoint in the final blank. You **do not** need to show your work.

At the endpoint $x =$ **-4**, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$

and that series **diverges**.

At the endpoint $x =$ **6**, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

and that series **converges**.

1. [4 points] Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges at $x = 6$ and diverges at $x = -2$. What can you say about the behavior of the power series at the following values of x? For each part, circle the correct answer. Ambiguous responses will be marked incorrect.

a. [1 point] At $x = -3$, the power series...

CONVERGES $\boxed{\textbf{DIVERGES}}$ **CANNOT DETERMINE**

b. [1 point] At $x = 0$, the power series...

CONVERGES **DIVERGES** $\boxed{\textbf{CANNOT DETERMINE}}$

c. [1 point] At $x = 8$, the power series...

CONVERGES **DIVERGES** $\boxed{\textbf{CANNOT DETERMINE}}$

d. [1 point] At $x = 2$, the power series...

$\boxed{\textbf{CONVERGES}}$ **DIVERGES** **CANNOT DETERMINE**

2. [5 points] Determine the **radius** of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}.$$

Justify your work carefully and write your final answer in the space provided. Limit syntax will be enforced.

Solution: For $n = 0, 1, \ldots$, let $a_n = \frac{(2n)!}{(n!)^2}$. We have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to 4$$

as $n \to \infty$. Hence the radius of convergence is $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Radius of convergence = $\frac{1}{2}$

12. [9 points] Three intervals are given below. In the space next to each interval, write the letter(s) corresponding to each power series (A)-(I) (below) whose interval of convergence is **exactly** that interval. There may be more than one answer for each interval. If there are intervals below for which none of the power series (A)-(I) converge on that interval, write "NONE" in the space next to the interval. You do **not** need to show your work.

a. [3 points] $(-2, 2)$: B

b. [3 points] $(0, 10]$: C

c. [3 points] $[0, \infty)$: NONE

(A) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n+2}}{n!}$ (B) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n(2x)^n}{4^n}$ (C) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{n5^n}$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{\sqrt{n}}$ (E) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (F) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$

(G) $\displaystyle\sum_{n=1}^{\infty} \frac{(\frac{1}{2}x)^n}{n^2}$ (H) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-5)^n}{5^n}$ (I) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$

12. [8 points] Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges when $x = 0$ and diverges when $x = 9$. In this problem, you do not need to show your work.

a. [4 points] Which of the following could be the interval of convergence? Circle **all** that apply.

$\boxed{[0,8]}$ $[0,7]$ $\boxed{(-1,9)}$ $(-2,10)$ $(0,8]$

b. [2 points] The limit of the sequence a_n is 0.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

c. [2 points] The series $\sum_{n=0}^{\infty}(-5)^n a_n$ converges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

5. [10 points]

a. [5 points] Determine the **radius** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{4n}}{n^5 (16)^n}.$$

Solution: The above series will converge for x values such that

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} < 1$$

by the ratio test. We have

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} = \frac{1}{16}|x-5|^4.$$

and so the desired inequality holds if $\frac{1}{16}|x-5|^4 < 1$. This is equivalent to $|x-5| < 2$. Thus, the radius of convergence is 2.

The radius of convergence is $\underline{2}$.

b. [5 points] The power series $\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ has radius of convergence 1. Determine the **interval** of convergence for this power series.

Solution:

For $x = 1$, we get the series $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$. We have that $\lim_{n\to\infty} \frac{\frac{n+2}{n^4+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4+2n^3}{n^4+1} = 1$.

The limit comparison test then tells us that the series $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges if and only if the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p > 1$ it converges, and so $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges.

For $x = -1$, we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{n^4+1}$. By the work shown above, this series converges absolutely.

The interval of convergence for $\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ is then $-1 \le x \le 1$

The interval of convergence is $\underline{[-1, 1]}$.

3. [12 points] Define a sequence a_n for $n \geq 1$ by $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1/3 & \text{if } n \text{ is even.} \end{cases}$

Note that this means the first three terms of this sequence are $a_1 = 0, \quad a_2 = 1/3, \quad a_3 = 0.$

a. [2 points] Does the sequence $\{a_n\}$ converge or diverge?

Circle one: **converges** $\boxed{\textbf{diverges}}$

Briefly explain your answer.

Solution: This sequence does not converge because the terms of the sequence alternate forever between 0 and 1/3, and, in particular, never approach a single value. In other words, $\lim_{n\to\infty} a_n$ does not exist.

b. [4 points] Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(3 \cdot a_{2n})}{n3^n} x^n$.

i. Write out the partial sum with three terms for this power series.

Solution: Note that $2n$ is even for all integers n, so a_{2n} is always equal to $1/3$. Therefore, the power series can be rewritten as $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

The partial sum with three terms is thus $\dfrac{x}{3} + \dfrac{x^2}{2 \cdot 9} + \dfrac{x^3}{3 \cdot 27} = \dfrac{x}{3} + \dfrac{x^2}{18} + \dfrac{x^3}{81}$.

Answer: $\dfrac{x}{3} + \dfrac{x^2}{18} + \dfrac{x^3}{81}$

ii. Give the exact value of this partial sum when $x = -\frac{1}{2}$.

Solution: Substituting $x = -1/2$ into the partial sum above gives

$$\frac{(-1/2)}{3} + \frac{(-1/2)^2}{18} + \frac{(-1/2)^3}{81} = -\frac{1}{6} + \frac{1}{72} - \frac{-1}{648} = -\frac{25}{162}.$$

Answer: $-\dfrac{25}{162}$

c. [6 points] Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$.

Show every step of any calculations and fully justify your answer with careful reasoning. Write your final answer on the answer blank provided.

Solution: To compute the radius of convergence we apply the Ratio Test to the general term of the power series, which we will call $b_n = \frac{x^n}{n \cdot 3^n}$.

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{n3^n}{(n+1)3^{n+1}} \cdot |x| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|x|}{3} = \frac{|x|}{3}.$$

By the Ratio Test, this power series will therefore converge if $|x|/3 < 1$, i.e. $|x| < 3$, and diverge if $|x| > 3$. This shows that the radius of convergence is $R = 3$, and the interval of convergence is $(-3, 3)$ together with possibly one or both of the endpoints. We still need to check convergence at the endpoints.
At $x = -3$ the power series reduces to

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot \frac{3^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This series converges by the alternating series test. (It is the alternating harmonic series.) This implies that $x = -3$ is in the interval of convergence.
Finally, at $x = 3$, the power series reduces to

$$\sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{3^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This series diverges by p-test with $p = 1$. (It is the harmonic series.) This implies that $x = 3$ is NOT in the interval of convergence.
Hence, the interval of convergence is $[-3, 3)$.

Answer: $[-3, 3)$