Veech group of regular n-gon

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1 Introduction

We will be assuming introductory knowledge about translation surfaces and Veech groups (see for example [MT02]). Consider the regular $2k$-gon in the plane with one side parallel to the x-axis. We glue parallel sides together to make a translation surface, which we call the **regular $2k$-gon**. For the odd case, we consider two copies of a regular $(2k+1)$-gon one of which is rotated by π radians with respect to the other (once again let a side by parallel to the x-axis). Glueing parallel sides by translation, we get a translation surface we call the **double $(2k+1)$-gon** (see Figure 1). In the paper [Vee89], Veech computed the Veech groups of the regular $2k$-gon and double $(2k+1)$-gon and found that they are lattices in $\mathrm{SL}(2,\mathbb{Z})$. However, Veech's original paper is difficult to read, so this article summarizes Veech's proof in modern language. The article [MT02, Section 5.1] covers much of the proof for the regular 8-gon. The paper [Hoo13] computes a different Veech group using similar techniques.

Definition 1. Let p, q, r be integers greater than 1. The **triangle group** $\Delta(p, q, r)$ is the group generated by reflections over the sides of a geodesic triangle (in $S^2, \mathbb{R}^2$, or $\mathbb{H}^2$) with angles $\pi/p, \pi/q, \pi/r$. We may also let p or q be ∞, which represents a triangle with angle 0. The group $\Delta^+(p, q, r) < \Delta(p, q, r)$ is the index 2 subgroup of orientation preserving isometries.

When p, q, r are chosen to have a triangle in $\mathbb{H}$, $\Delta^+(p, q, r)$ and $\Delta(p, q, r)$ are subgroups in $\mathrm{PSL}(2,\mathbb{R})$. As an abuse of notation, we will use the same notation to refer to the preimages of these groups under the quotient map $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$.

Theorem 2. *The Veech group of the regular $2k$-gon, for $k \geq 4$, is $\Delta^+(k, \infty, \infty)$ and the Veech group of the double $(2k+1)$-gon, for $k \geq 2$, is $\Delta^+(2, 2k+1, \infty)$.*

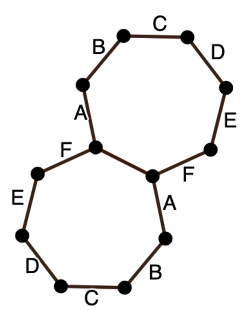

Figure 1: The double $(2k+1)$-gon. Sides with the same label are glued by translation.

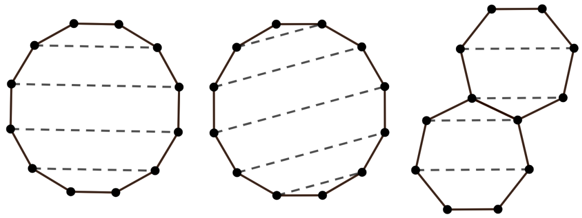

Figure 2: Cylinder directions

2 Proof

Proposition 3. *Consider the regular $2k$-gon and the cylinder directions shown in Figure 2. In each of these directions, either one cylinder has modulus $\tan\frac{\pi}{2k}$ and all other cylinders have moduli $\frac{1}{2}\tan\frac{\pi}{2k}$ or all cylinders have moduli $\frac{1}{2}\tan\frac{\pi}{2k}$. For a double $(2k+1)$-gon, the cylinders in the horizontal direction all have modulus $\frac{1}{2}\tan\frac{\pi}{2k}$.*

Proof. First we consider the left set of cylinders in Figure 2. This case further breaks up until 2 cases; we will only do the case for $k = 2l$. Label points $B_1, \ldots, B_l$ as in the Figure 3. Angle $\angle AOB_j = \frac{(2j-1)\pi}{2k}$. We label the cylinders C_j from 1 to l from top to the middle. Let C_j have height h_j and circumference w_j. Normalize the $2k$-gon so that segment OB_j has length 1. Then

$$h_j = \cos\frac{(2j-1)\pi}{2k} - \cos\frac{(2j+1)\pi}{2k}$$
$$w_j = 2\sin\frac{(2j-1)\pi}{2k} + 2\sin\frac{(2j+1)\pi}{2k}$$

for $j < l$. For the case $j = l$, w_l does not have the above factor of 2. We have the following trig identities

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

Thus, the moduli

$$m_j = \frac{h_j}{w_j} = \frac{1}{2}\frac{\sin\frac{\pi}{2k}}{\cos\frac{\pi}{2k}} = \frac{1}{2}\tan\frac{\pi}{2k}$$

When, $j = l$, the factor of $1/2$ disappears.

Now we consider the right set of cylinders in Figure 2. The heights and widths are

$$h_j = \cos\frac{(2j-2)\pi}{2k} - \cos\frac{2j\pi}{2k}$$
$$w_j = 2\sin\frac{(2j-2)\pi}{2k} + 2\sin\frac{2j\pi}{2k},$$

and the argument goes the same way. We leave as an exercise to the reader the remaining cases. □

Let (X_n, ω_n) denote the regular n-gon for n even and the double n-gon for n odd. Let Γ_n be the Veech group of (X_n, ω_n). We first show that Γ_n contains a copy of the relevant triangle group listed in Theorem 2. Let $r_\theta \in \mathrm{SL}(2, \mathbb{R})$ be the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

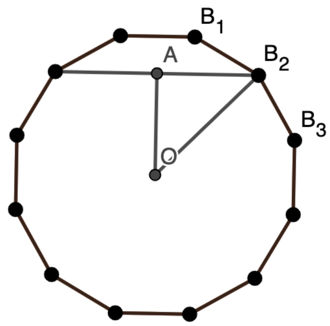

Figure 3: Heights and circumferences of the cylinders are calculated using trigonometry.

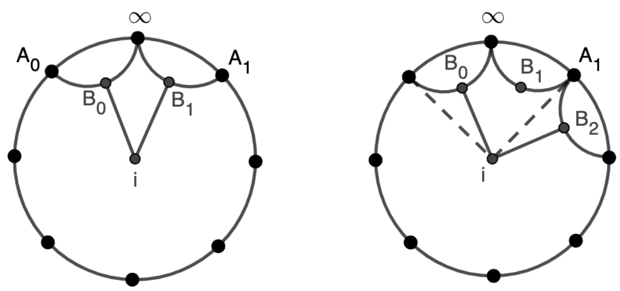

Figure 4: The figure is drawn for $n = 8$. Although the Poincare disk model is drawn, the points are labeled with the values of the corresponding points in the upper half plane model.

Corollary 4. *The matrices* $t_n = \begin{pmatrix} 1 & 2\cot\frac{\pi}{n} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ *and* $r_{\pi/n}^{-1} t_n r_{\pi/n}$ *are elements of* Γ_n.

Proof. Let $C_1, \ldots, C_l$ be the horizontal cylinders and m_i be the modulus of C_i. By Proposition 3, $2\cot\pi/n$ is an integer multiple of $1/m_i$ for every horizontal cylinder for n both even and odd, so $t_n \in \Gamma_n$ (see [Wri15, Proposition 3.4] for more details). For $r_{\pi/n}^{-1} t_n r_{\pi/n}$, we use the same argument after rotating the picture by π/n. $\square$

For even n, the regular n-gon has $r_{\frac{2\pi}{n}}$ in Γ_n. For odd n, the double n-gon has the smaller rotation $r_{\frac{\pi}{n}} \in \Gamma_n$ since it is the derivative of the affine homeomorphism that swaps the two polygons and rotates each by $\frac{\pi}{n}$. Thus, t_n and $r_{\pi/n}^{-1} t_n r_{\pi/n}$ are conjugate in Γ_n for odd n, but we will show later that they are not conjugate for even n. We first consider the group generated by $r_{\frac{\pi}{n}}, t_n$.

Proposition 5. *The group* $T_n := < r_{\frac{\pi}{n}}, t_n >$ *is isomorphic to the triangle group* $\Delta^+(2, n, \infty)$. *When n is even, the group* $S_n := < r_{\frac{2\pi}{n}}, t_n, r_{\frac{\pi}{n}} t_n r_{\frac{\pi}{n}}^{-1} >$ *is isomorphic to the triangle group* $\Delta^+(n/2, \infty, \infty)$.

Proof. We first try to construct a fundamental domain for $T_n \backslash \mathrm{SL}(2,\mathbb{R})$. We would like to caution the reader here as one normally thinks of a hyperbolic surface as a quotient of $\mathrm{PSL}(2,\mathbb{R})$. In this case, T_n contains $-\mathrm{Id}$, so $T_n \backslash \mathrm{SL}(2,\mathbb{R}) \cong (T_n/\{\pm \mathrm{Id}\}) \backslash \mathrm{PSL}(2,\mathbb{R})$.

In the upper half plane model, label $A_0 := -\cot\pi/n$ and $A_1 := \cot\pi/n$ (see Figure 4 left). Then $r_{\pi/n}$ takes A_0 to ∞ and ∞ to A_1. Note that $r_{\pi/n}$ corresponds to a $2\pi/n$ rotation. Let L_j be the geodesic connecting A_j and ∞ for $j = 0, 1$ and let B_j be the point on L_j closest to i. The rotation $r_{\pi/n}$ takes B_0 to B_1 and fixes i, so it must take the segments $\overline{iB_0}$ to $\overline{iB_1}$. In addition, t_n takes A_0 to A_1 and fixes ∞. Thus, it takes B_0 to B_1, so it takes the segment $\overline{\infty B_0}$ to $\overline{\infty B_1}$. Thus, by Poincare's Theorem on fundamental polygons (see [Bea95, Theorem 9.8.4]), the quadrilateral bounded by i, B_0, ∞, B_1 is a fundamental domain for the group T_n. This fundamental domain comes from unfolding a triangle with angles $(\frac{\pi}{2}, \frac{\pi}{n}, 0)$, so it is isomorphic to the group $\Delta^+(2, n, \infty)$.

We require n to be even for the second half of the proposition, so $\{\pm \mathrm{Id}\}$ is a subgroup of S_n and we can consider $S_n \backslash \mathrm{SL}(2, \mathbb{R})$. The argument for S_n is similar to the proof above. We will refer to the labeling scheme in Figure 4 right. The rotation $r_{\frac{2\pi}{n}}$ takes $\overline{iB_0}$ to $\overline{iB_2}$. The matrix t_n takes $\overline{B_0\infty}$ to $\overline{B_1\infty}$. The matrix $r_{\frac{\pi}{n}} t_n r_{\frac{\pi}{n}}^{-1}$ takes $\overline{B_1A_1}$ to $\overline{B_2A_1}$. Thus, by Poincare's theorem, the polygon with vertices i, B_0, ∞, A_1, B_2 is the fundamental domain of $S_n \backslash SL(2, \mathbb{R})$, which can be rearrange to form two copies of a triangle with angles $(\frac{2\pi}{n}, \infty, \infty)$. □

It is clear that $T_n < \Gamma_n$ for n odd and $S_n < \Gamma_n$ for all n, in particular for n even. We now show that these inclusions are actually isomorphisms.

Proposition 6. *Let $n \geq 8$ be even. The Veech group Γ_n has two conjugacy classes of parabolic elements represented by t_n and $t'_n := r_{\frac{\pi}{n}}^{-1} t_n r_{\frac{\pi}{n}}$.*

Proof. Assume by contradiction a conjugacy $t_n = a^{-1} t'_n a$ existed. An affine transformation ϕ with derivative a must map all of cylinders fixed by t_n to the cylinders fixed by t'_n. For $k = 2l + 1$, t_n and t'_n fix different numbers of cylinders. ϕ must also scale the moduli of the cylinders by a constant factor. However for $k = 2l$, t'_n fixes cylinders with all the same moduli while t_n fixes one cylinder with a different modulus from the rest. Thus, t_n and t'_n are not conjugate. □

The remainder of proof of Theorem 2 deviates from [Vee89] and follows that in [Hoo13].

Proof of Theorem 2. By Proposition 5, we have that $T_n < \Gamma_n$ for n odd. Recall the Gauss-Bonnet formula for a hyperbolic surface X with orbifold points with angles $\theta_1, \ldots, \theta_j$ is

$$area(X) = 2\pi(p + 2g - 2) + \sum_{i=1}^{j} (2\pi - \theta_i)$$

$\Delta^+(2, n, \infty) \backslash \mathrm{SL}(2, \mathbb{R})$ is genus 0 with 1 cusp and two orbifold points of angles π and $\frac{\pi}{n}$, so its area is $\frac{n-1}{n}\pi$. There is a covering map $\Delta^+(2, n, \infty) \backslash \mathrm{SL}(2, \mathbb{R}) \to \Gamma_n \backslash \mathrm{SL}(2, \mathbb{R})$ of finite degree. $\Gamma_n \backslash \mathrm{SL}(2, \mathbb{R})$ must still be genus 0 with 1 cusp. It must have positive area, so there must be at least two orbifold points. One orbifold point must have cone angle at most $\frac{\pi}{n}$, so the area of $\Gamma_n \backslash \mathrm{SL}(2, \mathbb{R})$ is at least $\frac{n-1}{n}\pi$. Thus, $\Gamma_n = T_n = \Delta^+(2, n, \infty)$.

By Proposition 5, $S_n < \Gamma_n$ for n even. The genus of $S_n \backslash \mathrm{SL}(2, \mathbb{R})$ is 0, there are 2 cusps and one cone singularity of area $\frac{2\pi}{n}$, so Gauss-Bonnet gives an area of $\frac{2n-2}{n}\pi$. By Proposition 6, $\Gamma_n \backslash \mathrm{SL}(2, \mathbb{R})$ also has genus 0 and 2 cusps. There also must be at least one orbifold point with angle at most $\frac{2\pi}{n}$. Thus, the area of $\Gamma_n \backslash \mathrm{SL}(2, \mathbb{R})$ is at least $\frac{2n-2}{n}\pi$, so $\Gamma_n = S_n = \Delta^+(n/2, \infty, \infty)$. □

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