Fukaya Categories and Intersection Numbers

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Submitted in partial fulfillment

of the requirements for the degree of

Bachelor of Arts

Department of Mathematics

Princeton University

May, 2018

This thesis represents my own work in accordance with University regulations.

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Abstract

We give an introductory survey of Floer homology and Fukaya categories assuming only basic symplectic geometry. The survey is meant to be targeted at a lower level than Auroux's survey [3]. We say very little about the analytical details involved in Fukaya categories and move quickly to discuss the algebraic side. We show an application of these algebraic techniques by proving a theorem by Keating about symplectic Dehn twists [14]. This theorem is a generalization of the theorem that if two curves α, β have minimal geometric intersection number ≥ 2, then the Dehn twists τ_α, τ_β generate a free subgroup of the mapping class group.

Acknowledgements

I would like to thank my advisor John Pardon for his explanations and guidance in helping me learn this material, as well as the care he has shown me in his advising. My mathematics have improved dramatically through my conversations with him. I would also like to thank my secondary advisor Henry Horton who has been exceedingly generous with his time and has helped me have significant development mathematically. Further, I am grateful to my mentors Peter Ozsváth, Dimitris Tsementzis, Ramon van Handel, and Lucas Culler who have all spent significant time helping me in various parts of my mathematical development. I'd like to thank Fine hall and the mathematical community around me, especially Maggie Miller, Dan Dore, and Semon Rezchikov. I'd like to thank my friends Joshua Wang and Roger Van Peski, who have been on this incredible mathematical and personal journey with me. I've learned more from them than I could have imagined. Lastly, I'd like to thank all of my friends and family who have supported me through my four years at Princeton and have made my time happy, enriching, and fulfilling.

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Chapter 1

Introduction

Gromov's paper "Pseudoholomorphic curves in symplectic manifolds" [11] introduced a drastically new way of studying symplectic manifolds by counting pseudoholomorphic curves. Floer found that counts of pseudoholomorphic curves form chain complexes defining what would become known as Floer homology [8],[9]. This provided the first techniques to prove the Arnold conjecture for general classes of symplectic manifolds. Further algebraic structure was discovered by Fukaya and others, and the Floer chain complexes were found to fit into the structure of A_∞-categories, algebra that had been studied in the context of homotopy theory. These algebraic structures, called Fukaya categories, could then be studied with a variety of techniques from homological algebra. Floer homology and Fukaya categories have a wide variety of applications to symplectic geometry and low-dimensional topology. Of significant importance, Fukaya categories are the language in which Kontsevich mathematically formalized the physical notions of mirror symmetry into the homological mirror symmetry conjecture [16]. Another remarkable application of Fukaya categories is the following:

Theorem 1.0.1 (Abouzaid and Kragh [2])**.** *A pair of lens spaces are diffeomorphic if and only if their cotangent bundles are symplectomorphic.*

The purpose of these notes is to introduce Floer homology and Fukaya categories to those with only basic knowledge of symplectic geometry, but [5] and [17] are great introductions to

symplectic geometry to be referred to as needed. In chapter 2, we describe Floer homology while avoiding the analytic technical difficulties. Instead, we focus on examples and figures. In section 2.1, we define pseudoholomorphic curves, and in sections 2.2 and 2.3, we discuss how counts of pseudoholomorphic curves come together to form Floer homology. Section 2.4 is an effort to understand the geometric content of Floer homology.

Chapter 3 is an introduction to Fukaya categories and some of the basic homological algebra used to study it. Section 3.1 is a geometric introduction to the A_∞ operations in Fukaya categories. Then, sections 3.2, 3.3, 3.4 become heavily algebraic. Our hope is that these sections can provide an overview of some of the homological algebra that the reader can then study in more depth in [23] or other references. In section 3.5, we apply this algebra to prove a concrete theorem 2.4.8.

The presentation is heavily influenced by the wonderful notes by Auroux [3].

Chapter 2

Floer Homology

Floer homology is inspired by Morse homology (see [19],[12],[21] for references on Morse homology). In particular, Floer homology is like infinite-dimesional Morse theory on the space of paths from one Lagrangian submanifold to another. Here the **action functional** acts like the Morse function, where the action is defined such that the difference in actions between two homotopic paths is the symplectic area that is sweeps out. The critical points of the action functional are intersections between these submanifolds, and the Morse trajectories are then pseudoholomorphic discs, which we define in this chapter. However, the stable and unstable manifolds of flows from a critical point may be infinite dimensional. Fortunately, the space of Morse flows lines from one critical point to another is finite dimensional under generic enough conditions. Still much work is needed to make these ideas rigorous. Lagrangian Floer homology can be thought of as a categorification of intersection number that is invariant under Hamiltonian isotopies. If two intersecting curves are isotoped to be disjoint, the intersection points must cancel along a shrinking pseudoholomorphic disc (see figure 2.0.1). In this chapter we will actually discuss Floer cohomology, which doesn't differ in any contentful way from Floer homology, but we will be using cohomology conventions throughout the paper.

References on symplectic topology include [5], [17]. References on Floer homology include [20], [18], [3] as well as Floer's original papers [8], [9].

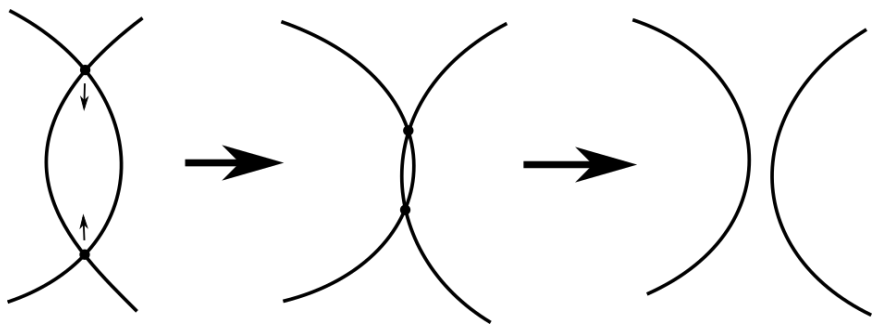


Figure 2.0.1: Two curves are isotoped to not intersection.

2.1 Pseudoholomorphic curves

Definition 2.1.1. Let M be a smooth manifold. An **almost complex structure** J is a smoothly varying choice of endomorphism $J_p : T_pM \to T_pM$ such that $J_p^2 = -\operatorname{id}_{T_pM}$ for all $p \in M$.

We use an abuse of notation and write $Ju = J_p u$ for $u \in T_pM$.

Example 2.1.2. A complex manifold M has a natural (almost) complex structure J on the tangent spaces. In coordinates $z_k = x_k + iy_k$, j is given by

$$J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}, \quad J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}$$

In this basis,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Definition 2.1.3. On a symplectic manifold (M, ω), an almost complex structure is **compatible** if for all $u, v \in T_pM$,

1. $\omega(u, Ju) > 0$
2. $\omega(Ju, Jv) = \omega(u, v)$

From a compatible almost complex structure, we will get a Riemannian metric

$$g(\cdot, \cdot) := \omega(\cdot, J\cdot)$$

g is clearly bilinear. g is positive definite by condition 1. By the second condition,

$$g(u,v) = \omega(u, Jv) = \omega(Ju, -v) = \omega(v, Ju) = g(v,u)$$

so g is symmetric.

Lemma 2.1.4. *Given a symplectic manifold (M, ω), there exists a compatible almost-complex structure.*

Proof. Choose a Riemannian metric g on M. We have isomorphisms $\tilde{g} : u \mapsto g(u, \cdot)$ and $\tilde{\omega} : u \mapsto \omega_p(u, \cdot)$ from $T_pM \to T_p^*M$. Define A such that at each point $p \in M$, $A := \tilde{g}^{-1} \circ \tilde{\omega} : T_pM \to T_pM$. This is a matrix on every tangent space. Then, for any $X, Y \in T_pM$, $g(AX, Y) = \tilde{g}(AX)(Y) = \tilde{\omega}_p(X)(Y) = \omega(X, Y)$. Furthermore,

$$g(A^*X, Y) = g(X, AY) = \omega(Y, X) = -\omega(X, Y) = g(-AX, Y)$$

so A is skew-symmetric. At every point, the matrix AA^* is symmetric and positive definite so it has a unique symmetric positive definite square root R. If $AA^* = BDB^{-1}$ is a diagonalization, then $R = B\sqrt{D}B^{-1}$. Since A is skew-symmetric, A and A^* are simulaneously diagonalizable. This implies $AR = RA$. Define $J = R^{-1}A$. This is a fiberwise orthogonal map because

$$\begin{aligned} g(JX, JY) &= g(R^{-1}AX, R^{-1}AY) = g(AX, (R^{-1})^* R^{-1} AY) \\ &= g(AX, (AA^*)^{-1} AY) = g(X, Y) \end{aligned}$$

Additionally,

$$J^* = (R^{-1}A)^* = A^*(R^{-1})^* = R^{-1}A^* = -R^{-1}A = -J$$

which shows that $J^2 = -1$. This is a compatiable almost complex structure because

$$\omega(JX, JY) = g(Ju, v) = g(u, -Jv) = \omega(u, v) \qquad \square$$

The space of almost complex structures on a manifold is contractable, so many constructions involving almost complex structures do not depend on which one we choose.

Definition 2.1.5. Let (M, ω) be a symplectic manifold with almost complex structure J, and let (Σ, j) be a Riemann surface. Then, a **psuedoholomorphic curve** (or J-holomorphic curve) in M is a map $\phi : \Sigma \to M$ s.t. $d\phi \circ j = J \circ d\phi$.

Example 2.1.6. Holomorphic curves are examples of pseudoholormorphic curves where the pseudoholomorphic structure comes from the structure of a complex manifold. For example, $\mathbb{C}P^2$ has a symplectic form called the Fubini-Study form compatible with the almost complex structure. Then projective algebraic curves are holomorphic curves in $\mathbb{C}P^2$.

Example 2.1.7. Let (M, ω, J) be a symplectic surface with compatible complex structure. For every closed topological disc D in M, up to conformal automorphism there is a unique holomorphic disc whose image is D by the following theorem.

Theorem 2.1.8 (Riemann mapping theorem)**.** *Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For every $J \in \mathcal{J}(\overline{\mathbb{D}})$, there is a unique orientation preserving diffeomorphism ψ s.t. $\psi^* i = J$ up to automorphisms of the disc.*

A proof of this theorem can be found in appendix C.5 of [18].

2.2 Moduli Spaces

Definition 2.2.1. Let (M^{2n}, ω) be a symplectic manifold. A submanifold $L^n \subset M$ is a **Lagrangian submanifold** or a Lagrangian if $\omega\big|_L = 0$.

Note that on symplectic surfaces all curves are Lagrangian since their tangent spaces are 1-dimensional and symplectic forms are skew-symmetric. See [5], [17] for more information about Lagrangian submanifolds.

Let L_0, L_1 be two transversely intersecting oriented compact Lagrangian submanifolds of (M, ω). We will define the cochain complex $CF(L_0, L_1)$ generated over $\mathbb{Z}/2\mathbb{Z}$ by the intersection points. The differential d will be given by counting pseudoholomorphic discs with certain boundary conditions, which we define below.

Definition 2.2.2. Let $p, q \in L_0 \cap L_1$ be two intersection points. Let $\mathbb{R} \times [0,1]$ have coordinates (s,t). A **pseudoholomorphic disc** (or pseudoholomorphic strip) from p to q is a smooth map $u : \mathbb{R} \times [0,1] \to (M, \omega)$ satisfying the Cauchy-Riemann equations

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0 \tag{2.1}$$

with the following boundary conditions:

1. $u(s,0) \in L_0$ and $u(s,1) \in L_1$ for all $s \in \mathbb{R}$
2. $\lim_{s\to+\infty} u(s,0) = p$, $\lim_{s\to-\infty} u(s,0) = q$

Remark 2.2.3. Condition (2.1) is equivalent to the definition of pseudoholomorphic curve given earlier

$$du \circ j = J \circ du \tag{2.2}$$

where j is the standard complex structure on the strip given by $j\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t}$ and $j\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial s}$. By plugging in $\frac{\partial}{\partial t}$, we get the Cauchy-Riemann equations:

$$du \circ j\left(\frac{\partial}{\partial t}\right) = du\left(-\frac{\partial}{\partial s}\right) = -\frac{\partial u}{\partial s} = J\frac{\partial u}{\partial s} \tag{2.3}$$

Plugging in $\frac{\partial}{\partial s}$ we similarly get the Cauchy-Riemann equations multiplied by J. Thus, these equations are equivalent.

Remark 2.2.4. By the Riemann mapping theorem, a pseudoholomorphic disc is the same as a mapping from a disc missing two boundary points to M. The boundary conditions imply that the map extends continuously to the whole closed disc, and the two points map to p, q respectively.

Let $p, q \in L_0 \cap L_1$ be intersection points between two Lagrangians. Define

$$\widehat{\mathcal{M}}(p, q; [u], J)$$

to be the space of all J-holomorphic discs from p to q that are homotopic to u relative to $L_0 \cup L_1$. It is a topological subspace of $\mathcal{C}^\infty(\mathbb{R} \times [0,1], M)$. Our hope is that $\widehat{\mathcal{M}}(p, q; [u], J)$

is a manifold of dimension $\mathrm{ind}([u])$, where $\mathrm{ind}([u])$ is an integer-valued invariant of the homotopy class $[u]$ called the **Maslov index**.

There is an $\mathbb{R}$-action on $\widehat{\mathcal{M}}(p,q;[u],J)$ given by

$$u(s,t) \mapsto u(s+r,t)$$

Like in Morse homology we will mod out by this action to get the unparametrized moduli spaces

$$\mathcal{M}(p,q;[u],J) = \widehat{\mathcal{M}}(p,q;[u],J)/\mathbb{R}$$

This space can be naturally compactified by the Gromov compactness theorem [11] to give a compact space $\overline{\mathcal{M}}(p,q;[u],J)$. There are many analytic technical difficulties to formalize these ideas that we do not discuss here (see [18],[8]).

If $\mathrm{ind}([u]) = 1$, then $\overline{\mathcal{M}}(p,q;[u],J)$ is a compact 0-dimensional manifold, so we can count the number of points mod 2, which we denote $\#\overline{\mathcal{M}}(p,q;[u],J)$. We would like to define a cochain complex $CF(L_0,L_1)$ generated by the intersection points $L_0 \cap L_1$ with a differential that counts pseudoholomorphic discs between these intersection points:

$$d(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]:\mathrm{ind}([u])=1}} \#\overline{\mathcal{M}}(p,q;[u],J)q$$

where the sum is over all intersection points $L_0 \cap L_1$ and all homotopy classes of discs with Maslov index 1. It still remains to prove $d^2 = 0$.

Example 2.2.5. Figure 2.2.1 shows examples of dimension 0 and dimension 1 moduli spaces. In the left picture, there is a pseudoholomorphic disc from p to q, but there is no pseudoholomorphic disc from q to p because the left side of the disc must be sent to L_0 (blue) and the right side of the disc must be sent to L_1 (red). To compute the Maslov index we have the following method. Parametrize the paths from p to q along L_0, L_1 respectively and count how many times the tangent spaces are not transverse at a particular time t. This computes the Maslov index. In this picture, the Maslov index is 1 because the tangent spaces are non-tranverse once when they both are both vertical (we choose a parametrization

of the curve s.t. this happens at the same time). Thus, there is one nonzero differential $d(p) = q$.

In the right picture, we have the following differentials

$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ a \arrow[dr] & & b \arrow[dl] \\ & q & \end{tikzcd}$$

The Maslov index of the heart-shaped region from p to q is 2 since the tangent spaces are non-transverse when the tangent spaces are both vertical and both horizontal. To see the 1-dimensional moduli space geometrically, we may make a slit in L_1 (shown in red in the picture) of various lengths, and this determines different pseudoholomorphic disc. We may lengthen the slit until it reaches point a, and the pseudoholomorphic disc breaks into two discs, one from p to a and one from a to q. We may also make slits with the L_0 curve. The ends of this moduli space are the two broken pseudoholomorphic discs from p to q. We notice in this example that indeed $d^2 = 0$.

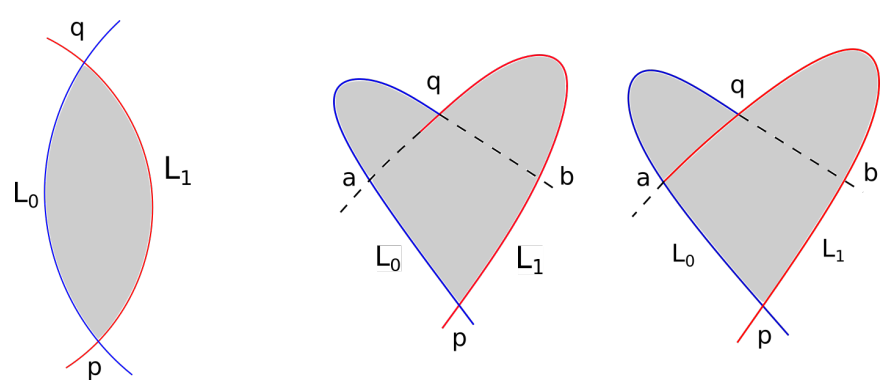


Figure 2.2.1: Left is a pseudoholomorphic disc from p to q. Right is a one-dimensional moduli space of discs from p to q, and one of the ends is shown.

To compute d^2, one counts broken strips like figure 2.2.2 left. Just like in Morse homology, we would like to prove that these broken strips are boundaries of the one dimensional parts of the compactified moduli spaces. By the Gromov compactness theorem, the moduli

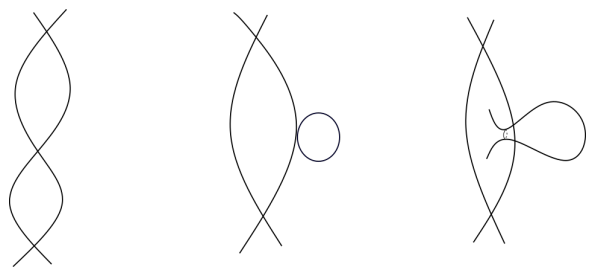


Figure 2.2.2: Left is a broken pseudoholomorphic strip. Center is a disc bubble. Right is a sphere bubble.

space can be compactified with three types of ends (see figure 2.2.2). The phenomenon that sequences of pseudoholomorphic curves can converge to ends that are not broken flow lines is called **bubbling**. We would like to rule out these other types of ends. The proof of $d^2 = 0$ can then be finished with Floer's gluing theorem which roughly states that each of these broken flow lines is an end of a 1-dimensional moduli space. All of this is discussed in detail in [18].

Example 2.2.6. Figure 2.2.3 illustrates an example where $d^2 \neq 0$. There is one pseudoholomorphic disc from p to q because the left boundary of the disc must map to the blue curve and the right boundary of the disc must map to the red curve. Thus, $d(p) = q$. Similarly, there is one pseudoholomorphic disc from q to p, so $d(q) = p$, so $d^2 = \mathrm{id}$. $d^2 \neq 0$ can happen when bubbling occurs. In this case, one end of the moduli space was a broken strip while the other was a disc whose boundary was the red curve. Thus, the broken strips are not matched, and there are an odd number of them.

2.3 Lagrangian Intersection Floer Cohomology

The simplest setting for Floer cohomology is that of exact symplectic manifolds.

Definition 2.3.1. (M, ω) is an **exact symplectic manifold** if $\omega = d\alpha$ for a 1-form α. $L \subset M$ is an **exact Lagrangian submanifold** if $\alpha\big|_L = df_L$ for some function $f_L : L \to \mathbb{R}$.

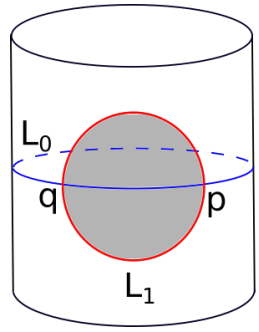


Figure 2.2.3: A case where $d^2 \neq 0$.

Example 2.3.2. The cotangent bundle T^*M of a manifold M^n has a canonical symplectic form given by $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$, where the q_i are coordinates on M and the p_i are standard coordinates on the fibers. This is an exact symplectic form $\omega = d\alpha$, where $\alpha = -\xi d\theta$. The zero section L is an exact Lagrangian since $\alpha\big|_L \equiv 0$.

The graph of a closed 1-form is also exact. Let η be a closed 1-form, and $L = \{(q, \eta_q) : q \in M\}$ be its graph. Then $\alpha = -\eta_q dq = -d(q\eta_q)$ on L, so it is an exact Lagrangian.

Example 2.3.3. A non-compact orientable surface has an exact symplectic structure because its second cohomology is trivial. A closed symplectic manifold cannot be exact since ω has a nontrivial de Rham cohomology class.

Theorem 2.3.4. *Let (M, ω) be an exact symplectic manifold and L_0, L_1 be tranverse oriented exact Lagrangians. Then the cochain complex:*

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}/2\mathbb{Z} \cdot p$$

is generated as a $\mathbb{Z}/2\mathbb{Z}$ vector space by the intersection points of L_0, L_1. It has a $\mathbb{Z}/2\mathbb{Z}$ grading where positive intersections $L_0 \cap L_1$ have degree 0 and negative intersections have degree 1. The Floer differential:

$$d(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]:\mathrm{ind}([u])=1}} \#\mathcal{M}(p, q; [u], J)q$$

is well-defined and satisfies $d^2 = 0$. The isomorphism class of the Floer cohomology

$HF(L_0, L_1) = H^*(CF(L_0, L_1, J), d)$ is independent of the chosen almost complex structure J. It is also invariant under Hamiltonian isotopies of L_0, L_1.

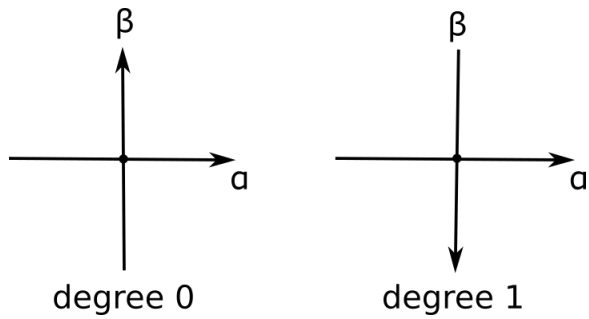


Figure 2.3.1: The degrees of the intersection points in $CF(\alpha, \beta)$

Remark 2.3.5. By invariance we mean that Hamiltonian isotopies of L_0, L_1 give chain homotopy equivalences between the Floer cochain complexes. The same is true when we change J.

Remark 2.3.6. Floer cohomology can be defined with $\mathbb{Z}$ coefficients by giving L_0, L_1 spin structures, which allows the moduli spaces $\mathcal{M}(p, q; [u], J)$ to be oriented.

Remark 2.3.7. We have defined a $\mathbb{Z}/2\mathbb{Z}$ grading on the Floer cochain complexes, but other gradings could be defined as well if the relative index between points p and q did not depend on the choice of pseudoholomorphic disc connecting them. Different homotopy classes of discs between p and q differ in Maslov class by multiples of $2c_1(M)$, where $c_1(M)$ is the first Chern class of $TM \to M$ thought of as a complex vector bundle, where the complex structure is given by J (see section 2.7 of [17]). Thus, we can define a $\mathbb{Z}/d\mathbb{Z}$ grading for a factor d of $2c_1(M)$. This will only be a relative grading i.e. only the relative index between two points p, q is defined.

Corollary 2.3.8. *The Euler characteristic of Floer cohomology is the algebraic intersection number.*

The case of surfaces can be made combinatorial due to the Riemann mapping theorem (theorem 2.1.8). In this case we can compute Floer cohomology by counting topological

discs with the right boundary conditions i.e. elements of $\pi_2(M, L_0 \cup L_1)$ see [1], [6].

Example 2.3.9. Consider two curves on a cylinder (see figure 2.3.2). There are two differentials from p to q that cancel. Thus, the differential is 0. We have that $HF(L_0, L_1) \cong (\mathbb{Z}/2\mathbb{Z})^2 \cong H^*(S^1; \mathbb{Z}/2\mathbb{Z})$.

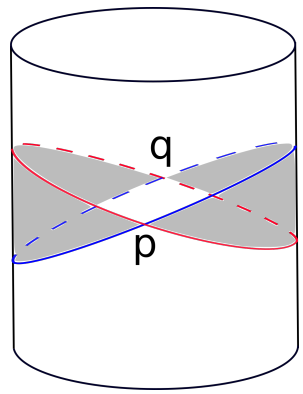


Figure 2.3.2: Two pseudoholomorphic discs from p to q of equal area.

There are cases when it is useful to consider nonexact Lagrangians. For example, compact symplectic manifolds are not exact. In this case we must keep track of the **energy** or **symplectic area** of a pseudoholomorphic disc $u : D \to M$,

$$E(u) := \int_D u^*\omega$$

This quantity is finite since D is compact, and it is an invariant of the homotopy class u. In general, there may be an infinite number of pseudoholomorphic discs coming from p, but there are only finitely many with energy bounded by a fixed constant C by Gromov compactness. Thus, we can use the following field of coefficients to avoid infinite sums in the Floer differential.

Definition 2.3.10. The **Novikov ring** over $\mathbb{K}$ is like a power series ring with formal variable T,

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

The **Novikov field** is the field of fractions of Λ_0 which is

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\}$$

In the nonexact setting, we use a cochain complex with coefficients in the Novikov field

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p$$

and the differential becomes

$$d(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]:\mathrm{ind}([u])=1}} \#\mathcal{M}(p, q; [u], J) T^{E(u)} q$$

Remark 2.3.11. In an exact symplectic manifold $(M, d\alpha)$ with exact Lagranagians $(L_0, df_0 = \alpha\big|_{L_0}), (L_1, df_1 = \alpha\big|_{L_1})$, the energy of a pseudoholomorphic disc between intersection points p, q is fixed:

$$\int_{\mathbb{R}\times[0,1]} u^*\omega = \int_{\mathbb{R}\times[0,1]} du^*\alpha = \int_{\partial(\mathbb{R}\times[0,1])} u^*\alpha = f_0(p) - f_0(q) + f_1(p) - f_1(q)$$

Therefore, there are therefore only finitely many such pseudoholomorphic discs by Gromov compactness all with the same energy. Thus, we do not need to worry about Novikov coefficients in the exact setting.

Example 2.3.12. Now we consider non-exact Lagrangians on the cylinder (figure 2.3.2). Then the differential is

$$d(p) = (T^{E(u_1)} + T^{E(u_2)})q$$

where u_1, u_2 are the two pseudoholomorphic discs from p to q. If the curves are Hamiltonian isotopic, they must sweep out 0 area i.e. the two gray regions must have the same area. Thus, the differential is zero. If the gray regions have different areas, then $T^{\omega(u_1)} + T^{\omega(u_2)}$ is an invertible element of the Novikov field, so there is a nonzero differential between two generators of the cochain complex. Thus, cohomology is zero. This makes sense because in this case these curves can be Hamiltonian isotoped to be disjoint. This example shows another reason to keep track of the energy of the pseudoholomorphic curves.

Using the fact that Floer cohomology is invariant under Hamiltonian isotopy, we can define Floer cohomology between non-tranverse Lagrangians. If L_0, L_1 are not transverse, we define $HF(L_0, L_1)$ by choosing a Hamiltonian isotopy L_1^+ of L_1 s.t. L_0, L_1^+ are transverse. The main reason for this is to define the self-Floer cohomology groups $HF(L, L)$.

Theorem 2.3.13 (Floer)**.** *If $[\omega] \cdot \pi_2(M, L) = 0$, the self Floer cohomology $HF(L, L) \cong H_*(L; \Lambda)$, the singular cohomology of L with Novikov coefficients.*

Remark 2.3.14. Example 2.3.9 is an instance of this theorem. Note that example 2.3.12 is not a counterexample to this theorem because a Hamiltonian isotopy must sweep out zero area.

2.4 Relations to intersection numbers

As we have seen the Euler characteristic of $CF(\alpha, \beta)$ is the algebraic intersection number of α, β (corollary 2.3.8), so Floer cohomology is some kind of categorification of intersection number. We would like to continue to explore this analogy. On surfaces, we will see that $\dim HF(\alpha, \beta)$ is the same as the following:

Definition 2.4.1. Let α, β be curves on a surface. The **geometric intersection number** $i(\alpha, \beta)$ is the minimal number of intersection points between all curves α', β' such that α' is isotopic to α and β' is isotopic to β.

By the Riemann mapping theorem 2.1.8, Floer cohomology of surfaces becomes combinatorial. This is worked out by de Silva, Robbin, and Salamon in [20]. They show that for two nonisotopic curves, their Floer cohomology is independent of the smooth isotopy class, not just the Hamiltonian isotopy class.

Theorem 2.4.2 (Theorem 9.2 in [6])**.** *Let α, β be non-isotopic curves in a surface Σ. Let α' be isotopic to α and β' be isotopic to β. Then*

$$HF(\alpha, \beta) \cong HF(\alpha', \beta')$$

On a seemingly unrelated note, the following is a criterion for whether two curves intersect minimally.

Theorem 2.4.3 (Proposition 1.7 in [7])**.** *α, β intersect minimally in their isotopy class iff they bound no bigon i.e. there is not embedded disc whose boundary is $\alpha \cup \beta$.*

We see that bigons are the same as pseudoholomorphic discs in this setting. Thus, we get the following corollary.

Corollary 2.4.4. *Let α, β be non-isotopic. Then*

$$i(\alpha, \beta) = \dim HF(\alpha, \beta)$$

Proof. α, β can be smoothly isotoped such that there are no embedded bigons. In that case, they are minimally intersecting and $i(\alpha, \beta) = \dim HF(\alpha, \beta)$. □

Thus, if $HF(\alpha, \beta) = 0$, then α, β may be a smoothly isotoped to be disjoint. However, they may not necessarily be made disjoint through a Hamiltonian isotopy.

Example 2.4.5. In figure 2.4.1, the two curves α, β separate the surface into four regions, two of which are labeled A and B. Region A is the single differential between the two intersection points between the curves. Thus, $HF(\alpha, \beta) = 0$. A Hamiltonian isotopy of the curves that reduces region A by a fixed area will reduce the area of region B by the same amount. Thus, if A has larger area than B, these curves cannot be Hamiltonian isotoped to not intersect, but clearly the curves can be smoothly isotoped to not intersect.

In the case of surfaces, the geometric intersection number is the same as the dimension of Floer cohomology. However, in higher dimensions it is unclear what the relationship between these quanities is since geometric intersection number is a smooth isotopy invariant, and Floer cohomology is a Hamiltonian isotopy invariant. Any interesting question may be if certain facts about surfaces that are based on geometric intersection number generalize to symplectic manifolds of dimension ≥ 4 using Floer cohomology. An example of this is the following:

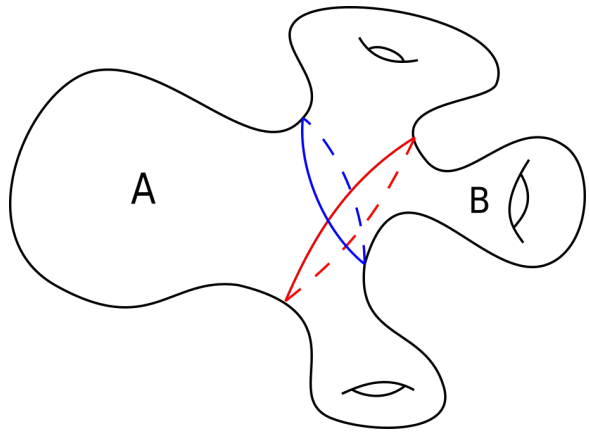


Figure 2.4.1: Curves that can be smoothly isotoped to be disjoint, but that cannot be Hamiltonian isotoped to be disjoint.

Theorem 2.4.6 (Ishida [13])**.** *Suppose α, β are a pair of simple closed curves on an oriented surface, and $i(\alpha, \beta) \geq 2$. Then the group generated by τ_α and τ_β, the Dehn twists around α and β, is a free subgroup of the mapping class group.*

To generalize this result, we first must define a **symplectic Dehn twist**, which is basically a Dehn twist that is a symplectomorphism. Let S be a Lagrangian sphere in a symplectic manifold M^{2n}. By Weinstein's tubular neighborhood theorem [5], there is a tubular neighborhood of S that is symplectomorphic to a neighborhood of the zero section of T^*S with canonical symplectic form. Thus, it suffices to consider the case of T^*S^n. Consider local coordinates q on S^n and let p be fiber coordinates of T^*S^n. We define $H(q,p) = h(\|p\|)$, where $h : [0, \infty) \to \mathbb{R}$ is a smooth function s.t. $h'(0) = \pi$, $h'' \leq 0$ and h is constant outside a small enough neighborhood of 0. Then, H is a Hamiltonian function on $T^*S^n \backslash S^n$. Applying the time-one Hamiltonian flow, we get a symplectomorphism on $T^*S^n \backslash S^n$ that extends to a symplectomorphism on T^*S^n as the antipodal map on S^n. The Hamiltonian isotopy class of the curve doesn't depend on the choices made in this construction. Thus, symplectic Dehn twists are elements of the following group:

Definition 2.4.7. The **symplectic mapping class group** of a symplectic manifold (M, ω) is the group of symplectomorphisms of M quotiented out by symplectic isotopies.

Now we can state the symplectic generalization of Ishida's theorem.

Theorem 2.4.8 (Keating [14]). *Let (M, ω) be an exact symplectic manifold in M, and let S_0, S_1 be exact Lagrangian spheres such that* $\dim HF(S_0, S_1) > 2$. *Then the group generated by the symplectic Dehn twists τ_{S_0} and τ_{S_1} is a free subgroup of the symplectic mapping class group.*

This result also holds under some mild conditions for $HF(S_0, S_1) = 2$. We will prove this theorem in section 3.5.

Chapter 3

Fukaya Categories

A_∞-algebra captures the idea of homotopy. Consider for example a topological space X and base point $*$, and let ΩX be the space of based loops $f : [0,1] \to X$. Let $a, b, c \in \Omega X$. There is an operation that we call $\mu^2(b,a)$ that goes twice as fast and traverses a in time $1/2$ and then b in the second half. Then, $\mu^2(c, \mu^2(b,a))$ and $\mu^2(\mu^2(c,b),a)$ are not the same loop (see figure 3.0.1), but there is a homotopy $\mu^3(c,b,a)$ that is the homotopy that relates them. Similarly, there may be a homotopy μ^4 that relates the μ^3 homotopies, and we can continue this process ad infinitum. As we have seen above μ^2 is not associative, but it is associative up to homotopy as we know $\pi_1(X,*)$ is a group. Similarly, these μ^k operations satisfy A_∞-relations that can be thought of as generalized associativity up to homotopy. Floer cochains has these μ^k operations that satisfy the A_∞-relations. The induced product on cohomology is associative. Under suitable hypotheses, we would like to associate to a

Figure 3.0.1: Left corresponds to $\mu^2(c, \mu^2(b,a))$ while right corresponds to $\mu^2(\mu^2(c,b),a)$.

symplectic manifold (M, ω) an A_∞-category such that the objects are oriented embedded Lagrangian submanifolds and the morphisms are Floer cochain complexes, although once again there are various technical difficulties in formalizing this idea. This is called the

Fukaya category Fuk(M). This object can then be studied with a variety of methods in homological algebra. In this section, we will use $\mathbb{Z}/2\mathbb{Z}$ coefficients for simplicity (see [23] for signs). Some references on Fukaya categories include [3], [4],[24], [23], [10]. References on A_∞-algebra include [15], [23].

3.1 Product Operations

There are operations on Lagrangian Floer cohomology

$$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)[2-k]$$

given by counting pseudoholomorphic polygons. Let $L_0, \ldots, L_k$ be pairwise transverse Lagrangians, and let $q \in L_0 \cap L_k$, $p_i \in L_{i-1} \cap L_i$ for $1 \leq i \leq k$. We will consider maps from the disc D to M where D has $k+1$ marked points $z_0, \ldots, z_k$ in that order around the boundary of D (see middle shape in figure 3.1.1). Define

$$\mathcal{M}(p_1, p_2, \ldots, p_k, q; [u], J)$$

as the moduli space maps from D to M homotopic to u, where z_0 maps to q, z_i maps to p_i and the corresponding regions on the boundary of the disc map to $L_0, \ldots, L_k$. The space of $k+1$ marked points on D mod $PSL(2, \mathbb{R})$ is a $k-2$ dimensional polytope (see figure 3.1.1), so

$$\dim \mathcal{M}(p_1, \ldots, p_k, q, [u]) = k - 2 + \mathrm{ind}([u])$$

Thus, we should count u of index $2-k$ mod 2.

$$\mu^k(p_k, \ldots, p_1) = \sum_{\substack{q \in L_0 \cap L_k \\ [u]:\mathrm{ind}([u])=2-k}} \#\mathcal{M}(p_1, \ldots, p_k, q, [u]) T^{\omega(u)} q$$

We encounter transversality and compactness difficulties similar to those before in order to ensure that the moduli spaces are compact manifolds, but we don't go into them here. The moduli space $\mathcal{R}^{k+1}$ of discs with $k+1$ marked points is not compact because for example two points can approach each other. The Deligne-Mumford compactification of this space

that associates nodal curves as these limit points (see figure 3.1.1). The Deligne-Mumford compactification $\overline{\mathcal{R}^{k+1}}$ are $k-2$ dimensional manifolds with corners known as the Stasheff associahedra $\mathcal{K}^k$ [4].

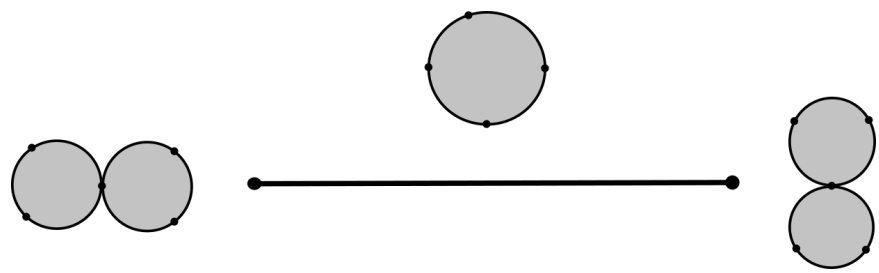


Figure 3.1.1: The 1-dimensional moduli space of 4 pointed discs. The ends are nodal discs.

The associahedra $\mathcal{K}^k$ are polytopes whose vertices are different ways to combine k points using a binary operation (see vertices of figure 3.1.2). These vertices can be represented by rooted trees s.t. each of the $n-1$ interior nodes has degree three. Two vertices are connected by an edge if there is an edge in each corresponding tree that can be shrunk to produce the same tree. This process similarly determines all n dimensional faces (figure 3.1.2).

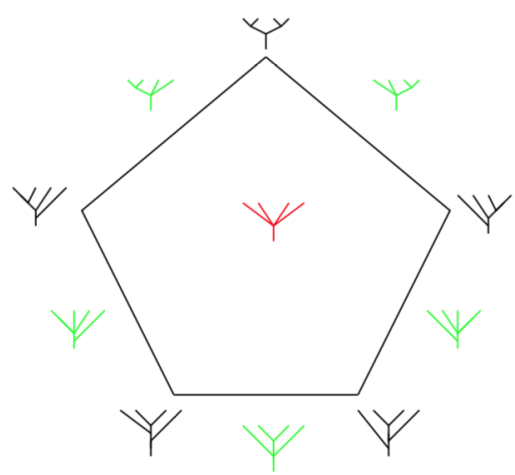


Figure 3.1.2: $\mathcal{K}^4$ and corresponding trees. Picture from [4].

The μ^k operations satisfy certain A_∞**-relations** because the ends of 1-dimensional moduli spaces have an even number of points. The μ^2 operation satisfies

$$d\mu^2(p_2, p_1) + \mu^2(p_2, dp_1) + \mu^2(dp_2, p_1) = 0 \tag{3.1}$$

because the ends in figure 3.1.3 are the only types of ends of pseudoholomorphic triangles when no bubbling occurs.

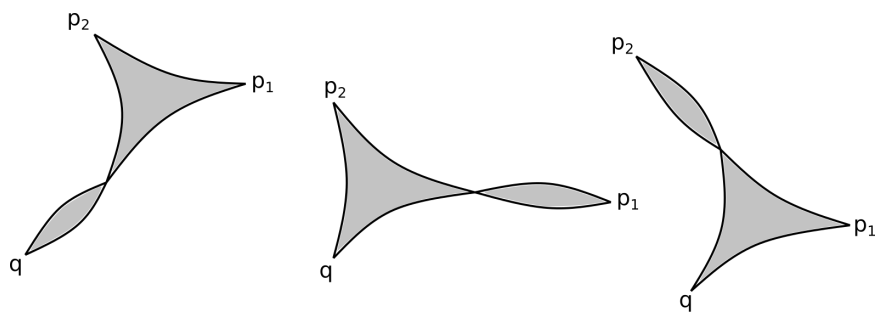


Figure 3.1.3: 3 types of ends of 1-dim moduli spaces of pseudoholomorphic triangles.

In this case, the only type of ends were ones with bigons at the vertices of the triangles. The ends of pseudoholomorphic squares include ones where a square breaks into two triangles as in figure 3.1.1, which correspond to $\mu^2(p_3, \mu^2(p_2, p_1)), \mu^2(\mu^2(p_3, p_2), p_1)$. In general these are the two possible types of ends. The latter case corresponds to codimension 1 faces of the Stasheff associahedra. The A_∞ relations come from summing all of the ends of the 1-dimensional moduli space (3.2).

Now we talk about a pictoral notation for the A_∞-relations. To denote the d and μ^k operations we use **rooted trees** with one central vertex. Here a rooted tree is a tree with a distinguished leaf (figure 3.1.4).

The ends that are polygons with a bigon on the corner are represented as a μ^k operation with an additional vertex on one of the leaves. The codimension 1 faces of the associahedra are represented by rooted trees with two degree ≥ 3 vertices next to each other as in the edges of figure 3.1.2. Thus, the A_∞-relations are obtained by adding together all such rooted trees (figure 3.1.6).

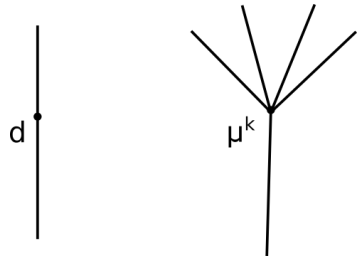


Figure 3.1.4: The d and μ^k operations depicted as trees.

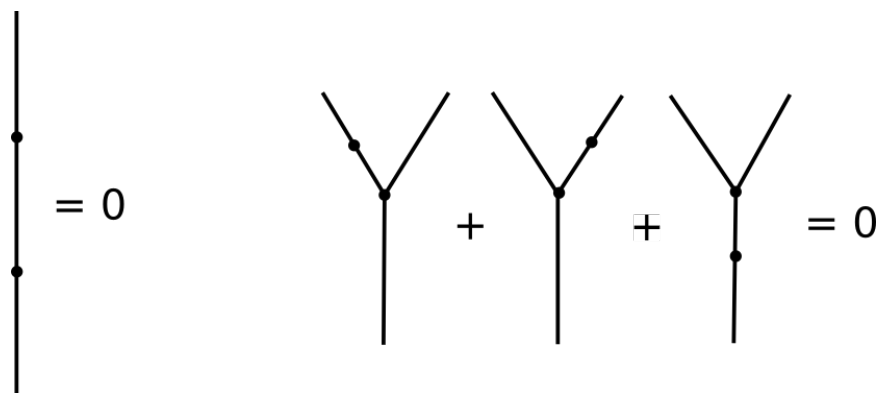


Figure 3.1.5: Left represents $d^2 = 0$. Right represents $\mu^2(db, a) + \mu^2(b, da) + d\mu^2(b, a) = 0$.

3.2 Defining the Category

Definition 3.2.1. A non-unital A_∞**-category** $\mathcal{A}$ is a collection of objects $Ob\mathcal{A}$ and for every pair of objects $X, Y \in \mathcal{A}$ a $\mathbb{Z}/2\mathbb{Z}$-graded group $\mathcal{A}(X, Y) = \hom_{\mathcal{A}}(X, Y)$. In addition, there are composition maps of every order $k \geq 1$,

$$\mu^k : \mathcal{A}(X_{k-1}, X_k) \otimes \mathcal{A}(X_{k-2}, X_{k-1}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_k)[2-k]$$

satisfying for each k the A_∞ relations (figure 3.1.6)

$$\sum_{m,n} \mu^{k-m+1}(a_k, \ldots, a_{n+m+1}, \mu^m(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1) = 0 \tag{3.2}$$

where the sum is over integers m, n s.t. $0 \leq n$, $1 \leq m$, and $m + n \leq k$. We use the notation $da = \mu^1(a)$.

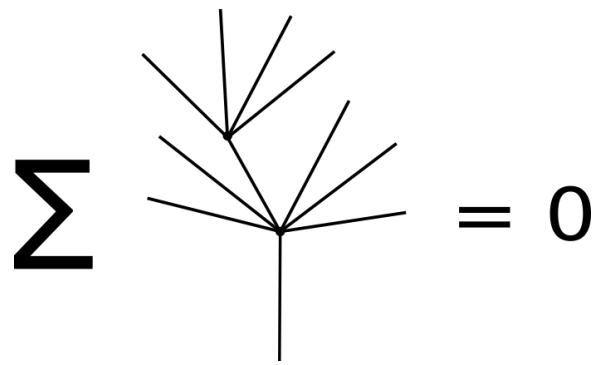


Figure 3.1.6: Sum over all two-vertex rooted trees is zero.

We look at this definition in more depth. For $k = 1$, we have that

$$d : \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_1)[1]$$

satisfies the relation $d^2 a_0 = 0$ for any $a_0 \in \mathcal{A}(X_0, X_1)$, so $\mathcal{A}(X_0, X_1)$ is a chain complex with d as its differential. The map

$$\mu^2 : \mathcal{A}(X_1, X_2) \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_2)$$

satisfies $d\mu^2(a_1, a_0) = \mu^2(da_1, a_0) + \mu^2(a_1, da_0)$. Thus, μ^2 satisfies a product rule.

$$\mu^3 : \mathcal{A}(X_2, X_3) \otimes \mathcal{A}(X_1, X_2) \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_3)[-1]$$

satisfies

$$\begin{aligned} &\mu^2(\mu^2(a_2, a_1), a_0) + \mu^2(a_2, \mu^2(a_1, a_0)) \\ &= d\mu^3(a_2, a_1, a_0) + \mu^3(da_2, a_1, a_0) + \mu^3(a_2, da_1, a_0) + \mu^3(a_2, a_1, da_0) \end{aligned}$$

This relation shows that μ^2 is associative up to homotopy, with the homotopy given by μ^3.

Thus, there is an associated (non-unital) cohomology category $H(\mathcal{A})$ with the same objects as $\mathcal{A}$ and morphism spaces $H(\mathcal{A})(X, Y) = H(\mathcal{A}(X, Y), d)$, with an associative operation given by μ^2:

$$[a_2] \cdot [a_1] := [\mu^2(a_2, a_1)]$$

We would like the **Fukaya category** of a symplectic manifold to be an A_∞-category whose objects are the oriented compact Lagrangian submanifolds and the morphism spaces are the Floer cochain complexes $CF(L_0, L_1)$. The A_∞ operations are the maps counting pseudoholomorphic polygons. We have already seen that these maps satisfy the A_∞ relations. In practice, this becomes difficult to define due to transversality issues. Abouzaid in [1] and Kontsevich in [16] consider **A_∞-precategories** where morphism spaces $CF(L_0, L_1)$ are only defined between transverse pairs of Lagrangians. However, they do not have as rich of an algebraic structure as bona fide A_∞ categories. In particular they do not have self-cochain complexes $CF(L_0, L_0)$. One way to solve these issues is to choose Hamiltonian perturbations for every n-tuple of Lagrangians to make them pairwise transverse. By choosing enough perturbations, we get an A_∞ category. The category is not canonical, but the quasi-equivalence class of the A_∞ category is determined by this construction. See [23] for a rigorous treatment of this in the exact case. When self-cochain complexes are defined, we can talk about units.

Definition 3.2.2. $\mathcal{A}$ is **c-unital** if $H(\mathcal{A})$ is a unital category.

Definition 3.2.3. Let $\mathcal{A}$ be a c-unital A_∞-category. A closed morphism $e \in \mathcal{A}(L_0, L_1)$ is a **quasi-isomorphism** if there is a closed morphism $e^{-1} \in \mathcal{A}(L_1, L_0)$ s.t.

$$[e^{-1}] \cdot [e] = 1_{L_0}, \quad [e] \cdot [e^{-1}] = 1_{L_1}$$

i.e. $[e]$ is an isomorphism in $H(\mathcal{A})$. Clearly quasi-isomorphism is an equivalence relation on objects.

Lemma 3.2.4. *Let $e \in \mathcal{A}(L_0, L_1)$ be a quasi-isomorphism. Then for any $N \in \mathcal{A}$,*

$$\mu_2(-, e) : \mathcal{A}(L_1, N) \to \mathcal{A}(L_0, N)$$
$$\mu_2(e, -) : \mathcal{A}(N, L_0) \to \mathcal{A}(N, L_1)$$

are quasi-isomorphisms of cochain complexes.

Proof. This follows from $[e]$ being an isomorphism in $H(\mathcal{A})$. □

Fortunately, Fukaya categories are c-unital [4], [23]. We now describe what these units look like. We fix a Hamiltonian isotopy $(L_t)_{0\leq t\leq 1}$ form L to L^+ s.t. L and L^+ intersect transversely, so that the Floer cochain group $CF(L, L^+)$ is the morphism space $\hom_{\mathrm{Fuk}(M)}(L, L)$. Let $q \in L \cap L^+$. We define

$$\mathcal{M}(q, [u])$$

as the moduli space of maps $u : D \to M$ of homotopy class $[u]$ from a disc D with marked point z to M with the following boundary condition. The induced map on the boundary minus the point can be thought of as a map $\hat{u} : \mathbb{R} \to M$

$$\hat{u}(s) \in \begin{cases} L_0, & s \leq 0 \\ L_t, & s \in (0,1) \\ L_1, & s \geq 1 \end{cases}$$

and

$$\lim_{s\to\pm\infty} \hat{u}(s) = q$$

$$e_L = \sum_{\substack{q\in L\cap L^+ \\ \mathrm{ind}([u])=0}} \#\mathcal{M}(q, [u]) T^{\omega(u)} q$$

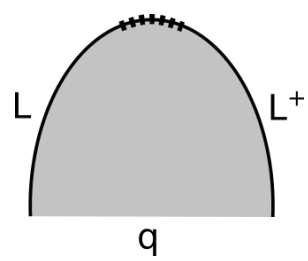


Figure 3.2.1: e_L counts pseudoholomorphic discs with these boundary conditions

By looking at ends of 1-dimensional moduli we get

$$0 = \sum_{\substack{p,q\in L\cap L^+ \\ \mathrm{ind}([u])=0, \mathrm{ind}([u'])=1}} \#\mathcal{M}(p, [u]) \#\mathcal{M}(p, q, [u']) T^{\omega(u)+\omega(u')} q = d(e_L)$$

so it is a closed morphism. Indeed, we can define these e_L morphisms for any Hamiltonian isotopy, and this will be a quasi-isomorphism.

Proposition 3.2.5. *Hamiltonian isotopic Lagrangians are quasi-isomorphic.*

In the case of surfaces, Abouzaid uses combinatorics to show that the e_L are units. In the following example, we use Abouzaid's proof for a related application.

Example 3.2.6. The cylinder T^*S^1 is an exact symplectic manifold (see example 2.3.2). The zero section is an exact Lagrangian manifold. To show that this is the only quasi-isomorphism class in $\mathrm{Fuk}(T^*S^1)$, we can show that all compact exact Lagrangians are Hamiltonian isotopic to this one. An algebraic proof is given in [1], which we describe below.

Proposition 3.2.7. *Any two exact simple closed curves L_0, L_1 on T^*S^1 are quasi-isomorphic.*

Idea of proof. Let $p_1, q_1, \ldots, p_k, q_k$ be the intersections between L_0, L_1 labeled in order such that the p_i's are the degree 0 intersections and q_i's are the degree 1 intersections. Define

$$e_{0,1} = \sum_{i=1}^{k} p_i$$

$de_{0,1}$ is a sum where each q_i is added twice, so $e_{0,1}$ is a closed morphism. Let L_0' be a C^1 close Hamiltonian isotopy of L_0 s.t. L_0 and L_0' only intersect in two points p, q, where p is degree zero and q is degree one. Then the simiarly constructed cochain $e_{0,0'} = p$. Since L_0' is C^1 close, the intersection points $p_i, q_i \in L_0 \cap L_1$ have corresponding intersection points $p_i', q_i' \in L_0' \cap L_1$. The element $e_{1,0'} = \sum_{i=1}^{k} q_i'$. Thus, $\mu^2(e_{1,0'}, e_{0,1}) = e_{0,0'}$ because the only triangle contributing to this product is $\mu^2(q_i', p_i) = p$ shown in figure 3.2.2. $[e_{0,0'}]$ can be thought of as the unit in $H^* \hom(L_0, L_0)$. Thus, we have shown that $e_{0,1}$ has a left quasi-inverse. We can similarly show that $e_{0,1}$ has a right inverse, to show that it is a quasi-isomorphism. $\square$

3.3 Some Homological Algebra

We will motivate the construction of twisted complexes with the mapping cone of cochain complexes. Many fundamental algebraic lemmas are stated that are proven in [23]. We will continue to use $\mathbb{Z}/2\mathbb{Z}$ coefficients for consistency.

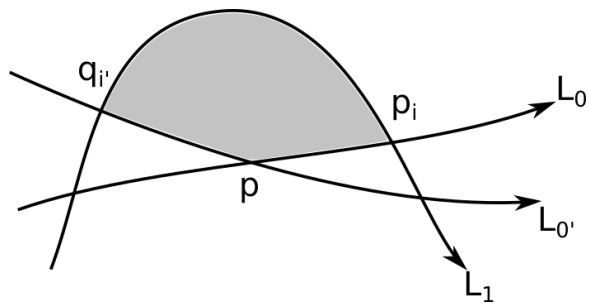


Figure 3.2.2: The triangle that shows $\mu^2(e_{1,0'}, e_{0,1}) = e_{0,0'}$

Definition 3.3.1. Let (A, d_A) be a cochain complex, where A^i is the degree i component of A. Then $A[1]$ is the cochain complex with the same differential with degree i component equal to A^{i+1}. [1] is called the **shift operator**.

Definition 3.3.2. Let (A, d_A) and (B, d_B) be cochain complexes, and let $f : A \to B$ be a chain map. Then the cochain complex $\mathrm{Cone}(f) = A[1] \oplus B$ with differential

$$d_C = \begin{pmatrix} d_A & 0 \\ f & d_B \end{pmatrix}$$

is the **mapping cone** of f.

Lemma 3.3.3. *$f : A \to B$ is a quasi-isomorphism of cochain complexes iff its mapping cone is acyclic (i.e. $H_*(\mathrm{Cone}\, f) = 0$).*

Proof. There is an exact sequence of cochain complexes

$$0 \to B \to \mathrm{Cone}(f) \to A[1] \to 0$$

where B maps to $\mathrm{Cone}(f)$ by inclusion and $\mathrm{Cone}(f) \to A[1]$ is the natural projection. Thus, there is a long exact sequence in cohomology

$$\cdots \to H^{i-1}(\mathrm{Cone}\, f) \to H^i(A) \xrightarrow{f^*} H^i(B) \to H^i(\mathrm{Cone}(f)) \to \cdots$$

We show below that the connecting homomorphism is f^*. Then we will have that f is a quasi-isomorphism iff f^* is an isomorphism iff $H^*(\mathrm{Cone}(f)) = 0$.

To compute the connecting homomorphism, we look at the chain level.

$$\begin{tikzcd}
0 \arrow[r] & B^{i-1} \arrow[r] \arrow[d] & C^{i-1} \arrow[r] \arrow[d] & A^{i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^{i} \arrow[r] & C^{i} \arrow[r] & A^{i+1} \arrow[r] & 0
\end{tikzcd}$$

Choosing a cocycle $\alpha \in A^i$, we chase the diagram to get

$$\begin{tikzcd}
 & (\alpha, \beta) \in C^{i-1} \arrow[r] \arrow[d] & \alpha \in A^{i} \arrow[d] \\
f\alpha \in B^{i} \arrow[r] & (0, f\alpha) \in C^{i} \arrow[r] & 0 \in A^{i+1}
\end{tikzcd}$$

Thus, we proved that f^* is the connecting homomorphism which completes the proof. $\square$

Now we return to the world of A_∞-categories. We would like a notion of direct sum and mapping cones for an A_∞-category, so we formally enlarge this category.

Definition 3.3.4. Let $\mathcal{A}$ be an A_∞ category. A **twisted complex** (E, δ) is

1. a formal direct sum $A = \bigoplus_{i=1}^{N} A^i[k_i]$ of objects, for some integer N
2. a **connection** which is a strictly lower triangular differential δ, i.e. a collection of maps $\delta_{ij} \in \mathcal{A}^{k_j - k_i + 1}(A^i, A^j)$, $i < j$ s.t.

$$\sum_{k \geq 1} \sum_{i = i_0 < i_1 < \cdots < i_k = j} \mu^k(\delta_{i_{k-1} i_k}, \ldots, \delta_{i_0 i_1}) = 0$$

for all $1 \leq i < j \leq N$.

Definition 3.3.5. Given an A_∞ category $\mathcal{A}$, $\mathrm{Tw}(\mathcal{A})$ is the A_∞ category whose objects are twisted complexes, and for two objects $A = \bigoplus A^i[k_i]$ and $B = \bigoplus B^j[k'_j]$ the morphism spaces are

$$\mathrm{Tw}(\mathcal{A})(A, B) = \bigoplus_{i,j} \mathcal{A}(A^i, B^j)[k_i - k'_j]$$

For $F = (f_{i,j}) \in \mathrm{Tw}(\mathcal{A})(A, B)$, the differential is given by

$$d_{Tw}(F)_{i,j} = \sum_{\substack{i < i_1 < \cdots < i_d \\ j_1 < \cdots < j_e < j}} \mu^{d+e}(\delta^B_{j_e, j}, \ldots, \delta^B_{j_1, j_2}, f_{i_d, j_1}, \delta^A_{i_{d-1} i_d}, \ldots, \delta^A_{i, i_1})$$

where the sum is over all d, e and $i < i_1 < \cdots < i_d, j_1 < \cdots < j_e < j$ that make sense. Note that δ^A is a closed morphism in $\mathcal{A}(A, A)$. The higher order operations can be found in [3], [23].

Lemma 3.3.6. *$\mathcal{A}$ embeds fully faithfully into* $\mathrm{Tw}(\mathcal{A})$.

Lemma 3.3.7. *If $\mathcal{A}$ is c-unital, then so is* $\mathrm{Tw}(\mathcal{A})$.

Definition 3.3.8. Given twisted complexes (A, δ^A) and (B, δ^B), and a closed degree 0 morphism $f \in \mathcal{A}^0(A, B)$, the **abstract mapping cone** is

$$\mathrm{Cone}(f) = \left(A[1] \oplus B, \begin{pmatrix} \delta^A & 0 \\ f & \delta^B \end{pmatrix} \right)$$

A **mapping cone** of f is an object of $\mathcal{A}$ that is quasi-isomorphic to the abstract mapping cone of f in $\mathrm{Tw}(\mathcal{A})$.

Example 3.3.9. Consider two simple closed curves α, β in the torus T^2 that intersect in one point p. Let p be a degree 1 morphism. Then

$$(\alpha \oplus \beta, p)$$

is the abstract mapping cone of p. By Seidel's theorem below (theorem 3.4.5) the Dehn twist $\tau_\alpha(\beta)$ a mapping cone of p. Thus, these objects are quasi-isomorphic in $\mathrm{Tw}(\mathrm{Fuk}(T^2))$.

Lemma 3.3.10. *Every object in* $\mathrm{Tw}(\mathcal{A})$ *is an iterated mapping cone of morphisms of $\mathcal{A}$.*

There is a natural notion of functors $\mathcal{F}$ between A_∞-categories A, B (see section 1.1 in [23]). $\mathcal{F} : A \to B$ is a **quasi-equivalence** if $\mathcal{F}$ induces an equivalence on cohomology categories. As expected, many properties of these categories are preserved under equivalences. We can always find quasi-equivalent A_∞ categories with some nice properties

Definition 3.3.11. An A_∞-category is **minimal** if μ^1 vanishes on every morphism space.

Definition 3.3.12. An A_∞-category is **strictly unital** if for each object $X \in \mathcal{A}$, there is a morphism $e_X \in \mathcal{A}^0(X, X)$ s.t.

1. $de_X = 0$

2. $\mu^2(e_{X_1}, a) = a = \mu^2(a, e_{X_0})$ for any $a \in \mathcal{A}(X_0, X_1)$

3. $\mu^k(a_{d-1}, \ldots, a_{n+1}, d, a_n, \ldots, a_1) = 0$ for any $d > 2$

Lemma 3.3.13. *Strictly unital A_∞-categories are c-unital with homological unit $[e_X]$.*

Lemma 3.3.14 (Lemma 3.1 in [14])**.** *For any c-unital A_∞-category, there is a minimal, strictly unital category that is quasi-equivalent to it.*

3.4 Seidel's Exact Triangle

In analogy to short exact sequences of chain complexes inducing long exact sequences of cohomology, exact triangles will induce long exact sequences on cohomology.

Let $A, B \in \mathcal{A}$ and $f \in \mathcal{A}^0(A, B)$ a closed morphism. The following is a prototypical exact triangle:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "i"] \\ & \mathrm{Cone}(f) \arrow[ul, "p", "{[1]}"'] \end{tikzcd}$$

Where the maps are the natural inclusion map $i : B \to \mathrm{Cone}(f)$ and the projection map $p : \mathrm{Cone}(f) \xrightarrow{[1]} A$, where $\xrightarrow{[1]}$ denotes that the map has degree 1. Thus we may make the following definition:

Definition 3.4.1. A diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "g"] \\ & C \arrow[ul, "h"] \end{tikzcd}$$

is an **exact triangle** if C is a mapping cone of f, and there is an quasi-isomorphism $b \in \hom_{\mathrm{Tw}\,\mathcal{A}}(C, \mathrm{Cone}(f))$ s.t. $i = \mu^2(b, g)$ and $h = \mu^2(p, b)$.

Lemma 3.4.2. *Let $A \to B \to C \to A[1]$ be an exact triangle. Then for any object D, there is a long exact sequence on cohomology:*

$$\cdots \to H\mathcal{A}^i(D, A) \to H\mathcal{A}^i(D, B) \to H\mathcal{A}^i(D, C) \to H\mathcal{A}^{i+1}(D, A) \to \cdots$$

is a long exact sequence and similarly for

$$\cdots \to H\mathcal{A}^i(C,D) \to H\mathcal{A}^i(B,D) \to H\mathcal{A}^i(A,D) \to H\mathcal{A}^{i+1}(C,D) \to \cdots$$

Corollary 3.4.3. *$f : A \to B$ is a quasi-isomorphism iff its mapping cone is acyclic.*

We now go on a diversion and describe two functors before stating Seidel's theorem at the end of the section. Let $\mathcal{A}$ be a strictly unital A_∞-category (definition 3.3.12), and $X \in \mathcal{A}$. Let Ch denote the category of finite-dimensional cochain complexes. There is a functor $\hom(X,-) : \mathrm{Tw}(\mathcal{A}) \to Ch$ that takes any morphism $f : Y \to Z$ of twisted complexes to

$$\hom(X,Y) \xrightarrow{\mu^2(f,-)} \hom(X,Z)$$

There is another functor $- \otimes X : Ch \to \mathrm{Tw}(\mathcal{A})$. Let $T : U^m \to V^n$ be a map between vector spaces. Choosing bases for U and V, T can be written as a matrix (a_{ij}). Then the twisted complex

$$U \otimes X = X \oplus \cdots \oplus X$$

where there are m copies of X, and the differential is identically zero. If $u_1, \ldots, u_m$ is a basis, then we may also write the copies of X are $u_i \otimes X$. T is mapped to the matrix (a_{ij}) times the strict unit e_X i.e. the map from $u_i \otimes X \to v_j \otimes X$ is $a_{ij} e_X$, where $v_1, \ldots, v_n$ is a basis for V. Since abstract mapping cones are preserved by functors, the functor $- \otimes X$ can then be extended to all of Ch by taking cones. Explicitly, $C \otimes X$ is the twisted complex with connection $d_C \otimes e_X$. These functors form an adjunction

$$- \otimes X \dashv \hom(X,-)$$

Definition 3.4.4. ev is the co-unit of the tensor-hom adjuction. Concretely given $A, B \in \mathcal{A}$, there is a morphism $ev : \mathcal{A}(A,B) \otimes A \to B$ maps to the identity under the isomorphism

$$\hom(\mathcal{A}(A,B) \otimes A, B) \cong \hom(\mathcal{A}(A,B), \mathcal{A}(A,B))$$

ev induces the map $\mu^2 : H\mathcal{A}(A,B) \otimes H\mathcal{A}(C,A) \to H\mathcal{A}(C,B)$ on cohomology.

In $\mathrm{Tw}(\mathrm{Fuk}\, M)$, the ev map is tautological. Let L_0, L_1 be Lagrangians. Then, $CF(L_0, L_1) \otimes L_0$ is a direct sum of twisted complexes $p \otimes L_0$ for intersection points $p \in L_0 \cap L_1$. On each of these complexes $ev : p \otimes L_0 \to L_1$ is the cochain p.

Theorem 3.4.5 (Seidel [22], [23])**.** *Let L be a Lagrangian and S a Lagrangian sphere. There is an exact triangle in* $\mathrm{Tw}(\mathrm{Fuk}(M, \omega))$

$$\begin{tikzcd} CF(S,L) \otimes S \arrow[r, "ev"] & L \arrow[d] \\ & \tau_S(L) \arrow[ul, "{[1]}"] \end{tikzcd}$$

i.e. $\tau_S(L)$ is the mapping cone of ev.

Corollary 3.4.6. *There is an exact sequence for any Lagrangian A*

$$\cdots \to HF^i(S, L) \otimes HF^i(A, S) \xrightarrow{\mu^2} HF^i(A, S) \to HF^i(A, \tau_S(L)) \to \cdots$$

3.5 Proof of Keating's Theorem

The following is a more precise statement of theorem 2.4.8:

Theorem 3.5.1 (Keating [14])**.** *Let S_0, S_1 be exact Lagrangian spheres in an exact symplectic manifold M, and assume S_0 and S_1 are not quasi-isomorphic in* $\mathrm{Fuk}(M)$*. If*

$$\dim HF(S_0, S_1) \geq 2$$

then the symplectic Dehn twists τ_{S_0} and τ_{S_1} generate a free subgroup of the symplectic mapping class group.

In this section, we develop the algebra needed to prove this theorem. For better pedogogy, we will only prove the case for $\dim HF(L_0, L_1) > 2$, but the following techniques can be modified slightly to cover the case for $\dim HF(L_0, L_1) = 2$ (see [14]). We first prove the analogue of this theorem for surfaces (theorem 2.4.6). Recall that $i(\alpha, \beta)$ is the geometric intersection number (definition 2.4.1).

Theorem 3.5.2. *Suppose α, β are a pair of simple closed curves on an oriented surface Σ, and $i(\alpha, \beta) \geq 2$. Then the group generated by τ_α and τ_β, the Dehn twists around α and β, is a free subgroup of the mapping class group.*

We use the proof in [7].

Lemma 3.5.3 (Ping pong lemma)**.** *Let G be a group and $g_1, g_2 \in G$. If there is a set X such that G acts on X and if X_1, X_2 are nonempty, disjoint subsets of X such that $g_1^j(X_2) \subset X_1$ and $g_2^k(X_1) \subset X_2$ for all nonzero integers j, k. Then the group generated by g_1, g_2 is a free subgroup of G.*

Proof. For any group element g generated by g_1, g_2, it can be put in the form

$$g = g_1^{j_1} g_2^{k_1} \cdots g_1^{j_n} g_2^{k_n} g_1^{j_{n+1}}$$

where all exponents are nonzero by conjugating by g_1 if necessary. Let $x \in X_2$. Then $g_1^{j_n+1} x \in X_1$, $g_2^{k_n} g_1^{j_{n+1}} x \in X_2$ etc. and $gx \in X_1$, so $gx \neq x$ since X_1, X_2 are disjoint. Thus, g doesn't fix x, so it is not the identity. Thus, there are no relations between g_1, g_2 in G. □

Proof of theorem. Let a, b be simple closed curves such that $i(a, b) \geq 2$. We have two elements τ_a, τ_b of the mapping class group $MCG(\Sigma)$. $MCG(\Sigma)$ acts on the isotopy classes of simple closed curves X. Let

$$X_a = \{c \in X : i(c, b) > i(c, a)\}$$
$$X_b = \{c \in X : i(c, a) > i(c, b)\}$$

These sets are clearly disjoint. $a \in X_a$ and $b \in X_b$, so these sets are nonempty. By the ping pong lemma, it is sufficient to check that $\tau_a^j(X_b) \subset X_a$ and $\tau_b^k(X_a) \subset X_b$ for all nonzero integers j, k. By symmetry, it is sufficient to check the first inclusion.

Lemma 2.1 in [13] (see also proposition 3.4 in [7]) is the following inequality:

$$i(\tau_a^k(c), b) + i(b, c) \geq |k| i(a, b) i(a, c) \tag{3.3}$$

Using this, we have

$$\begin{aligned} i(\tau_a^k(c), b) &\geq |k| i(a,b) i(a,c) - i(b,c) \\ &\geq 2|k| i(a,c) - i(b,c) \\ &> 2|k| i(a,c) - i(a,c) \\ &\geq i(a,c) = i(\tau_a^k(a), \tau_a^k(c)) \\ &= i(a, \tau_a^k(c)) \end{aligned}$$

□

A crucial component of the proof is the inequality (3.3) above. Actually it suffices for this proof to have the inequality,

$$i(c,b) + i(\tau_a^n(c), b) \geq i(c,a) i(a,b)$$

If S is a Lagrangian sphere and L_0, L_1 are Lagrangians, then Seidel's long exact sequence (corollary 3.4.6) gives us an analogous inequality for $n = 1$,

$$\dim HF(\tau_S(L_0), L_1) + \dim HF(L_0, L_1) \geq \dim HF(S, L_0) \cdot \dim HF(S, L_1)$$

Thus, $\dim HF$ seems to be an appropriate generalization of geometric intersection number for this purpose. Keating proves the following:

Theorem 3.5.4. *Let M be a symplectic manifold. Let S be a Lagrangian sphere, and L_0, L_1 be Lagrangians. Then,*

$$\dim HF(\tau_S^n(L_0), L_1) + \dim HF(L_0, L_1) \geq \dim HF(S, L_0) \cdot \dim HF(S, L_1)$$

We first explain a few preliminaries needed to prove this theorem.

Definition 3.5.5. An **A_∞-algebra** is an A_∞-category with one object. Given an A_∞-algebra A, a right **A_∞-module** over A is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with maps

$$\mu_M^n : M \otimes A^{\otimes(n-1)} \to M$$

for all $n \geq 1$ such that the A_∞ relations hold:

$$\sum \mu_M^{r+t+1}(\mathrm{id}^r \otimes \mu^s \otimes \mathrm{id}^t) = 0$$

where the sum is taken over all r, s, t such that $r+s+t=n$, $r, t \geq 0$, $s > 0$, and μ^s denotes μ^s_M when $r=0$ and μ^s_A otherwise. We define left A_∞-modules similarly.

An example of an A_∞ algebra is $E = \mathbb{F}_2[\epsilon]/\epsilon^2$, where $\mathbb{F}_2$ is the field with two elements. The products are given by μ^2 is multiplication and all other μ^k maps are zero. Let (ϵ) to denote the one-dimensional vector space generated by ϵ. This is a right and a left A_∞-module over E. Another set of right A_∞-modules is the following. For $k \geq 2$, let $\mathcal{R}_k$ be a $\mathbb{Z}/2\mathbb{Z}$ vector space with 2 generators r^0_k and r^1_k viewed as a right A_∞-module over E such that $\mu^k(r^0_k, \epsilon, \ldots, \epsilon) = r^1_k$ is the only nontrivial product. The left A_∞-module $\mathcal{L}_k$ is defined similarly. The following is a classification of finite-dimensional A_∞-modules over E.

Lemma 3.5.6 (Lemma 5.3 in [14])**.** *Let M be a strictly unital, finite dimensional right (resp. left) A_∞-module over E. Then M is quasi-isomorphic to a module N that decomposes into a finite direct sum of A_∞-modules of the following forms:*

1. *copies of $\mathbb{Z}/2\mathbb{Z}$ with trivial A_∞ actions*
2. *copies of $\mathcal{R}_k$ (resp. $\mathcal{L}_k$)*

Definition 3.5.7. $C^n_S L_0$ is the twisted complex

$$\begin{aligned} L_0 &\oplus \hom(S, L_0) \otimes S \\ &\oplus \hom(S, L_0) \otimes (\epsilon) \otimes S \\ &\quad \vdots \\ &\oplus \hom(S, L_0) \otimes \underbrace{(\epsilon) \otimes \cdots \otimes (\epsilon)}_{n-1} \otimes S \end{aligned}$$

We do not write out the connection, but it can be deduced from the μ^1 map below. For any object $L_1 \in \mathrm{Fuk}(M)$,

$$\begin{aligned} \hom(L_1, C^n_S L_0) = \hom(L_1, L_0) &\oplus \hom(S, L_0) \otimes \hom(L_1, S) \oplus \cdots \\ &\oplus \hom(S, L_0) \otimes \underbrace{(\epsilon) \cdots (\epsilon)}_{n-1} \otimes \hom(L_1, S) \end{aligned}$$

The nonzero morphisms in μ^1 are

$$\mu^2 : \hom(S, L_0) \otimes \hom(L_1, S) \to \hom(L_1, L_0)$$

$$(\mu^3 + \mathrm{id} \otimes \mu^2 + \mu^2 \otimes \mathrm{id}) : \hom(S, L_0) \otimes (\epsilon) \otimes \hom(L_1, S) \to \hom(L_1, L_0)$$

$$\vdots$$

$$\sum_{\substack{i+j=r,\\ j>1}} (\mathrm{id}^{\otimes i} \otimes \mu^j + \mu^j \otimes \mathrm{id}^{\otimes i}) : \hom(S, L_0) \otimes \underbrace{(\epsilon) \cdots (\epsilon)}_{r-2} \otimes \hom(L_1, S) \to \hom(L_1, L_0)$$

Lemma 3.5.8 (proposition 6.3 in [14])**.** *There is a morphism of twisted complexes $C_S^n L_0 \to L_0$ given by $ev : \hom(S, L_0) \otimes S \to L_0$ and zero on the other summands. $\tau_S^n L_0$ is quasi-isomorphic to the cone of this map in* $\mathrm{Tw}(\mathrm{Fuk}\, M)$.

The idea of the proof is to first replace $\mathrm{Tw}(\mathrm{Fuk}\, M)$ by a quasi-isomorphic category that is minimal and strictly unital (lemma 3.3.14). Then inductively use Seidel's theorem 3.4.5.

Definition 3.5.9. We use the following notation to denote **bar complexes**. Let M be a right A_∞-module and N be a left A_∞-module. Define

$$(M \otimes_E N)_n := M \otimes N \oplus M \otimes (\epsilon) \otimes N \oplus \cdots \oplus M \otimes \underbrace{(\epsilon) \cdots (\epsilon)}_{n-1} \otimes N$$

Now we have the preliminaries for the proof.

Proof of theorem 3.5.4. Let M be a symplectic manifold, S a Lagrangian sphere, and L_0, L_1 Lagrangians. By lemma 3.5.8 we have the exact triangle

$$\cdots \to \hom(C_S^n L_0, L_1) \to \hom(L_0, L_1) \to \hom(\tau_S^n L_0, L_1) \to \cdot$$

Taking the rank we have

$$\dim HF(\tau_S^n L_0, L_1) + \dim HF(L_0, L_1) \geq \mathrm{rk}(\hom(C_S^n L_0, L_1))$$

and comparing the definitions

$$\mathrm{rk}(\hom(C_S^n L_0, L_1)) = \mathrm{rk}(CF(S, L_0) \otimes_E CF(L_1, S))_n$$

which is isomorphic to $(M \otimes_E N)_n$ where M, N are minimal right and left modules respectively by lemma 3.5.6. $(M \otimes_E N)_n$ is a direct sum of items of the following form

(a) $(\mathbb{Z}/2\mathbb{Z} \otimes_E \mathbb{Z}/2\mathbb{Z})_n$

(b) $(\mathcal{R}_k \otimes_E \mathbb{Z}/2\mathbb{Z})_n$

(b') $(\mathbb{Z}/2\mathbb{Z} \otimes_E \mathcal{L}_k)_n$

(c) $(\mathcal{R}_k \otimes_E \mathcal{L}_k)_n$

It suffices to show $\operatorname{rk} H(M \otimes_E N)_n \geq \operatorname{rk} M \cdot \operatorname{rk} N$ for each of the cases above.

1. We want to show that the rank of cohomology of $(\mathbb{Z}/2\mathbb{Z} \otimes_E \mathbb{Z}/2\mathbb{Z})_n$ is at least 1. The differential is identically zero, so the cohomology has rank n.

2. We want to show that the ranks of cohomology of $(\mathcal{R}_k \otimes_E \mathbb{Z}/2\mathbb{Z})_n$ and $(\mathbb{Z}/2\mathbb{Z} \otimes_E \mathcal{L}_k)_n$ are each at least two. These cases symmetric, so we do the former case. There are differentials $r_k^0 \otimes \underbrace{\epsilon \cdots \epsilon}_{m} \otimes u \to r_k^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{m-k+1} \otimes u$. Thus, there is no differential going into or coming out of $r_k^0 \otimes u$ or $r_k^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1} \otimes u$. This gives at least two elements in cohomology.

3. We want to show that the cohomology of $(\mathcal{R}_j \otimes_E \mathcal{L}_k)_n$ has rank at least 4. Without loss of generality we assume $j \leq k$. $r_j^0 \otimes l_k^0$ and $r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1} \otimes l_k^1$ will survive in cohomology for the same reason as above.

   If $n < k$, $r_j^0 \otimes l_k^1$ and $r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1} \otimes l_k^0$ will also survive in cohomology.

   If $n \geq k$,

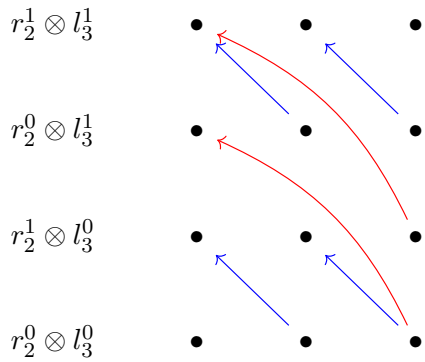
$$d(r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1} \otimes l_k^0) = d(r_j^0 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1-k+j} \otimes l_k^1) = r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-k} \otimes l_k^1$$

   So $r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1} \otimes l_k^0 + r_j^0 \otimes \underbrace{\epsilon \cdots \epsilon}_{n-1-k+j} \otimes l_k^1$ is in the kernel of d and it isn't in the image of d. Both $r_j^0 \otimes l_k^1$ and $r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{k-j} \otimes l_k^0$ are in the kernel of d but the only differential that maps to them is

$$d(r_j^0 \otimes \underbrace{\epsilon \cdots \epsilon}_{k-1} \otimes l_k^0) = r_j^0 \otimes l_k^1 + r_j^1 \otimes \underbrace{\epsilon \cdots \epsilon}_{k-j} \otimes l_k^0$$

   so we get the fourth generator of cohomology.

As an illustrative example, the differential for $(\mathcal{R}_2 \otimes_E \mathcal{L}_3)_3$ is computed below.



The three columns represent elements of the form $r \otimes l$, $r \otimes \epsilon \otimes l$, and $r \otimes \epsilon \otimes \epsilon \otimes l$ respectively. The blue arrows come from the $\mu^2(r_2^0, \epsilon)$ operations and the red arrows come from the $\mu^3(\epsilon, \epsilon, l_3^0)$ operations. $\square$

The proof of theorem 3.5.1 now follows analogously to the proof of 3.5.2. We apply the ping pong lemma to

$$X_{S_1} = \{L \in X : \dim HF(S_2, L) > \dim HF(S_1, L)\}$$
$$X_{S_2} = \{L \in X : \dim HF(S_1, L) > \dim HF(S_2, L)\}$$

We note that the proof uses the fact

$$\dim HF(\tau_S^k(L_0), \tau_S^k(L_1)) = \dim HF(L_0, L_1)$$

which is true since symplectomorphisms preserve Floer homology. We also note that we need $\dim HF(S_2, S_1) > 2$ to show that $\dim HF(S_2, S_1) > \dim HF(S_1, S_1)$ so that $S_1 \in X_{S_1}$ and similarly to show X_{S_2} is not empty. Keating proves the stronger inequality

$$\dim HF(\tau_S^n(L_0), L_1) + \dim HF(L_0, L_1) \geq 2 \dim HF(S, L_0) \cdot \dim HF(S, L_1)$$

for $|n| \geq 2$, to deal with the case $\dim HF(S_2, S_1) = 2$.

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