MULTI-DIMENSIONAL FLUCTUATION SPLITTING SCHEMES FOR THE EULER EQUATIONS ON UNSTRUCTURED GRIDS

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To my one love:
James Matthew.
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CHAPTER I

INTRODUCTION

The most successful numerical schemes developed for computational fluid dynamics in the past have been based upon physics. While this success is readily seen in one-dimensional flows, there is a definite need for improvement in the approximation of solutions in higher dimensions. Not only do we seek a more accurate method of discretizing the equations, but issues of code robustness and rate of convergence are also important areas that need improvement. The focus of this work is the development of a two-dimensional scheme for approximating steady solutions of the inviscid fluid equations which attempts to make the numerics of the method as harmonious with the physics as possible. Throughout the work contained in this thesis, it has been realized that the extension of the schemes successful in one dimension to higher dimensions is not straight-forward, because new physical properties present themselves as the dimensions are increased. While upwinding in one dimension has met with great success, and its simple extension to higher dimensions has also proved effective for some purposes within the aerospace community, we have determined that, in two dimensions, upwinding alone is not a sufficient basis upon which to build a numerical model. We propose that it is necessary to incorporate, within the method, a system of elliptic origin, to account for the omni-directional distribution
of acoustical information within the subsonic regime. As the road to this conclusion has not been straight, we will also present some of the other paths taken in search of a truly multi-dimensional solver, eventually leading to the current method. As will be shown in the thesis, improved accuracy over existing methods is realized with the numerical method presented. However, the issues of robustness and convergence rate are still in need of attention.

A very successful class of schemes in one dimension, providing both accuracy and robustness, are Godunov based schemes [9], known as flux-difference splitting. However, the current accepted practice to extend these one-dimensional ideas to higher dimensions is based on one-dimensional physics. It leads to cell-centered schemes, where interactions between cells take place at the interfaces defined by the grid. At each interface, the states on either side, referred to as the left and right states, can either simply be given by the states at the corresponding cell centers or can be calculated from some higher-order extrapolations. Either way, this defines a one-dimensional Riemann problem at each interface, which can be approximately solved, for example, utilizing the Roe method. This method provides an exact solution to the linearized equations, and decomposes the flux difference at the interface into a uniquely defined set of simple waves, all aligned with the grid face [24]. This is then utilized to evolve the solution within the cells by sending the disturbance caused by each individual wave to the cell toward which it is propagating.

Although successful codes have been developed utilizing this strategy, clearly the true physics of the multi-dimensional system are not modeled correctly, as waves are only allowed to propagate in directions defined by the grid. This causes an inconsistency in the decomposition when the dominant characteristic present in the flow field lies oblique to the grid. A classic example of this is the case of a simple
shear flow oriented oblique to the grid, as shown in Figure 1.1. Because the Riemann

\[
\begin{align*}
\begin{array}{c}
\vec{q}_L \quad \vec{q}_R
\end{array}
\end{align*}
= \begin{array}{c}
shear \\
\text{acoustic pair}
\end{array}
\]

Figure 1.1: Shear flow oblique to grid recognized as shear parallel to grid face plus a pair of acoustic waves.

problem is solved in the direction normal to the grid, the oblique shear will be “recognized” as a shear aligned with the grid, in order to account for the jump in the velocity component aligned with the grid face, plus a pair of acoustic waves, to account for the jump in the velocity component normal to the grid face. The strength of these acoustic waves will depend upon the angle between the physical shear wave and the grid face, allowing for rather strong spurious acoustics for large angles. These spurious acoustic waves bring along unnecessary numerical dissipation, causing the oblique shear layer to diffuse over many cells. It is this dependence of the solution on the structure of the grid that demands a more physically correct algorithm for approximating multi-dimensional flows.

Over the past decade a substantial effort has been put into the development of a truly multi-dimensional method for approximating solutions to the Euler equations,
eliminating the dependence of the accuracy on the grid structure. Unlike in one dimension, where the decomposition into simple waves is unique, in two dimensions there are infinitely many unsteady planar waves present, as there are an infinite number of directions in which the waves can propagate, see Roe [25]. The first use of these ideas in a numerical code was in the framework of what are referred to as Rotated Riemann Solvers. In this approach, fluxes are still obtained by solving Riemann problems at grid interfaces, but now in directions not dependent on the grid normals, but rather determined by the physical nature of the local flow [30, 8]. While these methods were successful in improving accuracy, substantial losses of robustness and convergence speed prohibited the realization of a universal scheme surpassing existing methods.

It thus became clear that a radically new approach had to be sought in order to model the multi-dimensional physics adequately. Because the Riemann methods only provide two states at each interface, the true two-dimensional character of the flow can not be realized. For example, we do not know if a measured one-dimensional gradient is balanced by a gradient in some other direction. We need information from at least three points and thus the Riemann methods have to be abandoned.

We consider the family of schemes, known as fluctuation-splitting schemes. In one dimension, Roe showed that these schemes provide an alternative method of implementing the flux-difference scheme [24]. In this cell-vertex approach, fluctuations are defined as the integral of the time derivative over each cell. In one dimension, these fluctuations, as in the flux-difference method, can be decomposed into a unique set of linear waves each governed by an advection operator. The decomposition can be thought of as “explaining” what the time derivative is due to. Each wave then updates the solution at the endpoints of the cell, updating more strongly the point
toward which it moves.

The advantage of the fluctuation-splitting approach is that the natural extension of the intervals in one dimension leads to triangles in two dimensions and tetrahedra in three. In two dimensions, this cell-vertex method operates on triangular grids with the data stored at the vertices. As in one dimension, fluctuations are determined on each mesh element, defined as the integral of the now two-dimensional time derivative over each triangle. Each cell fluctuation then needs to be distributed back to the vertices of the given triangle, providing the changes to the nodal states, used to evolve the solution in time. This thesis is focused on finding a method of distributing the cell fluctuations to the vertices of the local simplex, which simulates the physics of the equations as closely as possible.

There are several important ingredients in this family of schemes: one is a conservative linearization of the Euler equations utilized in calculating the fluctuation within the local simplex. The known Roe parameter vector in one dimension provides an appropriate linearization, and is shown to have a natural extension to higher dimensions [4]. With the resulting linearized system, a decomposition is performed; in one dimension this led to a set of simple waves, but in two dimensions other decompositions are possible and seem more natural. Finally, utilizing the resulting subproblems, which in one dimension are advection operators, the fluctuation is distributed to the vertices of the local simplex. The very compact nature of the distribution process creates the potential for efficient coding, and will be adhered to in everything that follows.

This extension of the fluctuation-splitting schemes to higher dimensions has been carried out in a strong collaboration between the University of Michigan and the Von Kármán Institute for Fluid Dynamics in Brussels. The first success was realized
for scalar advection operators, in the development of upwind biased discretizations on unstructured triangular grids. Original work by Roe, Struijs, et. al. [26, 33] provides nearly-second-order compact schemes, with the ability to capture discontinuities sharply, without overshoots even when these lie oblique to the grid. Recently, multi-dimensional upwinding methods developed by Sidilkover, originally based on structured finite-volume grids but extended to unstructured cell-vertex grids, have been unified with the fluctuation-splitting schemes [32]. The formulation of Sidilkover utilizes symmetric limiting functions, for which any existing one-dimensional limiting method can be substituted, providing an elegant framework which encompasses a wide range of schemes. It is within this methodology that the work in this thesis is presented.

The success of the scalar advection schemes appeared to confirm that the natural extension from one dimension would continue to be the decomposition into advection operators. From the beginning of the present work, up until the past year, the breaking of the two-dimensional cell fluctuations into advection problems has dominated the research path. In all flow domains, subsonic as well as supersonic, the decomposition of the Euler system was based on the idea that all local flow gradients and fluctuations present within a mesh simplex could be modeled as a finite set of unsteady planar waves. Thus, the disturbances present within the mesh simplex were discretely modeled as linear waves governed by advection operators, based on the original wave model approach of Roe [25]. Various assumptions have been used to determine the finite set of waves. The assumptions are required to select the propagation directions, out of infinitely many possibilities, providing a unique set of waves for given local flow conditions.

While the linearization of the system, as well as the upwind discretizations for
scalar advective operators, appear to be adequate, solutions obtained with this wave modeling approach showed a clear lack of accuracy; its cause appeared to lie in the decomposition of the Euler system into a set of planar wave equations. Several wave models within this framework were developed and tested, all of which produced results with too much numerical dissipation [34]. The basic difficulty was that even steady flow was often interpreted by the model as resulting from cancelation between a number of unsteady waves.

In order to avoid this dissipation, Roe proposed that the scheme should be able to recognize existing steady patterns in the field. It is easily determined that in solutions of the steady system there exist patterns, given by shear and entropy layers, plus potential-flow solutions in subsonic regimes, or acoustic waves defined along the Mach lines in supersonic flows. Following this reasoning, a combination of both steady and unsteady wave patterns were used to model the spatial gradients, with the time-dependent planar waves accounting for the cell fluctuations [27]. In numerical experiments second-order-accurate results, with low numerical error, were obtained, but convergence was hindered by the ambiguity of wave orientation for cells with small residuals [20]. It is also noted that this decomposition only proved stable in subsonic regimes; the development of a supersonic counterpart is a major obstacle yet to be overcome. While the Steady/Unsteady Splitting was not the answer to the multi-dimensional modeling problem, it did show the value of a necessary condition to be imposed on any scheme for the sake of accuracy. This rather obvious condition, not satisfied by the original wave models, says that, in the instance of vanishing cell fluctuations, the wave strengths of all unsteady waves must vanish [18]. This guarantees that the scheme will recognize and maintain a given steady solution.

It appeared that a major problem contained within both versions of the wave-
modeling schemes discussed was in the use of unsteady planar waves whose orientation was sometimes ill-defined. Therefore, in the present work, the decomposition of the Euler system into advective operators is abandoned following the lack of success of the methods discussed.

Abandoning the planar wave models calls for a new approach in determining the appropriate decomposition of the Euler system into subproblems. Contrary to the original approach of recognizing simple waves, and therefore scalar subproblems, it attempts to reduce the steady system to a set of subproblems that are decoupled from each other as completely as possible. These are similar to the canonical forms first proposed by Ta'asan [35], where the subsonic steady Euler Equations are reduced into hyperbolic and elliptic parts. The biggest difference between the work of Ta'asan and the method discussed here is that the canonical forms are different. In this work we obtain the unique complete decoupling of the linearized system. Also, the extension of this method to supersonic regimes is presented, while not yet addressed by Ta'asan.

Because most work in truly multi-dimensional schemes to date has been in the development of methods for obtaining steady-state solutions, we return to the steady equations for the determination of the correct decomposition, while simply utilizing the time dependence as a method of marching to the final solution. It is well known that the two-dimensional steady Euler equations in supersonic flow are equivalently represented by four characteristic equations. While these characteristic equations are decoupled in the linear system, non-linear coupling is present through the coefficients of the derivatives. This is similar to the plane-wave decomposition methods because four clearly defined propagation directions exist. In subsonic flows, however, only two of these characteristics remain, namely for advection of enthalpy and entropy along
streamlines. The acoustic characteristics from the supersonic regime become coupled in the subsonic regime, representing the spreading of subsonic acoustical information without preferential directions. Therefore, the steady Euler equations are now represented by two advective equations, providing the hyperbolic part, plus a system of two coupled equations whose characteristics are imaginary, thus representing the ellipticity of the steady acoustical field.

This decomposition of the steady system exhibits the different mechanisms for propagating information; that is one-directional convection and omni-directional wave propagation. In order to solve for the steady state, a “pseudo-time” is introduced. Since we are not concerned with time accuracy, the unsteady Euler system is modified via preconditioning techniques similar to those developed by Turkel and van Leer et al. [39, 41]. It is desired to preserve the known decoupled characteristic form of the steady equations even in the unsteady system. It was discovered that the original preconditioner of Van Leer already provided this property, allowing the isolation of the distinct information propagation mechanisms in the unsteady system.

This decomposition results in unique advective operators for which (as discussed for the wave modeling schemes) second-order, compact upwind schemes providing non-oscillatory sharp discontinuities already exist, (see Roe, Struijs, Sidilkover et al. [26, 33, 32]). However, in the subsonic regimes, the coupled acoustic system calls for the development of accurate discretizations. This development can be carried out for an alternate model problem, specifically the Cauchy-Riemann equations augmented with time terms. For this system, where there is no preferential direction of information propagation, central discretizations can be utilized, converged using a cell-vertex Lax-Wendroff scheme similar to those developed by Ni, Hall, Morton et al. [17, 10, 13].
It is possible to incorporate both the advective and omni-directional discretizations, including the nonlinear limiting required to avoid shock oscillations, within a compact stencil that permits very effective parallelization [37]. Moreover, the decomposition, or generalization of the earlier Riemann problem, needs to be made only once per cell in this approach, rather than once per interface.

The fundamental concept put forth in this work is the separation of the elliptic and hyperbolic components present within the system. Utilizing the characteristic form of the steady system to predict supersonic flows accurately is not a new concept, but the present work has implemented the ideas in a conservative way. It is found that the subsonic decomposition into hyperbolic and elliptic parts leads to accurate solutions, characterized by very low drag for subcritical cases and very low spurious entropy production. Nevertheless, issues concerning robustness and, most importantly, efficiency still exist.

There are numerous details that arise within these splitting schemes which are likely to be important due to the high ambition of increasing efficiency and robustness as well as accuracy, over existing methods. In this thesis we only present the underlying framework with current results. We do not claim that all or any of the details are yet perfect, and it is noted that quite different approaches may still be taken [19]. We believe that the success of the splitting scheme will depend on finding the proper method of solving the elliptic system that produces textbook convergence rates, whether with single-grid or multi-grid relaxation. This goal is not yet realized with the current use of a Lax-Wendroff method for solving the Cauchy-Riemann system [16].

The focus of this thesis is the development and testing of various methods of decomposing the Euler system, incorporated in a cell-vertex fluctuation-splitting
scheme. The issues of robustness and convergence performance are touched upon but not fully addressed. It is clear, however, that an advance has been made toward the eventual scheme that will surpass the current standard of industry for computing multi-dimensional flows, with finally a prevailing feeling that the answer is just around the corner.
CHAPTER II

FLUCTUATION SPLITTING SCHEMES

2.1 Fluctuation Splitting for the Euler Equations in One Dimension

We start by reviewing the concept of fluctuation splitting as proposed by Roe in 1981, in developing an upwind solver for the Euler Equations in one dimension [24, 23]. We analyze any set of hyperbolic partial differential equations

\[ U_t + F_x = 0, \quad (2.1) \]

in quasi-linear form:

\[ U_t + AU_x = 0, \quad (2.2) \]

where

\[ A = \frac{\partial F}{\partial U} \]

is the Jacobian of the flux function \( F \). Given a solution at any time level where data are stored at nodal points, a piecewise continuous linear representation of the data can be constructed based on the idea of linear finite elements in space, (see Figure 2.1). From this representation, piecewise constant spatial gradients can be calculated in each interval.

In the same notation as that first used by Roe [23], we then define the framework for fluctuation splitting schemes:
Figure 2.1: Linear Data Reconstruction in 1D.

- **Fluctuation:** the time residual over a mesh element; in 1D this gives fluctuations on each interval. We denote the fluctuation by $\Phi$, given by:

$$
\Phi_{i-\frac{1}{2}} = -\int_{x_{i-1}}^{x_{i}} U_t \, dx
$$

$$
= \int \mathbf{A} \mathbf{U}_x \, dx
$$

$$
= \mathbf{A} (U_i, U_{i-1}) [U_i - U_{i-1}].
$$

To create a conservative scheme a conservative linearization is necessary to evaluate $\mathbf{A}(U_i, U_{i-1})$ such that

$$
\mathbf{A}(U_i, U_{i-1}) [U_i - U_{i-1}] = F_i - F_{i-1}.
$$

This is known for the Euler equations as the Roe linearization; it is achieved through a parameter vector $\mathbf{Z}$, such that if $\overline{\mathbf{A}}$ is evaluated at the algebraic average of the parameter vector, the linearization is conservative. This algebraic average is calculated from the assumption of piecewise linear variation in $\mathbf{Z}$ over the interval, and thus

$$
\mathbf{Z}_{i-1/2} = \frac{1}{2} (\mathbf{Z}_{i-1} + \mathbf{Z}_i).
$$

The parameter vector is given by, $\mathbf{Z} = \{\sqrt{\rho}, \sqrt{\rho} u, \sqrt{\rho} H\}$, where $H = \frac{\gamma - 1}{\gamma - 1} \frac{\rho u^2}{2} + \frac{1}{2} u^2$ is the total enthalpy.
- **Signal**: the action performed on the data to bring it to equilibrium. This represents the distribution or mapping of the cell fluctuations back to the nodes. The distribution schemes discussed within this thesis are local and therefore the fluctuation within each cell is mapped only to the vertices which define that cell. In 1D this defines a mapping of an interval fluctuation to the two nodes which define its endpoints, i.e.

\[
\Phi_{i-\frac{1}{2}} = \Phi_{i+\frac{1}{2}}^+ + \Phi_{i-\frac{1}{2}}^-.
\]  

(2.7)

where \(\Phi^+\) denotes the portion of the fluctuation sent in the positive \(x\) direction to \(x_i\) and \(\Phi^-\) the portion sent to \(x_{i-1}\).

- **Residual**: the measure of the distance of the solution from equilibrium. The residuals are the accumulated signals at each of the nodes after looping over all cells and distributing the fluctuations. In one dimension each node receives a signal from the two intervals of which it is a vertex, and the residual \(\Delta U_i\) is given by their sum:

\[
\Delta U_i = \Phi_{i+\frac{1}{2}}^+ + \Phi_{i-\frac{1}{2}}^-.
\]  

(2.8)

The time evolution of the solution is then given by the following update:

\[
U_i(t + \Delta t) = U_i(t) - \Delta U_i \frac{\Delta t}{\Delta x_i},
\]  

(2.9)

where \(\Delta x_i\) is the area weighted to node \(i\), given by the sum of half the length of the two intervals bordering it, \(\Delta x_i = \frac{1}{2}(x_{i+1} - x_{i-1})\).

The main issue then within these types of distribution schemes is how to map the interval fluctuations to the nodes to create an efficient method of obtaining accurate steady state solutions. In 1D, the most successful method is the upwind Roe scheme,
where the Euler system is broken into a unique set of linear simple waves, of which each can be independently upwinded.

By diagonalizing the time dependent system we can uniquely determine the set of waves. We start with the quasi linear system, where we can break $\mathbf{A}$ into $\mathbf{RAL}$, where $\mathbf{R}$ and $\mathbf{L}$ are the matrices containing the right and left eigenvectors of $\mathbf{A}$ and $\mathbf{A}$ the corresponding eigenvalues:

\begin{align}
\mathbf{U}_t + \mathbf{A} \mathbf{U}_x &= 0, \\
\mathbf{U}_t + \mathbf{RAL} \mathbf{U}_x &= 0.
\end{align}

Multiplying through by $\mathbf{L}$,

\begin{align}
\mathbf{LU}_t + \mathbf{ALU}_x &= 0,
\end{align}

gives a set of nonlinearly coupled scalar advection equations for the characteristic variables, $\mathbf{W}$, defined by $\partial \mathbf{W} = \mathbf{L} \partial \mathbf{U}$,

\begin{align}
\mathbf{W}_t + \mathbf{A} \mathbf{W}_x &= 0.
\end{align}

Now that the system has been broken into scalar advection problems, we can easily upwind each. If we denote $\mathbf{A} = \text{diag}(\lambda_k)$, the sign of $\lambda_k$ determines the direction each wave, $k$, is moving. We therefore split $\mathbf{A}$ into

\begin{align}
\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-
\end{align}

where $\mathbf{A}^+$ contains the positive values, $(\lambda_k \geq 0)$, representing those waves traveling in the positive $x$ direction, and $\mathbf{A}^-$ contains the negative values, $(\lambda_k < 0)$ representing the waves moving in the negative $x$ direction. From this we can find the splitting of the flux difference by determining the split Jacobian matrix $\mathbf{A}$, given by:

\begin{align}
\mathbf{A}^+ = \mathbf{RA}^+ \mathbf{L} \quad \mathbf{A}^- = \mathbf{RA}^- \mathbf{L},
\end{align}
where

\[ A^+ + A^- = A, \]  \hspace{1cm} \text{(2.16)}

and

\[ \Phi^+ = A^+ \Delta U \quad \Phi^- = A^- \Delta U. \]  \hspace{1cm} \text{(2.17)}

It was shown by Roe [23] that the fluctuation splitting scheme described is equivalent to a finite-difference scheme utilizing a flux function based on the above wave decomposition. Therefore, in one-dimension fluctuation splitting offers nothing new. Fortunately, the extension of this idea to higher dimensions does. The standard extension of the flux-based scheme to higher dimensions is to continue to solve one-dimensional Riemann problems in given directions, the most successful being in directions determined by the grid structure. This of course does not allow for the truly multi-dimensional flavor of the flow to be correctly modeled. Although various methods of finding more realistic directions in which to solve the Riemann problem were tried [8, 30], all still resorted to one-dimensional physics and none met with increased success. It is thus the topic of this work to develop, at least in two dimensions, a scheme which more accurately solves the inviscid gas-dynamic equations by incorporating a more physically correct model for two-dimensional flow. The framework within which these schemes operate is the extension of the fluctuation-splitting idea to higher dimensions.

2.2 Extension to Two Dimensions?

The natural extension of the assumption of piecewise linear data to two dimensions gives rise to triangular elements with data points at the vertices.

Using the same terminology, cell fluctuations, \( \Phi \), are defined as the integral over the triangular elements, \( \tau \), which construct the domain, similar to one-dimensional
splitting schemes where fluctuations are defined on intervals. Integrating a two-dimensional partial differential equation:

\[ \mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = 0, \tag{2.18} \]

over a triangle \( \tau \),

\[ \Phi_\tau = -\iint \mathbf{U}_t \, dA = \iint (\mathbf{F}_x + \mathbf{G}_y) \, dA, \tag{2.19} \]

and applying Gauss’ Theorem, the evaluation of the fluctuation reduces to a path integral,

\[ \Phi_\tau = \oint (\mathbf{F} \, dy - \mathbf{G} \, dx), \tag{2.20} \]

which is easily computed assuming linear variation of data over the cells. A detailed discussion of this computation concerning conservation is given in Chapter VI.

Given the cell fluctuation, a distribution or mapping back to the vertices is needed. In order to maintain a compact stencil, the distribution step must only utilize information within the local simplex, and distribute signals only to the vertices which define the cell. This allows the numerical scheme to be implemented as a loop over mesh simplices, thus making the algorithm very efficient for parallel applications as shown by Tomaich [37]. The basis of the splitting schemes is the distribution of the fluctuation to the three vertices of the local triangle, (see Figure 2.2). Each vertex receives a portion of the residual, denoted by \( \Phi^i_\tau \), \( i = 1, 2, 3 \), such that:

\[ \Phi_\tau = \Phi^1_\tau + \Phi^2_\tau + \Phi^3_\tau. \tag{2.21} \]

Each of these signals then make a contribution to the nodal residuals. After looping over all triangles, and mapping cell fluctuations to nodes, each node has an accumulated residual consisting of signals gathered from all triangles for which it is
Figure 2.2: Fluctuation splitting in two dimensions.

a vertex. These residuals are then used to evolve the solution in time. Considering the residual contribution from one triangle, the following conservative evolution in time is used:

\[
S_i U_1 \leftarrow S_i U_1 - \Delta t \Phi_i^1 + T.F.O.T \tag{2.22}
\]

\[
S_2 U_1 \leftarrow S_2 U_1 - \Delta t \Phi_2^2 + T.F.O.T \tag{2.23}
\]

\[
S_3 U_1 \leftarrow S_3 U_1 - \Delta t \Phi_3^3 + T.F.O.T \tag{2.24}
\]

where \( S_i \) is the area weight of vertex \( i \), equal to one third of the area of those triangles surrounding \( V_i \). And \( T.F.O.T \) denotes \textit{Terms From Other Triangles}, as each vertex may be receiving signals from other triangles surrounding it.

The main focus is again on the ideal way to distribute the cell fluctuations in order to obtain accurate solutions with sharp discontinuities. In the one-dimensional decomposition of the Euler equations, the system is diagonalizable and thus results in a set of scalar advection operators. However, this is not directly extensible to higher dimensions because the Jacobian matrices \( A \) and \( B \) are not simultaneously diagonalizable, and thus the system can not be reduced in the same manner. The
issue then becomes what is the appropriate extension. How do we correctly decompose the two dimensional system into recognizable subproblems, and then determine the proper discretizations? This is the focus of the remainder of this thesis.
CHAPTER III

THE TWO DIMENSIONAL EULER EQUATIONS

The Euler equations in two dimensions are given by:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

(3.1)

where the conservative state vector $U$ and the flux functions $F$ and $G$ are defined as:

$$U = \begin{cases} \rho \\ \rho u \\ \rho v \\ \rho E \end{cases}, \quad F = \begin{cases} \rho u \\ \rho u^2 + p \end{cases}, \quad G = \begin{cases} \rho v \\ \rho u v \end{cases}.$$

(3.2)

In this system, $\rho$ is the density, $u$ and $v$ are the velocity components in the $x$ and $y$ direction, respectively, and $p$ is the static pressure. The specific energy and enthalpy, denoted by $E$ and $H$, respectively, are given by:

$$E = \frac{1}{\gamma - 1} \rho + \frac{1}{2}(u^2 + v^2),$$

(3.3)

$$H = \frac{\gamma}{\gamma - 1} \rho + \frac{1}{2}(u^2 + v^2).$$

(3.4)

In one dimension, as discussed in the previous chapter, there is a unique decomposition of the system into scalar subproblems, leading to independent advection
operators, but this is not true for the two-dimensional system. We now discuss extensions of these 1D decomposition ideas to two-dimensional flows. Again, the goal is to reduce the system to recognizable subproblems for which accurate discretizations are available.

Unfortunately this issue is non-trivial and the path has been long and tedious. We first present the various methods which have unsuccessfully been employed over the past five years. The framework for most of the work in multidimensional methods to date is based on the discrete wave models first presented by Roe in 1986 [25]. The original idea was to break the system of equations into an equivalent set of scalar advection operators representing the wave behavior inherent in fluid flow. This came from considering the simple wave solutions of the Euler system.

3.1 Simple Wave Solutions

First, for simplicity of the analysis, we will transform the Euler system into primitive variable form:

\[ \mathbf{V}_t + \mathbf{A} \mathbf{V}_x + \mathbf{B} \mathbf{V}_y = 0, \]  

(3.5)

where,

\[ \mathbf{V} = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}, \]

(3.6)

and

\[ \mathbf{A} = \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \frac{\partial \mathbf{F}}{\partial \mathbf{V}}, \quad \mathbf{B} = \frac{\partial \mathbf{V}}{\partial \mathbf{U}} \frac{\partial \mathbf{G}}{\partial \mathbf{V}} \]

are the flux Jacobians transformed into primitive-variable form.

The basis of the “wave-modeling” schemes is that the local flow gradients are
modeled as the result of a set of simple waves. These simple waves are of the form:

\[ \mathbf{V}(\bar{x}, t) = \mathbf{V}(\zeta) = \mathbf{V}((\bar{x} - \bar{\lambda}t) \cdot \bar{m}) = \mathbf{V}(\bar{x} \cdot \bar{m} - \lambda_m t), \quad (3.7) \]

where \( \bar{m} \) is a unit vector in the direction of the gradient imposed by the wave. We define the characteristic velocity (or ray velocity), representing the physical propagation of information, as \( \bar{\lambda} \), with the frontal speed (projection of ray velocity in gradient direction), \( \lambda_m \), given by

\[ \lambda_m = \bar{\lambda} \cdot \bar{m}. \quad (3.8) \]

In assuming local linear variation in \( \mathbf{V} \), we recover plane simple waves which are given by:

\[ \mathbf{V}(\bar{x}, t) = \mathbf{V}(\zeta) = W(\zeta)\mathbf{r} + \mathbf{V}_0 = \alpha(\bar{x} \cdot \bar{m} - \lambda_m t)\mathbf{r} + \mathbf{V}_0, \quad (3.9) \]

representing a traveling wave of constant strength \( \alpha \), orientated such that its normal is defined by \( \bar{m} = \{ \cos \theta, \sin \theta \} \). The projection of this wave onto the primitive variables, \( \mathbf{V} \), is given by \( \mathbf{r} \).

By definition, the simple wave supports a gradient field:

\[ \vec{\nabla} \mathbf{V} = \frac{\partial W}{\partial \zeta} \bar{m} \mathbf{r} = \alpha \bar{m} \mathbf{r}, \quad (3.10) \]

with a time derivative:

\[ \mathbf{V}_t = -\frac{\partial W}{\partial \zeta} \lambda_m \mathbf{r} = -\alpha \lambda_m \mathbf{r}. \quad (3.11) \]

Substituting Equations (3.10) and (3.11) in the Euler system, Equation (3.5), gives the following eigenvalue problem,

\[ \alpha(A \cos \theta + B \sin \theta - \lambda_m t)\mathbf{r} = 0. \quad (3.12) \]

The solution of this problem has the frontal speeds \( \lambda_m \) as the eigenvalues, with the corresponding eigenvectors providing \( \mathbf{r} \); \( \mathbf{r} \) represents the projection of the waves onto the primitive variables.
The solution of the eigenvalue problem yields four waves given by a pair of acoustics: \( \lambda_m^{a\pm} = \vec{q} \cdot \vec{m} \pm a \), shear: \( \lambda_m^s = \vec{q} \cdot \vec{m}^s \) and entropy: \( \lambda_m^e = \vec{q} \cdot \vec{m}^e \) with the corresponding eigenvectors:

\[
\mathbf{r}^{a\pm} = \begin{bmatrix}
\rho \\
\pm a \cos \theta^a \\
\pm a \sin \theta^a \\
\rho \alpha^2
\end{bmatrix},
\mathbf{r}^s = \begin{bmatrix}
0 \\
-a \sin \theta^s \\
\alpha \cos \theta^s \\
0
\end{bmatrix},
\mathbf{r}^e = \begin{bmatrix}
\rho \\
0 \\
0 \\
0
\end{bmatrix},
\]

(3.13)

where \( a \) is the local speed of sound:

\[
a^2 = \frac{\gamma \rho}{\rho}.
\]

(3.14)

To determine the advection operator for the waves, we can equivalently write Equation (3.12) as:

\[
\left( \frac{\partial W}{\partial t} + \alpha (\mathbf{A} \cos \theta + \mathbf{B} \sin \theta) \right) \mathbf{r} = 0.
\]

(3.15)

This can be reduced to an advection equation for \( W \) by utilizing the following relationship

\[
\alpha^k (\mathbf{A} \cos \theta + \mathbf{B} \sin \theta) = \alpha^k \lambda_m^k \mathbf{r}^k = \frac{\partial W}{\partial \zeta} \lambda_m^k \mathbf{r}^k,
\]

(3.16)

where \( \lambda_m^k \) and \( \mathbf{r}^k \) are a corresponding eigenvalue/eigenvector pair of the eigenvalue problem given in Equation (3.12). Incorporating the characteristic speed \( \bar{\lambda} \) by utilizing Equation (3.8), we recover an advection operator for \( W \):

\[
\left( \frac{\partial W}{\partial t} + \frac{\partial W}{\partial \zeta} (\bar{\lambda} \cdot \mathbf{m}) \right) \mathbf{r} = 0,
\]

(3.17)

\[
\left( \frac{\partial W}{\partial t} + \bar{\lambda} \cdot \frac{\partial W}{\partial \zeta} \mathbf{m} \right) \mathbf{r} = 0,
\]

(3.18)

\[
\left( \frac{\partial W}{\partial t} + \bar{\lambda} \cdot \nabla W \right) \mathbf{r} = 0.
\]

(3.19)
The ray speeds \( \vec{\lambda} \), shown pictorially in Figure 3.1 for an acoustic front, are given in the Euler system by:

\[
\begin{align*}
\vec{\lambda}^+ &= \vec{q} \pm a\vec{m}, \\
\vec{\lambda}' &= \vec{\lambda}^\epsilon = \vec{q},
\end{align*}
\]

representing the advection of entropy and shearing waves along streamlines, with acoustic disturbances propagating radially with the speed of sound, relative to the local flow velocity.

![Figure 3.1: Schematic of simple wave displaying orientation and corresponding ray speed \( \vec{\lambda} \).](image)

Any disturbance in the flow can be considered as due to the superposition of each of the four wave types \( k \) over all possible orientations:

\[
\vec{V} = \sum_{\theta=0}^{2\pi} \sum_k \alpha_k(\theta)\vec{r}_k(\theta)\vec{m},
\]

where the local fluctuation imposed by these waves is given by:

\[
\vec{V}_f = -\sum_{\theta=0}^{2\pi} \sum_k \alpha_k(\theta)(\vec{\lambda} \cdot \vec{m}_k)\vec{r}_k.
\]

Although infinitely many waves can exist, the basis of the discrete wave models is the choice of a finite number of simple waves to model local flow behavior. Essentially,
if a valid set of waves is determined, each of these waves is governed by an advection operator. These advection operators can be independently discretized with an upwind biased stencil utilizing schemes developed for scalar advection operators as presented in Chapter IV. Unfortunately the problem lies in choosing the finite set of waves.

3.1.1 Six-Wave Models

The original models proposed by Roe in 1986 [25] were based on the representation of local flow gradients by a discrete set of \( \mathcal{K} \) unsteady simple waves, thus

\[
\mathbf{V}_x = \sum_{k=1}^{\mathcal{K}} \alpha^k \cos \theta^k r^k, \quad \mathbf{V}_y = \sum_{k=1}^{\mathcal{K}} \alpha^k \sin \theta^k r^k.
\] (3.24)

For two-dimensional flow the spatial gradients provided eight degrees of freedom in choosing the waves: two degrees of freedom for each wave, its strength \( \alpha^k \) and its orientation defined by an angle \( \theta^k \). Roe proposed that any flow could be modeled by two pairs of mutually orthogonal acoustics (4 strengths plus 1 direction), a shear (1 strength and 1 direction) and an entropy wave (1 strength and 1 direction). This results in 9 unknowns, providing a one-parameter family of wave-models, expressed here in terms of the shear angle, \( \theta^s \).

The direction and strength of the entropy wave are:

\[
\theta^e = \tan^{-1} \frac{\rho_y - \frac{p_y}{\rho a^2}}{\rho_x - \frac{p_x}{\rho a^2}}, \quad \alpha^e = \sqrt{\left(\frac{\rho_x - \frac{p_x}{\rho a^2}}{\rho_y - \frac{p_y}{\rho a^2}}\right)^2 + \left(\frac{\rho_y - \frac{p_y}{\rho a^2}}{\rho_x - \frac{p_x}{\rho a^2}}\right)^2},
\] (3.25)

the strength of the shear wave:

\[
\alpha^s = \frac{v_x - u_y}{a},
\] (3.26)

and the four acoustics are given by:

\[
\theta^a = \frac{1}{2} \tan^{-1} \frac{v_x + u_y - a\alpha^s \cos(2\theta^s)}{u_x - v_y + a\alpha^s \sin(2\theta^s)},
\] (3.27)
\[ R = \sqrt{\left(\frac{u_x + u_y}{a} - \alpha^s \cos (2\theta^s)\right)^2 + \left(\frac{u_x + v_y}{a} + \alpha^s \sin(2\theta^s)\right)^2}, \quad (3.28) \]

\[ \alpha^{s_1} + \alpha^{s_2} = \frac{u_x + v_y}{a} + R, \quad \alpha^{s_1} - \alpha^{s_2} = \frac{p_x \cos \theta^s + p_y \sin \theta^s}{\rho a^2}, \quad (3.29) \]

\[ \alpha^{s_3} + \alpha^{s_4} = \frac{u_x + v_y}{a} - R, \quad \alpha^{s_3} - \alpha^{s_4} = \frac{p_y \cos \theta^s - p_x \sin \theta^s}{\rho a^2}. \quad (3.30) \]

An example displaying the notation used in identifying the six discrete waves is given in Figure 3.2. The two orthogonal acoustic pairs, defined by one angle \( \theta^s \), travel with the speed of sound in the direction normal to their orientation relative to the local flow speed. The shear wave, also depicted in the figure, is orientated at a given angle \( \theta^s \), and travels at the local flow speed. The entropy wave, not displayed, also travels at the local flow speed with its orientation aligned with the local entropy gradient.

![Wave orientation diagram of acoustics and shear for discrete six-wave models (entropy wave not shown).](image)

Numerous models were developed and tested within this family, differing within
their choice of the shear angle, $\theta'$. Some examples include:

- **Wave Model B** proposed by Roe [25], where the shear wave propagates perpendicular to the flow direction. In this model, the shear does not provide a contribution to the cell fluctuation, i.e., $\lambda' \cdot \vec{m}' = 0$.

- **Wave Model C**, of De Palma et al. [3], with the shear aligned along the pressure gradient. This model has problems in recognizing isolated shear layers through which pressure is constant.

- **Wave Model D** of Roe [28] is the most physical model. It is based on the strain-rate axes and couples the shear and acoustic wavefronts in the following form,

$$\theta' = \theta' + \frac{\pi}{4} \text{sgn} (v_x - u_y).$$

(3.31)

This model produces acoustic waves aligned with the principal strain-rate tensor.

In numerical experiments the best results were obtained with Model D, although all wave models in this family had excessive numerical dissipation. This led to results similar to a conventional first-order solver. It was found by Paillère et al. [18] (generalizing analysis of the supersonic case by Rudgyard [29]) that Linearity Preservation, $(LP)$ within the wave-model is required to avoid unnecessary dissipation. This property requires that the numerical scheme can recognize any cell in equilibrium, i.e., a cell having zero residual. In the context of the wave modeling schemes this requires that in the case of a cell in equilibrium, all wave strengths must be zero. The six-wave models do not possess this property. If spatial gradients exist in an equilibrium cell, the model represents these gradients as a set of unsteady waves, although they cancel each other in the total fluctuation. However, since each
wave is propagated independently, the resulting change to the vertices of the cell will destroy the original equilibrium state. Thus, the local steady solution is smeared. Except under special constraints, there can be no more unsteady waves in the model than there are cell residuals; in two dimensions this limit is four.

Other models falling within this framework were also developed by Rudgyard [29] and Parpia and Michalek [21], neither meeting with universal success. Rudgyard formulated a supersonic splitting, which placed pairs of acoustic wavefronts defined along the Mach lines. This defines that one acoustic wave in each pair represents the steady characteristic and thus makes no contribution to the cell residual. Although the model was not formulated in a linearity preserving framework it did give better results than any of the other wave models employed to date. It also presented some motivation for the necessity to represent steady patterns present within the flowfield as found by Roe. Therefore, in order to reduce the numerical error in mainly smooth subsonic flows, Roe proposed to model the local flow as a combination of both steady patterns and unsteady waves[27], as detailed in the following section.

3.1.2 Steady/Unsteady Decompositions

This splitting was developed to allow discrete decompositions to recognize and keep steady solutions in subsonic regimes, in hopes of reducing the numerical error. It should be noted here that there is currently no supersonic counterpart for which a stable numerical code is achievable. Contrary to the six-wave model schemes, where the steady solution is determined by the cancelation of unsteady waves, it was hoped that by recognizing steady patterns present in the flow, at convergence the unsteady waves would be small in amplitude, and the solution would be represented by a combination of the steady patterns.
First, to determine the patterns present in the two-dimensional Euler system, we study the steady equations in primitive-variable form,

\[ \mathbf{A} \mathbf{V}_x + \mathbf{B} \mathbf{V}_y = 0. \quad (3.32) \]

Substituting linear simple waves of the form: \( \mathbf{V} = \zeta(y - \lambda'x)\mathbf{r}' \), results in an eigenvalue problem for \( \mathbf{A}^{-1} \mathbf{B} \), with the resulting eigenvectors \( \mathbf{r}' \) representing the steady patterns given by:

\[ \mathbf{P} = \{ \mathbf{V}_x, \mathbf{V}_y \} = \{ -\lambda' \mathbf{r}', \mathbf{r}' \} \quad (3.33) \]

where \( \lambda' \) are the corresponding eigenvalues. In supersonic regimes this provides four simple wave patterns. However, in subsonic regimes two of the eigenvalues are complex, representing the elliptic nature of the acoustic front, with the patterns, which are not in simple wave form, resulting from taking the real and imaginary parts. These four patterns \( \mathbf{P} \) are given by two potential-flow solutions:

\[
\begin{align*}
\mathbf{P}^1_{\text{pot}} &= \begin{bmatrix}
\rho v & -\rho u \\
-2uv/M^2 & (u^2 - v^2)/M^2 \\
(u^2 - v^2)/M^2 & 2uv/M^2 \\
\rho a^2 v & -\rho a^2 u
\end{bmatrix} \\
\mathbf{P}^2_{\text{pot}} &= \begin{bmatrix}
\rho u & \rho v \\
-(u^2 + \beta^2 v^2)/M^2 & uv(\beta^2 - 1)/M^2 \\
uv(\beta^2 - 1)/M^2 & -(v^2 + \beta^2 u^2)/M^2 \\
\rho a^2 u & \rho a^2 v
\end{bmatrix}
\end{align*}
\] \quad (3.34)
plus steady shear and entropy solutions:

\[
\mathbf{P}_{\text{shear}} = \begin{bmatrix}
0 & 0 \\
-uv & u^2 \\
v^2 & uv \\
0 & 0
\end{bmatrix}, \quad \mathbf{P}_{\text{ent}} = \begin{bmatrix}
v & -u \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

where \( M = \sqrt{\frac{u_\infty^2 + v_\infty^2}{a}} \) is the Mach number and \( \beta^2 = M^2 - 1 \).

The discrete modeling of the local flow is now considered to be a combination of these steady patterns with a set of unsteady simple waves, such as those found in the previous section, i.e.,

\[
\tilde{\mathbf{V}} = \sum_k \alpha^k(\theta) \mathbf{r}^k(\theta) \tilde{m} + \sum_{\text{steady}} \alpha^\text{steady} \mathbf{P}
\]

We again have eight degrees of freedom in choosing a set of unsteady and steady waves to represent the local spatial gradients. Roe proposed to utilize three steady patterns: both potential-flow patterns plus the shear. This gives three unknowns representing their strengths. For the other five degrees-of-freedom we choose a pair of unsteady acoustics, providing three unknowns: an orientation angle plus their strengths, and an unsteady entropy wave, giving two unknowns: its orientation and strength. The resulting simple expressions for the strengths and orientations of the unsteady waves needed to evolve the solution are:

\[
\theta^a = \tan^{-1} \frac{v_\infty}{u_\infty},
\]

\[
\alpha^a(a^+ \cdot \tilde{m}^a) = \frac{1}{2} \left[ \sqrt{u_\infty^2 + v_\infty^2} - p_\infty/\rho \right],
\]

\[
\alpha^a(a^- \cdot \tilde{m}^a) = \frac{1}{2} \left[ \sqrt{u_\infty^2 + v_\infty^2} + p_\infty/\rho \right],
\]

where \( \tilde{\lambda}^a = q \pm a\tilde{m}^a \). The entropy wave strength and orientation are, as in the six
wave models,

\[ \theta^e = \tan^{-1} \frac{\rho u - \frac{\rho u}{\rho a^2}}{\rho \pm \frac{\rho u}{\rho a^2}}, \quad \alpha^e = \left| \frac{\rho h - \rho_i}{\rho a^2} \right|. \] (3.41)

By definition, the steady patterns do not alter the solution in time. Only the unsteady waves contribute to the fluctuation and because only three such waves are utilized in this model, the model is LP satisfying. This is easily seen, since, in the case of vanishing cell fluctuation, all of the wave amplitudes \( \alpha^k \lambda^k_m \) also vanish.

In numerical experiments solutions with very low numerical diffusion were obtained. However, problems with robustness and convergence were encountered in nearly all test cases performed. Because we attempt to minimize the unsteady cancellation of waves, the cell residuals become small as the steady solution is approached. The wave directions, which depend on these residuals, thus become ill defined. This hindered and usually prevented convergence in all test cases performed.

The original concept of the steady/unsteady splitting allows for alternate sets of waves to model the local flow than those originally proposed by Roe. Therefore other models within this context were experimented with including a model based on minimum-path-lengths similar to that developed by Rumsey for rotated Riemann solvers [30].

In the minimum-path-length model, the two steady potential-flow solutions plus a steady shear wave are utilized, as in the original Roe splitting. However, this model differs in the choice of the three unsteady waves. For the minimum-path-length model, the combination of unsteady waves is either given by:

- a pair of acoustics
- one entropy wave

or
one acoustic

one shear wave orientated perpendicular to the acoustic

one entropy wave.

The choice between the two possible sets of waves is given by the set for which the path-length in \((u_t, v_t, p_t)\)-space is minimized. The resulting mathematical constraints are:

- If
  \[
  (p_t)^2 \geq [\rho a (u_t \cos \theta^a + v_t \sin \theta^a)]^2
  \]
  for
  \[
  \theta^a = \tan^{-1} \frac{v_t}{u_t},
  \]
  then the first option of a pair of acoustics plus an entropy wave provides the shortest path-length.

- If
  \[
  (p_t)^2 \leq [\rho a (u_t \cos \theta^a + v_t \sin \theta^a)]^2,
  \]
  again for
  \[
  \theta^a = \tan^{-1} \frac{v_t}{u_t},
  \]
  the second option consisting of one acoustic, one shear perpendicular to the acoustic and an entropy wave provides the minimum path-length.

This alternate splitting did produce smooth solutions with low numerical dissipation, just as the original splitting. Unfortunately, it did not show improvement over the original model of Roe for robustness. For all test cases analyzed, this alternate model resulted in the same inability to converge to a steady solution.
It is also noted, that other LP models have been developed by the VKI, mainly within the context of the characteristic decompositions; the most recent is the Pseudo Mach Angle Model Method (PMA) [18]. In this method the supersonic solver is based on the steady characteristics, while in the subsonic regime four acoustic waves are employed to model the omni-directional acoustical propagation of information. Therefore the method still attempts to reduce the system to scalar advection operators. While this produced the best results to date, it utilizes many waves and therefore we seek an alternate decomposition which truly takes into account the omni-directional behavior of acoustical waves in subsonic regimes.

### 3.2 Hyperbolic/Elliptic Equation Decomposition

It appears that the use of time-dependent waves to represent steady solutions, or even the errors in steady solutions, is not an appropriate method. In order to develop a better discretization method for solving steady-state problems we first study the steady Euler system.

Considering the steady system in primitive variable form:

\[
A V_x + B V_y = 0, \quad (3.42)
\]

we desire to reduce this system to a set of simplified problems for which we know good discretization techniques. We find that for an invertible matrix \(A\):

\[
V_x + A^{-1} B V_y = 0 \quad (3.43)
\]

a decomposition into advection operators again reduces to an eigenvalue problem for \(A^{-1}B\), where we can equivalently represent \(A^{-1}B\) by \(L^{-1}AL\):

\[
V_x + L^{-1} AL V_y = 0. \quad (3.44)
\]
If we multiply through by the left eigenvector matrix $L$:

$$LV_x + ALV_y = 0 \quad (3.45)$$

the system then reduces to four decoupled equations for the characteristic variables

$$\partial \mathbf{W} = L \partial \mathbf{U};$$

$$\mathbf{W}_x + A \mathbf{W}_y = 0. \quad (3.46)$$

These four “advective” equations, equally represented as

$$\tilde{\lambda} \cdot \nabla \mathbf{W} = 0, \quad (3.47)$$

are given by:

- Two streamline characteristics representing constant entropy and enthalpy along streamlines,

$$\tilde{\lambda}_{s,c} = \bar{q}, \quad \partial W_c = \partial p - a^2 \partial \rho$$

$$\partial W_s = \partial p + \rho q \partial q$$ \quad (3.48)

where $q$ is the local flow velocity, given by:

$$q = \sqrt{u^2 + v^2} \quad \partial q = \frac{u \partial u + v \partial v}{q}. \quad (3.49)$$

- Acoustic characteristics which are real in supersonic flow, but imaginary in subsonic regimes, given by:

$$\tilde{\lambda}_\pm = (u \beta \mp v, v \beta \pm u) \quad \partial W^\pm = \partial p \pm \frac{\rho q^2}{\beta} \partial \theta \quad (3.50)$$

where $\beta = \sqrt{M^2 - 1}$ and $\theta$ is the local flow direction given by:

$$\tan \theta = \frac{v}{u} \quad \partial \theta = \frac{u \partial v - v \partial u}{q^2}. \quad (3.51)$$
This is the characteristic form of the system. The streamline characteristics are real, yielding hyperbolic equations governed by advective operators independent of the flow regime. However, the acoustic characteristics clearly differ between supersonic and subsonic flows, as \( \beta = \sqrt{M^2 - 1} \) becomes imaginary for \( M < 1 \). In supersonic flows the acoustic portion is hyperbolic and also given by advection operators: in subsonic regimes the acoustics are imaginary, yielding an elliptic “sub-system.” We thus discuss the two regions independently.

- **Supersonic Flow**

  In supersonic flows, the Euler system is hyperbolic in space, and thus is equivalently represented by four decoupled advection operators: the two streamline characteristics, plus the acoustic characteristics. The acoustic characteristics are defined along the Mach lines, shown pictorially in Figure 3.3. The acoustic characteristic variables represent the relationship between the pressure field gradients and the turning of the flow.

  This yields four uniquely defined directions on which upwind biased stencils can be based, representing the physical propagation of information in space. Since the equations are completely decoupled, in the linear system, we have no source terms and the reduced equation which we need an accurate discretization for is simply the scalar advection operator \( \bar{\lambda} \cdot \nabla u = 0 \).

- **Subsonic Flow**

  In subsonic regimes, the equations are of mixed hyperbolic/elliptic type. The streamline characteristic equations, equivalent to those in supersonic flow, are hyperbolic, and upwind discretizations are used. However, the acoustic equations are now complex, physically representing the isotropic spreading of infor-
Figure 3.3: Characteristics of steady Euler system in supersonic flow.

mation in space. By requiring both the real and imaginary parts of the acoustic equations to be zero, the following coupled system is recovered:

\[
\begin{bmatrix}
-\beta_s^2 & 0 \\
0 & \rho q^2
\end{bmatrix}
\begin{bmatrix}
p \\
\theta
\end{bmatrix}_s +
\begin{bmatrix}
0 & \rho q^2 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
p \\
\theta
\end{bmatrix}_n = 0.
\] (3.52)

For simplicity we use streamline coordinates, denoted by \( \partial_s \) and \( \partial_n \), representing derivatives along and normal to the streamline, respectively, and \( \beta_s = \sqrt{1 - M^2} \). By transforming variables with \( \partial u = \rho q^2 \partial \theta \) and \( \partial v = \beta_s \partial p \), and transforming into a stretched coordinate system given by \( \partial \xi = \beta_s \partial_s \) and \( \partial \eta = \partial_n \), we recover the Cauchy-Riemann equations:

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
u \\
\xi
\end{bmatrix} +
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}_\eta = 0.
\] (3.53)

This system, unlike the hyperbolic operators, does not possess a preferential direction along which information is sent, but rather represents an omnidirectional behavior. The elliptic nature of the system is clearly seen in the
Cauchy-Riemann equations, where the solution field is made of harmonic functions $u$ and $v$, i.e. $\nabla^2 u = \nabla^2 v = 0$. Central discretizations are used, deviating from the past ideas where upwind techniques were utilized for all portions of the flow. Therefore, the subsonic equations have been reduced to two types of simplified problems: advection operators which we discretize with upwind biased stencils as in the supersonic regime, and a $2 \times 2$ Cauchy-Riemann system, for which symmetric stencils are used.

3.2.1 “Time-Dependent” System

Now that we have successfully decomposed the steady system into recognizable subproblems, we need to incorporate time-dependent terms into the system in order to converge the solution by marching in “pseudo-time.” We propose to do this with preconditioning techniques similar to those developed by Van Leer/Lee/Roe [41] and Turkel [39]. These preconditioners were originally developed as convergence acceleration techniques, where the time-dependent Euler equations are modified in the following form:

$$U_t + P (AU_x + BU_y) = 0. \tag{3.54}$$

This alters the transient system but leads to an equivalent steady state, for positive-definite $P$. Although we still utilize the preconditioner for producing a “pseudo” time-dependent system with near-optimized condition numbers, now, more importantly, we demand that the preconditioner shall preserve, even in the time-dependent system, the unique complete decoupling found in the steady equations.

For simplicity in discussion we transform the Euler equations into “symmetrizing-variable” form in streamline coordinates, given by:

$$\frac{\partial U_{sym}}{\partial t} + A \frac{\partial U_{sym}}{\partial s} + B \frac{\partial U_{sym}}{\partial n} = 0$$
\[ \partial U_{\text{sym}} = \left\{ \frac{\partial p}{\partial a}, \partial q, \, q \partial \theta, \, \partial p - a^2 \partial \rho \right\}^T. \]

where again \( q \) is the local flow speed and \( \theta \) is the flow direction, with \( \partial s \) and \( \partial n \) representing derivatives along and normal to the streamline direction, respectively.

We note here that for the remainder of this chapter we will denote \( U_{\text{sym}} \) by \( U \) for brevity.

Because the information mechanisms vary between subcritical and supercritical flow we again analyze each region independently and then ensure their analytical continuity through the sonic line.

### 3.2.1.1 Supersonic Regime

We know that the steady supersonic Euler Equations are convective by nature, however the time-dependent Euler Equations recouple the characteristic equations, even in the linear system. What we attempt to do is find an altered time-dependent system which is also represented by four scalar convective equations. We also wish to optimize the condition number of the equations, requiring all unsteady waves to propagate at the same speed. To do this we consider the quasi-linear symmetrizing variable form:

\[ U_t + P (A U_s + B U_n) = 0 \quad (3.55) \]

where \( P \) is some positive-definite matrix that will diagonalize the system while equalizing speeds. Performing the following steps:

\[
\begin{align*}
U_t + P (A U_s + B U_n) &= 0, \\
P^{-1} U_t + \, A U_s + B U_n &= 0, \\
A^{-1} P^{-1} U_t + U_s + A^{-1} B U_n &= 0, \\
A^{-1} P^{-1} U_t + U_s + L^{-1} \Lambda U_n &= 0,
\end{align*}
\]
\[ LA^{-1}P^{-1} U_t + LU_s + ALU_n = 0, \]
\[ LA^{-1}P^{-1}L^{-1} U_t + LU_s + ALU_n = 0, \]
\[ LA^{-1}P^{-1}L^{-1} W_t + W_s + AW_n = 0, \]

converts the system to characteristic form. This again defines the characteristic variables \( W \) such that \( \partial W = L \partial U \), where \( L \) contains the left eigenvectors of \( A^{-1}B \), giving:

\[
\partial W = \begin{bmatrix}
\partial p - a^2 \partial \rho \\
\frac{1}{\rho \beta} \partial p + \partial q \\
\frac{2}{\rho \beta} \partial p + q \partial \theta \\
\frac{2}{\rho \beta} \partial p - q \partial \theta
\end{bmatrix},
\]

the same as in the steady system. And \( A \) is the diagonal matrix of the corresponding eigenvalues, given by:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\beta} & 0 \\
0 & 0 & 0 & -\frac{1}{\beta}
\end{bmatrix},
\]

which are now given in the streamline coordinate system. To diagonalize the system, we require that:

\[ D = LA^{-1}P^{-1}L^{-1} \]

be a diagonal matrix, yielding four independent advective equations for the characteristic variables, \( W \):

\[ DW_t + W_s + AW_n = 0. \]

The other constraint placed upon the preconditioner is that it perfectly conditions the system, i.e. it makes all waves travel at the same speed. This is achieved by re-
quiring each of the diagonal elements in the matrix $D$ to normalize the corresponding characteristic speed to some chosen rate, $\lambda_i$. Thus the elements of $D$ are,

For: $D = diag(d_i)$ and $\Lambda = diag(\lambda_i)$, $d_i = \frac{\sqrt{1 + \lambda_i^2}}{\lambda_i}$.

If we choose to advect all waves at the local flow velocity, giving $\lambda_c = q$ then we define

$$D = \begin{bmatrix}
\frac{1}{q} & 0 & 0 & 0 \\
0 & \frac{1}{q} & 0 & 0 \\
0 & 0 & \frac{1}{\beta_s} & 0 \\
0 & 0 & 0 & \frac{1}{\beta_s}
\end{bmatrix}.$$  \hspace{1cm} (3.60)

We can now determine the unique preconditioner $P$ given by:

$$P = L^{-1} D^{-1} \Lambda A^{-1} = \begin{bmatrix}
\frac{M}{\beta} & -\frac{1}{\beta} & 0 & 0 \\
\frac{1}{\beta} & 1 + \frac{1}{M\beta} & 0 & 0 \\
0 & 0 & \frac{\beta}{\sqrt{\eta}} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$  \hspace{1cm} (3.61)

This is equivalent to the supersonic preconditioner originally derived by Van Leer/Lee/Roe [41] for convergence acceleration.

The altered Euler system can now be represented by four scalar advective equations all with a characteristic speed equal to the local flow speed:

$$(\partial_t + q \partial_s) \left( \partial p - a^2 \partial \rho \right) = 0 \hspace{1cm} (3.62a)$$

$$(\partial_t + q \partial_s) \left( \partial p + \rho q \partial q \right) = 0 \hspace{1cm} (3.62b)$$

$$\left( \partial_t + \frac{\beta q}{M} \partial_s + \frac{q}{M} \partial_n \right) \left( \partial p + \frac{\rho q^2}{\beta} \partial \theta \right) = 0 \hspace{1cm} (3.62c)$$

$$\left( \partial_t + \frac{\beta q}{M} \partial_s - \frac{q}{M} \partial_n \right) \left( \partial p - \frac{\rho q^2}{\beta} \partial \theta \right) = 0. \hspace{1cm} (3.62d)$$

These advection equations clearly satisfy the decoupled steady-state operators that we previously derived, simply adding the appropriate time terms providing a mecha-
nism for marching to convergence. Each of the advection equations is thus independently updated by a forward Euler time integration scheme for the scalar advection equation:

\[ u_t + \bar{\lambda} \cdot \nabla u = 0, \quad (3.63) \]

providing a preferred direction on which to base an upwind bias. The details covering convergence and discretization properties of the now reduced system are shown in the following chapter.

3.2.1.2 Subsonic Regime

For subsonic flows, we attempt to utilize the same preconditioning technique to most importantly preserve the decomposition into scalar subproblems found for the steady-state equations. We also make the additional constraint of optimizing the condition number of the system, although perfect conditioning is not possible in the subsonic regime. In this case, the steady system is of mixed hyperbolic/elliptic type, and therefore we hope to extract two advection operators of the form:

\[ u_t + \bar{\lambda} \cdot \nabla u = 0. \quad (3.64) \]

For the remaining equations, we wish to extract a coupled acoustic system of two equations, which, under the same variable and coordinate transformations as given in the steady elliptic system, will reduce to the Cauchy-Riemann equations augmented by time terms in the following way:

\[
\begin{align*}
\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_y &= 0.
\end{align*}
\quad (3.65)
\]

This time-dependent system is hyperbolic, allowing for a forward Euler integration method consistent with that used for the advection operators. However, it does result
in an elliptic steady state. From here on, we refer to this hyperbolic sub-system as the “elliptic-based” system because it is based on the original elliptic acoustical system in the subsonic steady Euler equations.

We again start with the time-dependent Euler equations in symmetrizing variable form, modified with a preconditioning matrix \( P \). However, instead of diagonalizing the system, we desire the equations to be of the following form:

\[
W_t + A^w W_x + B^w W_y = 0; \quad A^w = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}, \quad B^w = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.
\]

This provides the first two characteristic/advective equations plus the isolated \( 2 \times 2 \) acoustic system.

We first assume the resulting characteristic variables will be the same as those found for the steady system, i.e.

\[
\partial W = L' \partial U = \begin{bmatrix} \partial p - a^2 \partial \rho \\ \partial p + \rho q \partial q \\ \frac{1}{\mu} \partial \rho \\ q \partial \theta \end{bmatrix}, \quad (3.66)
\]

and the matrices

\[
A^w = L' P A (L')^{-1} \quad (3.67)
\]

and

\[
B^w = L' P B (L')^{-1}. \quad (3.68)
\]

Thus the preconditioner is given by:

\[
P = (L')^{-1} A^w L'A^{-1} \quad (3.69)
\]
and

\[ P = \left( L^T \right)^{-1} B^w L^T B^{-1}. \]  

(3.70)

Satisfying both equations for \( P \), while requiring the proposed block form of \( A^w \) and \( B^w \) defined previously, we find that \( P \) must be of the following form:

\[
P = \begin{bmatrix}
-M_{p_{1,2}} & p_{1,2} & -M_{p_{2,3}} & 0 \\
 p_{1,2} & p_{2,2} & p_{2,3} & 0 \\
 -M_{p_{3,2}} & p_{3,2} & p_{3,3} & 0 \\
 0 & 0 & 0 & p_{4,4}
\end{bmatrix}.
\]

(3.71)

It appears that this subsonic preconditioner is not unique, and therefore we must make some assumptions. The first simplification comes from assuming the same sparseness of the preconditioning matrix as that found for the supersonic regime. This is advantageous when considering that the two preconditioners must merge at the sonic point. However, it is noted here that other terms could be present while still allowing for sonic line continuity as long as those terms contain a factor of \((1 - M^2)\).

This constraint requires \( p_{3,2} = p_{2,3} = 0 \), and results in a symmetric preconditioning matrix:

\[
P = \begin{bmatrix}
-M_{p_{1,2}} & p_{1,2} & 0 & 0 \\
p_{1,2} & p_{2,2} & 0 & 0 \\
 0 & 0 & p_{3,3} & 0 \\
 0 & 0 & 0 & p_{4,4}
\end{bmatrix}.
\]

(3.72)

We now wish to optimize the condition number of the system, and find that the subsonic preconditioner developed by Van Leer/Lee/Roe [41] fits the desired form
given in Equation (3.72) given as:

\[
P = \begin{bmatrix}
\frac{M^2}{\beta_*} & -\frac{M}{\beta_*} & 0 & 0 \\
-\frac{M}{\beta_*} \frac{1}{\beta_*} + 1 & 0 & 0 \\
0 & 0 & \beta_* & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]  

(3.73)

The resulting decoupled system produced with this preconditioner is given by two streamline advection equations for entropy and enthalpy, consistent with the supersonic case:

\[
(\partial_t + q \partial_z) \left( \partial p - a^2 \partial \rho \right) = 0, 
\]  

(3.74)

\[
(\partial_t + q \partial_z) (\partial p + \rho q \partial q) = 0. 
\]  

(3.75)

Again both quantities are convected with the local flow speed \( q \). The now coupled acoustic system is given by:

\[
\left\{ \begin{array}{c}
\beta_* \frac{\partial p}{\rho q} \\
q \partial \theta
\end{array} \right\}_t + \left[ \begin{array}{c}
-q \beta_* \\
0
\end{array} \right] \left\{ \begin{array}{c}
\beta_* \frac{\partial p}{\rho q} \\
q \partial \theta
\end{array} \right\}_z + \left[ \begin{array}{c}
0 \\
q
\end{array} \right] \left\{ \begin{array}{c}
\beta_* \frac{\partial p}{\rho q} \\
q \partial \theta
\end{array} \right\}_n = 0. 
\]  

(3.76)

This system, as desired, reduces to the time-dependent Cauchy-Riemann system given in Equation (3.53). The time-dependent system is hyperbolic producing an elliptical domain of influence, centered about the point of origin, with a streamwise range of \( q \beta_* \) and a cross-stream range equal to the local flow speed \( q \). This is considered to be the optimal conditioning of the subsonic Euler system, with a condition number of \( 1/\beta_* \).

Unfortunately there are some issues recently discovered which pertain to this preconditioned system of equations. When this system was utilized in a numerical code, stability problems in stagnation regions hindered convergence, usually preventing well-converged solutions from being obtained. It was discovered by Darmofal and
Schmidt [2] that a degeneracy in the system eigenvectors allows for transient energy growth which is unbounded in the limit $M \to 0$. Another problem with the preconditioner is the sensitivity to the flow angle in the same stagnation point limit [42]. While removal of the eigenvector degeneracy is nontrivial, the flow-angle sensitivity is easily removed by requiring that in the preconditioning matrix the $p_{2,2}$ element approaches the $p_{3,3}$ element in the $M \to 0$ limit. This is easily seen by rotating the preconditioner into Cartesian coordinates by the following transformation,

$$
R(\theta) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
$$

which is a function of the local flow direction. The preconditioner in Cartesian coordinates is then given by:

$$
P^c = R^{-1}(\theta)PR(\theta).
$$

Substituting in the general form of the preconditioner given in Equation 3.72, we find

$$
P^c = \begin{bmatrix}
-Mp_{1,2} & p_{1,2}\cos \theta & p_{1,2}\sin \theta & 0 \\
p_{1,2}\cos \theta & p_{2,2}\cos^2 \theta + p_{3,3}\sin^2 \theta & \cos \theta \sin \theta(p_{2,2} - p_{3,3}) & 0 \\
p_{1,2}\sin \theta & \cos \theta \sin \theta(p_{2,2} - p_{3,3}) & p_{2,2}\sin^2 \theta + p_{3,3}\cos^2 \theta & 0 \\
0 & 0 & 0 & p_{4,4}
\end{bmatrix}.
$$

From this matrix it is easily seen that only if element $p_{2,2}$ approaches $p_{3,3}$ in the limit of $M \to 0$, will the two central diagonal terms given by $p_{2,2}\cos^2 \theta + p_{3,3}\sin^2 \theta$ approach a constant; if not, the remaining functions of the flow angle can have large variations within a stagnation region.
We again attempt to optimize the condition number, under the same sparseness consideration plus this added flow-angle sensitivity constraint [42]. The following family of preconditioners is recovered:

\[
P = \begin{bmatrix}
\frac{cM^2}{\beta_*} & -\frac{cM}{\beta_*} & 0 & 0 \\
-\frac{cM}{\beta_*} & \frac{1}{\beta_*} + \epsilon & 0 & 0 \\
0 & 0 & \beta_* & 0 \\
0 & 0 & 0 & \epsilon
\end{bmatrix}
\]

where \(\epsilon = fcn(M)\) such that \(\epsilon = \frac{1}{2}\) in the \(M \to 0\) limit satisfying the flow angle sensitivity constraint. Also, \(\epsilon = 1\) in the transonic limit gives the smooth transition of the preconditioning matrices between the two regimes. The slightly altered decoupled system of equations is now:

\[
(\partial_t + \epsilon q \partial_s)(\partial p - a^2 \partial \rho) = 0
\]

\[
(\partial_t + \epsilon q \partial_s)(\partial p + \rho q \partial q) = 0
\]

\[
\left\{ \begin{array}{c}
\frac{\beta_* \partial p}{\rho q} \\
q \partial \theta
\end{array} \right\}_i + \left[ \begin{array}{ccc}
-\epsilon q \beta_* & 0 & 0 \\
0 & q \beta_* & \epsilon q
\end{array} \right] \left\{ \begin{array}{c}
\frac{\partial p}{\rho q} \\
q \partial \theta
\end{array} \right\}_s = 0.
\]

The characteristic variables remain the same, however the wave-speeds have been altered.

First we determine a valid function for \(\epsilon(M)\) with the appropriate limits, we choose the following:

- **Incompressible Region**, \(M \leq \frac{1}{3}\) \(\quad \epsilon = \frac{1}{2}\).

- **Transition Region**, \(\frac{1}{3} \leq M \leq \frac{2}{3}\) \(\quad \epsilon = \frac{1}{2} + \frac{27}{2}(M - \frac{1}{3})^2 - 27(M - \frac{1}{3})^3\).

- **Transonic Region**, \(M \geq \frac{2}{3}\) \(\quad \epsilon = 1\).
This provides for a cubic function within the transition region giving a smooth variation between the incompressible and transonic limits. Many other functions would probably suffice but were not dealt with in this work.

The advective equations now have characteristic speeds given by \( \epsilon q \). The acoustic domain of influence is shown in Figure 3.4. While it remains elliptical in shape, as the original van Leer preconditioner, the center has moved to \( \frac{1}{2} \beta_\epsilon (1 - \epsilon) q \), which is away from the origin for \( \epsilon \neq 1 \). The stream-wise range consists of a forward wave moving at \( \beta q \) and a backward wave traveling at \( \epsilon \beta q \). The cross-stream waves travel at a rate of \( \pm \sqrt{\epsilon} q \) normal to the flow. This produces a ray speed given as

\[
\lambda = \left( \sqrt{\frac{1}{4} \beta_\epsilon^2 (1 - \epsilon)^2 + \epsilon} \right) q. \tag{3.81}
\]

Clearly the backward acoustic is the slowest wave, giving \( \lambda_{\text{min}} = \epsilon \beta_\epsilon q \) with the fastest wave given by the maximum of the forward acoustic and the ray speed for the transverse waves given in Equation (3.81). For \( M \leq 0.5 \) the forward acoustic is the fastest giving a condition number of two, which is twice that of the unmodified preconditioner due to the added stagnation limit constraint. Therefore, for \( M \geq 0.5 \) the transverse acoustics are the fastest giving a condition number of

\[
\frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} = \frac{\sqrt{\frac{1}{4} \beta_\epsilon^2 (1 - \epsilon)^2 + \epsilon}}{\epsilon \beta_\epsilon q}. \tag{3.82}
\]

The transonic limit of the condition number, \( \frac{2}{3} \), is equivalent to that of the original Van Leer preconditioner because the modified preconditioner reverts to its original form for \( M \geq \frac{2}{3} \).

Although this modification has not overcome the problem of degenerating eigenvectors, also determined by Darmofal and Schmid [2], it has alleviated the stagnation point stability issue for the discretizations discussed in this work. Essentially, non-conservative formulations of the method appear stable for decreasing grid sizes, even
Figure 3.4: Domain of influence of acoustic waves for preconditioned system near sonic line.

In low speed flows. However, conservative formulations do appear to suffer from the unstable transient growth due to the degeneracy, hindering convergence. More about this will be discussed in the Numerical Algorithm and Results chapters, VI and VII respectively.

In employing this preconditioning technique the elliptic-based system is isolated from the hyperbolic components even in the transient system. Because the acoustical elliptic-based system no longer has a preferential direction of propagation but rather represents the isotropic spreading of information, damping schemes are appropriate to converge the discrete approximation. This of course eliminates the need to choose advection directions, necessary in the past methods, from the infinitely many that exist.

There is another very useful benefit in utilizing this form of preconditioning,
similar to work of Van Leer/Lee/Roe and Turkel: the preservation of accuracy in the compressible solver even in the incompressible limit [43, 40]. Standard computational methods used to approximate the compressible equations generally suffer from a loss of accuracy as the freestream Mach number becomes small, i.e., in the limit of incompressible flow. This is because the numerical scheme does not approach an acceptable method for approximating incompressible flow, as do the differential equations governing the system. Fortunately, with the use of the preconditioning matrices shown here, the numerical approximation does remain valid in low-speed flows, approaching the correct incompressible limit. This allows for accurate solutions to be obtained with the compressible solver, even in nearly incompressible flows. The main advantage of this is in situations of low-speed flow over airfoils at high angles of attack, where isolated compressible regions are generated near the leading edge, or in propulsion systems, where flow conditions may range from stagnation in a reservoir to supersonic outflow.

3.2.1.3 Sonic Line Continuity

It is clear that the convection of entropy and enthalpy along streamlines is continuous between the subsonic and supersonic regimes. The continuity of the acoustic behavior is seen by examining the domain of influence of the acoustic waves in the transonic limit, shown pictorially in Figure 3.5. In approaching the limit from the subcritical side the domain of dependence of the acoustic disturbances is an ellipse whose maximum stream-wise extent is \( q\beta_s \), while the cross-stream extent is the flow velocity, \( q \). It is clear that as \( M \to 1, \beta_s \to 0 \), and the ellipse flattens out into a line of length \( q \) from the origin. The domain of influence for the characteristics defining the acoustic disturbances in the supersonic regime, shown in Figure 3.5 b).
are the Mach lines with length equal to the local flow speed. As the sonic point is approached the Mach lines open up eventually becoming normal to the streamline, giving the same limit as the subcritical case.

![Diagram](image)

(a) Subsonic Acoustic Wavefront  (b) Supersonic Acoustic Wavefront

Figure 3.5: Domain of influence of acoustic waves for preconditioned system near sonic line.

### 3.2.2 Summary of Decomposition Method

The decomposition presented here is the unique decoupling of the linear Euler system, giving four independent advective operators in supersonic flows, and two advective operators plus an isolated elliptic-based system in subsonic regimes. Through preconditioning techniques, this decoupling was preserved even in the “pseudo” time-dependent equations, providing an iterative strategy for solving the decoupled system. However, while the linear system is decoupled, for non-linear problems coupling between the independent systems will occur via the coefficients.
In comparison, the canonical form advocated by Ta’asan [35]:

\[
\begin{bmatrix}
q \frac{\partial}{\partial s} & 0 & 0 & 0 \\
0 & q \frac{\partial}{\partial s} & 0 & 0 \\
0 & 0 & (a^2 - q^2) \frac{\partial}{\partial s} & a^2 q \frac{\partial}{\partial n} \\
\frac{1}{\gamma} \frac{\partial}{\partial n} \frac{q}{(\gamma - 1) \rho} \frac{\partial}{\partial n} & \frac{1}{\gamma - 1} \frac{\partial}{\partial n} & -\frac{\partial}{\partial n} & q \frac{\partial}{\partial s}
\end{bmatrix}
\begin{bmatrix}
H \\
S \\
q \\
\theta
\end{bmatrix} = 0,
\]  

(3.83)

with

\[
\partial S = \partial p - a^2 \partial \rho,
\]  

(3.84)

and

\[
\partial H = \frac{\gamma}{\gamma - 1} \left( \frac{\partial p}{\rho} - p \frac{\partial \rho}{\rho^2} \right) + q \partial q,
\]  

(3.85)

results in direct coupling between the “elliptic” and hyperbolic advective operators even in the linearized system. With regard to convergence rates and accuracy it is not clear if any advantage is gained by limiting the coupling to the coefficients alone. Currently different discretization techniques based on staggered grids are utilized by Ta’asan, making comparisons between the canonical forms alone impossible.
CHAPTER IV

UPWINDING SCHEMES FOR ADVECTION OPERATORS IN TWO DIMENSIONS

In the previous chapter, three methods of reducing the Euler equations into sub-problems were presented: simple-wave models, steady/unsteady splitting and hyperbolic/elliptic decomposition. In the “wave models”, the equations were represented by simple waves governed by advection operators, where the Hyperbolic/Elliptic splitting resulted in hyperbolic advection operators for enthalpy, entropy and supersonic acoustic disturbances. In the last decade, significant progress has been made in the development of truly multidimensional upwind schemes for advection operators in two and three dimensions. These upwinding schemes are formulated in a fluctuation-splitting context, operating on compact stencils, and producing accurate monotone solutions. Presented in this section will be the formulations for various schemes developed in two dimensions, including their recent unification with finite volume formulations of Sidilkover [32].

4.1 Linear Schemes

In analyzing a two dimensional advection equation of the form

\[ u_t + \bar{\lambda} \cdot \nabla u = 0, \]  

(4.1)
the fluctuation in a given triangle, $\tau$,
\[
\Phi_\tau = \int \int (\vec{\nabla} \cdot \vec{u}) \, dA,
\]  
(4.2)
is calculated from the basic assumption of piecewise linear data, $u$, over the element. This signal then must be distributed among the three vertices, $V_i$, where we denote the signal sent to each vertex $i$ in a given triangle $\tau$ by $\Phi^i_\tau$ for $i = 1...3$ and the sum of the signals equals the fluctuation calculated within the element, i.e.
\[
\sum_{i=1}^{3} \Phi^i_\tau = \Phi_\tau.
\]  
(4.3)
A conservative evolution of the solution is then given by:
\[
S_1 u_1^{n+1} = S_1 u_1^n - \Delta t \Phi^1_\tau + TFOT
\]  
(4.4)
\[
S_2 u_2^{n+1} = S_2 u_2^n - \Delta t \Phi^2_\tau + TFOT
\]  
(4.5)
\[
S_3 u_3^{n+1} = S_3 u_3^n - \Delta t \Phi^3_\tau + TFOT,
\]  
(4.6)
where $TFOT$ refers to terms from other triangles, because each node can receive a signal from each triangle of which it is a vertex. The area weight $S_i$ assigned to each node is its median dual area given as one third of the area of the triangles of which it is a vertex,
\[
S_i = \sum_{\forall r, i \in V_r} \frac{1}{3} S_r.
\]  
(4.7)
In developing upwind schemes, the velocity of advection, $\vec{\lambda}$, defines the direction of flow, or direction in which information is propagated. In any given triangle, the local flow direction $\vec{\lambda}$ results in two types of triangles as shown in Figure 4.1.

- Type I refers to elements with only one inflow side.
- Type II triangles have two inflow sides and therefore the cell fluctuation must be split between the two opposite vertices.
All schemes are equivalent in the type I case with the entire fluctuation sent to the one downstream vertex. The various upwind schemes used throughout this work differ in how the fluctuation within the type II triangle is distributed between the two downstream vertices.

Before presenting the various distribution methods, some desirable properties of the resulting update schemes are presented here to clarify notation.

- **LP Property (Linearity Preservation)**

  A scheme which is LP satisfying is globally able to obtain, for any triangulation, an exact linear solution to the advection problem for which the solution is linear in space. This constraint locally reduces to the need for vanishing signals sent to the cell vertices in the case of vanishing cell fluctuation, i.e.

  \[ \Phi_r \to 0 \Rightarrow \Phi_r^i \to 0 \quad (\forall i). \]

  (4.8)

  For regular Cartesian grids broken into triangles using equivalent diagonals all linear schemes which satisfy LP are at least second order accurate [5]. For
arbitrary grids, on which accuracy is difficult to analyze, numerical tests presented later have shown that schemes satisfying LP are more accurate than those which do not, emphasizing the important correlation between property LP and accuracy.

- **Positivity Property**

For a linear scheme, where the solution at a vertex $i$ is given as a function of those points surrounding it, with constant coefficients, i.e.

$$u_i = \sum_{k=1}^{N} c_{i,k} u_k,$$

(4.9)

the update scheme is said to be positive if all of the coefficients are non-negative, i.e. $c_k \geq 0, \ k = 1...N$. This prohibits extrema from developing in the interior of the solution, guaranteeing stability and thus convergence of the scheme. This constraint enforces positivity at each vertex, called global positivity from here on, but actually a stronger constraint of local positivity is desired in order to maintain the compactness of the scheme. By ensuring that the update sent to each vertex from each triangle has positive coefficients, it is guaranteed that the global update at a vertex, i.e. the total update to that vertex from all triangles of which it is a member, will satisfy positivity constraints. This local enforcement allows the algorithm to process each mesh simplex individually, without the need for information about updates to the local vertices from other triangles of which they are members.

The first schemes discussed are linear distribution schemes for which the coefficients $c_{i,k}$ in Equation (4.9) are constants, i.e. do not depend upon the data.
4.1.1 Low Diffusion Scheme

A linear splitting scheme which satisfies property LP but is not positive is the Low Diffusion Scheme (LDA). The cell fluctuation is split into two parts, sent to the downstream vertices, that are proportional to the two areas cut from the triangle by the advection vector, as shown in Figure 4.2. The area $A_2$ weights the signal to node 2, with $A_3$ weighting the signal given to node 3. The resulting split signals are

$$
\Phi_2 = \frac{A_2}{A_r} \Phi, \quad \Phi_3 = \frac{A_3}{A_r} \Phi_r.
$$

(4.10)

This scheme clearly satisfies property LP, given that both $\Phi_2$ and $\Phi_3$ go to zero as the cell fluctuation vanishes. Unfortunately, local positivity constraints are violated within each two-target triangle and in fact even the total updates at each vertex cannot be made positive. This is consistent with a proof given by Deconinck, Struijs et al. [5] showing that a linear scheme cannot be both linearity preserving and locally positive, which is rooted in Godunov’s theorem.

Figure 4.2: Splitting of fluctuation for LDA scheme
4.1.2 N Scheme

Another linear splitting of the fluctuation, known as the N Scheme (because of its Narrow stencil and Narrow discontinuity capturing), results from splitting the advection vector into two components parallel to the sides opposite the downstream vertices as shown in Figure 4.3. We set

$$\bar{\lambda} = \bar{\lambda}_2 + \bar{\lambda}_3,$$  \hspace{1cm} (4.11)

where $\bar{\lambda}_2$ lies parallel to the side opposite vertex 3 and $\bar{\lambda}_3$ lies parallel to the side opposite vertex 2.

![Figure 4.3: Splitting of advection vector for N Scheme](image)

The signals sent to the two vertices are the new fluctuations that are calculated from the now split advection problem:

$$\Phi_2 = \int \int \bar{\lambda}_2 \cdot \nabla u \, dA$$  \hspace{1cm} (4.12a)

$$\Phi_3 = \int \int \bar{\lambda}_3 \cdot \nabla u \, dA$$  \hspace{1cm} (4.12b)

where

$$\Phi_2 + \Phi_3 = \Phi_r.$$  \hspace{1cm} (4.13)
Unlike LDA, this scheme will satisfy local positivity constraints on each triangle, thus ensuring global positivity. The necessary time-step constraint within a given triangle is:

$$\Delta t \leq \min\left( \frac{2S_2}{\lambda \cdot \vec{n}_2}, \frac{2S_3}{\lambda \cdot \vec{n}_3} \right).$$

(4.14)

The normals, $\vec{n}_2$ and $\vec{n}_3$, are the inward normals of the sides opposite vertex 2 and 3 respectively, scaled to the length of the side, as shown in Figure 4.4.

![Figure 4.4: Schematic of normals needed in time-step constraint.](image)

However, this distribution does not satisfy property L.P. The only condition placed upon the signals in the case of vanishing cell fluctuation is that they must be equal and opposite, i.e.

$$\Phi_2 = 0 \Rightarrow \Phi_2 = -\Phi_3,$$

(4.15)

but the signals are not necessarily equal to zero.

In order to enforce positivity while maintaining accuracy, by Godunov’s theorem we must utilize nonlinear distribution methods. It was first proposed by Sidilkover [32] that by employing non-linear limiting functions in the N Scheme formulation,
the accuracy could be increased while maintaining the ability to preserve monotonicity. These limiting schemes, although originally developed for a cell-centered finite-volume scheme [31], can be applied in a fluctuation-splitting context on random triangular grids, providing an elegant way of formulating the upwind distribution method.

4.2 Nonlinear Schemes via Limiting Functions

In this approach, the signals calculated from the N Scheme are modified by subtracting from one signal and adding to the other a non-linear function of the two signals. The resulting new signals are of the form:

\[
\{\Phi_2\}_{\tilde{a}m} = \Phi_2 - \Psi(\Phi_2, -\Phi_3) \quad (4.16a)
\]

\[
\{\Phi_3\}_{\tilde{a}m} = \Phi_3 + \Psi(\Phi_2, -\Phi_3). \quad (4.16b)
\]

The sum of the new signals is still the cell fluctuation and therefore the conservation property of the distribution scheme is preserved. The desired conditions placed on the function \(\Psi(a, b)\) are:

- **Linearity Preservation (LP):** producing vanishing signals in the case of vanishing cell fluctuation:

  \[
  \Psi(a, a) = a. \quad (4.17)
  \]

- **Symmetry:** in two dimensions neither edge should be favored over the other, and therefore the resulting signals should be independent of which edge is chosen first, giving the condition:

  \[
  \Psi(a, b) = \Psi(b, a). \quad (4.18)
  \]
Some well-known limiters which meet the above constraints are MinMod, Harmonic and Superbee.

\[ \text{MinMod : } \Psi(a,b) = \text{sgn}(a) \max \left[ 0, \min (|a|, \text{sgn}(a)b) \right] \]
\[ \text{Harmonic : } \Psi(a,b) = \frac{1}{2} \left( 1 + \text{sgn}(ab) \right) \frac{2ab}{a + b} \quad (4.19) \]
\[ \text{Superbee : } \Psi(a,b) = \frac{1}{2} \left( 1 + \text{sgn}(ab) \right) b \max \left[ \min \left( \frac{2a}{b}, 1 \right), \min \left( \frac{a}{b}, 2 \right) \right]. \]

All of the resulting limiting schemes are linearity preserving and therefore will produce nearly second-order schemes. The other desired property for an upwind scheme, positivity, can also be preserved. However, only the minmod limiter meets local positivity constraints, which are preserved for the same time-step restriction as that given for the unlimited N Scheme. While local positivity is not guaranteed on all triangulations for the other harmonic limiter and the compressive superbee limiter, it is shown by Sidilkover and Roe that local positivity can be achieved for sufficiently regular grids [32]. Specifically, if a type II triangle has type I triangles as neighbors across each of its inflow sides, then a bound can be placed on the degree of compression, \([|\Psi|/\min(a,b)]\), allowed to the limiter. This bound usually exceeds one. Also, the timestep constraint used for stabilizing the N scheme, coupled with the compressive limiters, is the same as that given for the unlimited N Scheme.

It is easily seen that the N Scheme, limited with minmod, is equivalent to the Positive Streamwise Invariant (PSI) Scheme developed by Struijs et. al. [33], again unifying the finite-volume work of Sidilkover with the original fluctuation-splitting methods. We have chosen to present the schemes in the limiter format because of the elegant formulation, providing numerous distribution schemes within one simple framework.
4.3 Numerical Test Cases

In order to compare the upwinding schemes discussed previously including:

- Linear LDA Scheme,
- Linear N Scheme,
- N Scheme with various limiting functions: minmod, harmonic and superbbee,

three types of test cases are studied. First, the advection of a smooth profile is used to determine the accuracy of the schemes. Next, the ability of the various methods to capture discontinuities, including both shear layers and shocks, is discussed.

4.3.1 Accuracy Study

The first test case consists of a circular convection problem where a given smooth inlet profile is convected about the origin with $\bar{x} = \{y, -x\}$. In Figure 4.5 the domain is shown with the imposed boundary conditions given by:

- The inflow profile of $u_{\text{inlet}} = e^{2x} \sin^2(\pi x)$ on $y = 0$.
- $u = 0$ on $x = -1$.

The other two boundaries, which have outflow, need no extra condition and their solution is determined solely by the interior scheme. In order to determine numerically the order of accuracy of the various upwinding schemes, the error of the solution is computed on several different grids. For this study the error of the scheme is determined by the $L_2$ norm defined by the integral of the error over the computational domain:

$$
||error||_2 = \sqrt{\int \int (u - u_{\text{exact}})^2} \, dA. \quad (4.20)
$$
This integration is numerically computed by assuming piecewise-constant data over the median dual mesh. We choose piecewise-constant data because the integral norm reduces to an area-weighted point norm, providing equivalent error values to those found for linear elements but at a lesser cost.

\[ u = e^{2x} \sin(\pi x) \]

Figure 4.5: Test case of circular convection of inlet profile.

The solutions were computed on isotropic grids which consist of a structured square grid broken into triangles along alternating left and right diagonals, e.g. see the $11 \times 11$ grid shown in Figure 4.6. Test cases were performed on $N \times N$ grids for $N = 11, 15, 21, 31, 41, 61$. The log(error) plotted vs. the log(h), where h denotes the grid spacing, is shown in Figure 4.7. To determine the order of accuracy of the schemes, $m$, defined by \( error = Ch^m \), a least-squares fit of a straight line is made through the data points for the four largest grids. The calculated values are given in Table 4.1.
Figure 4.6: $11 \times 11$ isotropic grid for circular convection test case.

Figure 4.7: Log-log plot of $L_2$ norm of Error vs. grid spacing for circular convection test case on isotropic grid.
### Scheme | Order of accuracy
--- | ---
Low Diffusion | 1.98
Nscheme | 0.91
Nscheme (minmod) | 1.71
Nscheme (harmonic) | 1.97
Nscheme (superbee) | 1.88

**Table 4.1**: Order of accuracy for circular convection of smooth profile on isotropic grid.

![Grid Plot](image)

**Figure 4.8**: $11 \times 11$ right-running-diagonal grid for circular convection test case.

The importance of property LP is clear from the low accuracy demonstrated by the unlimited N Scheme. When the limiting is included the schemes gain nearly an order of accuracy.

To see the sensitivity of the accuracy of the schemes to the grid, we now perform this same analysis on a grid consisting of quadrilaterals broken into triangles with right-running diagonals, (see the $11 \times 11$ grid shown in Figure 4.8). Also, to determine the sensitivity of the accuracy for irregular grids, the regular grid is perturbed by
allowing each interior point to move a given fraction of the regular grid spacing in a random direction. Results from grids perturbed by 0%, 10%, 20%, and 30%, are given in Table 4.2. The $11 \times 11$, 30% perturbed grid is shown in Figure 4.9. The solutions using LDA and the limited N Scheme with both the minmod and harmonic limiters, are computed for grids of size $N = 11, 21, 31, 41$.

The unperturbed grid shows order-of-accuracy values very close to those given for the isotropic grid. For the LDA and N Scheme limited with minmod, the error norms for grids constructed with right-running diagonals are compared with the

<table>
<thead>
<tr>
<th>Distribution Scheme</th>
<th>Order of accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>unperturbed</td>
</tr>
<tr>
<td>Low Diffusion</td>
<td>2.14</td>
</tr>
<tr>
<td>N scheme (minmod)</td>
<td>1.67</td>
</tr>
<tr>
<td>N scheme (harmonic)</td>
<td>1.85</td>
</tr>
</tbody>
</table>

Table 4.2: Order of accuracy for circular convection of smooth profile on perturbed right-running-diagonal grids.
error norms from the isotropic grids and are shown in Figure 4.10. Clearly the right-running-diagonal grids provide more accurate solutions which is understandable as less dissipation is introduced for diagonals aligned with the direction of advection. Also, it is noted from Table 4.2 that the order of accuracy of the schemes does not degrade much with growing perturbation, still providing nearly second-order solutions even on fairly random grids.

4.3.2 Discontinuity Capturing

Another important feature of the upwind methods is their ability to sharply capture discontinuities. Two types of discontinuities exist: shear layers, where the characteristics lie along the discontinuous surface; and shocks, where the characteristics come into the discontinuity from both sides. Both types are studied here to
compare the various schemes.

The first test case, a shear case, consists of a discontinuity advected at 26.6°. The left state is given by \( u = 1.0 \), the right state by \( u = 2.0 \). The case was analyzed on a nearly equilateral \( 31 \times 31 \) grid, shown in Figure 4.11, produced with an advancing front Delaunay grid generation method developed by J.-D. Müller [15].

![Figure 4.11: 31 x 31 nearly equilateral grid for the discontinuity test cases.](image)

The solutions obtained with the various upwind schemes are shown in Figure 4.12, with a cross-cut at \( x = 0 \) shown in Figure 4.13. It is clear that the harmonic and compressive superbee limiters are very effective in hindering the inherent numerical spreading of a contact surface, producing solutions superior to the LDA scheme as well. In the test case shown, most of the grid does meet the regularity requirement for positivity found by Sidilkover [32]. Due to this, the overshoots for both limiters are very small. Unfortunately these overshoots cannot be predicted and therefore in true applications enforcement of positivity will be necessary. However, considering the increase in sharpness of the shear layer, the benefit of utilizing a weaker of even
Figure 4.12: Contour plots for 26.6° shear flow case (28 contours with $u_{\text{min}} = 0.98$, $\Delta u = .04$)
compressive limiter may outweigh the added cost of ensuring grid regularity within the vicinity of the discontinuity.

![Graph showing convergence histories of shear flow cases.](image)

**Figure 4.13:** Cross-cut at x=0 for 26.6° shear flow cases.

The convergence histories of the shear flow cases are shown in Figure 4.14. The test cases were initialized with $u = 0$ everywhere except on the two inflow boundaries where the discontinuity is defined. All of the methods converge quickly. The N Scheme limited with minmod is the quickest, taking less than 100 iterations to reach machine zero. The slowest scheme is the superbee limiter, for which the residual hangs up at a slightly higher level than for the other limiters. The probable cause of this is that superbee, unlike the harmonic limiter, is not a smooth limiter. This is known to be a source of limit cycles and general convergence slow-down.

The last test case consists of a shock flow. This is based on a scalar nonlinear
advection equation related to Burgers' Equation given by

\[ u_t + uu_x + u_y = 0, \quad (4.21) \]

which supports steady discontinuities in \( u \). The specific case analyzed is referred to as the "Tree Burgers Case" which is defined by the following inflow boundary conditions:

- \( u = 1.5 \) on \( x = -1 \)
- \( u = -.5 \) on \( x = 0 \)
- \( u = 1.5 - 2(x + 1) \) on \( y = 0 \).

The boundary \( y = 1 \) is an outflow boundary where the solution is again determined by the interior scheme. Results of the LDA scheme and the N Scheme limited with
minmod, harmonic and superbee, performed on the same grid as in the shear flow case, are shown in Figure 4.15. For this type of discontinuity, all of the schemes appear to perform equally well, capturing the shock over 2-3 cells without spreading. It therefore appears for these discontinuities, similar to those determined by acoustics in the Euler system, where the characteristics run into the shock hindering spreading, the locally positive minmod scheme is sufficient for obtaining quality solutions.

Figure 4.15: Contour plots for Burgers test case (*16 contours with u_{min} = -0.8, \Delta u = 0.18.*
CHAPTER V

CELL VERTEX DISCRETIZATIONS FOR ELLIPTIC-BASED OPERATORS

In Chapter III, the steady subsonic Euler Equations were reduced to hyperbolic and elliptic components. The hyperbolic portion of the system is governed by advection operators, whose discretization and time marching strategies were discussed in the previous chapter; the elliptic portion of the system was shown to be equivalent to the Cauchy-Riemann Equations. With the use of preconditioning matrices, this reduced system was augmented with time terms producing the following equivalent system

\[
\begin{align*}
&\begin{cases}
  u \\
  v
\end{cases}_t + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{cases}
  u \\
  v
\end{cases}_x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{cases}
  u \\
  v
\end{cases}_y = 0. \\
\end{align*}
\]  

This elliptic-based system represents the distribution of information via the subsonic acoustical properties of the flow. An appropriate discretization technique for this system is needed in order to build a truly two-dimensional inviscid flow solver.

Because this system is elliptic in the steady state, central discretizations are appropriate. There exists no preferential direction upon which to base a bias in the weighting functions. In the current work, a forward Euler integration scheme is chosen, for simplicity in combining the marching to steady state of both the hyperbolic and elliptic-based components. A viable choice then for approximating the system
is an unstructured cell-vertex implementation of a Lax-Wendroff scheme in the same style as that proposed by Ni in 1982 [17] and later developed by Hall [10] and Morton et. al.[13]. Employing these schemes on triangular grids poses problems different to those encountered by the above authors on quadrilateral grids. Rather than posing, as they did, the under-determined problem of driving cell fluctuations to zero, giving rise to various spurious mode patterns, we only enforce fluctuation balance at vertices. In order to predict the accuracy and convergence rates for the acoustic Euler system we first study this family of Lax-Wendroff schemes applied to the linear Cauchy-Riemann model system.

The standard Lax-Wendroff scheme has the equivalent equation:

$$U_t + A U_x + B U_y = \frac{\tau}{2} \left( A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} \right) (AU_x + BU_y),$$

(5.2)

where $\tau = \Delta t$ produces the time-accurate version, and we note that the right-hand side, which represents the artificial dissipation, should vanish in the steady state. Assuming linearity and expanding the dissipation term, we get

$$U_t + A U_x + B U_y = \frac{\tau}{2} \left( A^2 U_{xx} + (AB + BA)U_{xy} + B^2 U_{yy} \right).$$

(5.3)

For the Cauchy Riemann system,

$$AB = -BA,$$

and $A^2 = B^2 = I$, resulting in:

$$u_t + u_x + v_y = \frac{\tau}{2} \Delta u$$

$$v_t - v_x + u_y = \frac{\tau}{2} \Delta v.$$

The dissipation terms are therefore the Laplacian operators acting on $u$ and $v$. 
The method is easily implemented in a fluctuation splitting format thus maintaining the favorable compactness of the scheme. Given the fluctuation in a cell,

$$\Phi^r = \int \int (\mathbf{A}u_x + \mathbf{B}u_y) dx dy,$$  \hspace{1cm} (5.4)

it is distributed to the three vertices of the triangle. We define the proportion or signal received by each vertex by matrix weights, $\mathcal{A}_i$, and thus,

$$\Phi^r_i = \mathcal{A}_i \Phi^r, \quad i = 1..3,$$  \hspace{1cm} (5.5)

is the signal sent to node $i$. There are two terms,

$$-u = (\mathbf{A}u_x + \mathbf{B}u_y) + \frac{r}{2} \left( \mathbf{A} \frac{\partial}{\partial x} + \mathbf{B} \frac{\partial}{\partial y} \right) (\mathbf{A}u_x + \mathbf{B}u_y)$$

in the equivalent equation. For the first term a central distribution sends equal proportions to the three vertices, each receiving one third of the total residual. The dissipation or second-order term in the Lax-Wendroff scheme is discretized in the fluctuation-splitting framework based on a Galerkin Finite-Element method\[38\], resulting in total weights given by:

$$\mathcal{A}_i = \frac{1}{3} + \frac{r}{2S_r} (\mathbf{A}, \mathbf{B}) \cdot \frac{1}{2} \vec{n}_i,$$  \hspace{1cm} (5.6)

where $\vec{n}_i$ is the inward scaled normal to the side opposite the vertex $i$. Conservation of the method is ensured since,

$$\sum_{i=1}^{3} \vec{n}_i = \vec{0},$$

therefore

$$\sum_{i=1}^{3} \mathcal{A}_i = 1.$$

An analysis of a one-dimensional Lax-Wendroff scheme to approximate the full transonic Euler equations was performed by Morton/Rudgyard/Shaw [14]. Although
this one-dimensional analysis displayed the desirability of optimizing the convective properties of the scheme to eliminate errors through boundaries, the extension of this analysis to higher dimensions is not so straightforward. Here we shall discuss issues raised by a two-dimensional analysis of the model system, adopting the notation of Morton et. al.

When time-accuracy is not a concern, there exist two distinct time terms in the Lax-Wendroff scheme; a nodal time-step, $\Delta t$, which drives the overall evolution of the scheme; and a cell time scale, $\tau$, which determines the amount of dissipation. These time scales are given by:

$$\Delta t = \nu_n \frac{h}{\lambda}, \quad \tau = \nu_c \frac{h}{\lambda},$$

with $\nu_n$ and $\nu_c$ representing the nodal and cell CFL numbers respectively. The local length scale is denoted by $h$, with $\lambda$ representing the maximum local wave speed. For Cauchy-Riemann augmented as in Equation (5.1) the local wave speed is unity in all directions.

For simplicity we analyze a uniformly spaced, structured, rectangular grid, broken into triangular elements along right-running-diagonals. A section of the grid, shown in Figure 5.1, provides a stencil of 6 neighbors about all interior points.

Applying the Lax-Wendroff method to the model system, an update for point $U_0$ is given by:

$$\Delta U_0 = \sum_{i=0}^{6} M_i U_i,$$  \hspace{1cm} (5.7)

where $M_i$ is the matrix weight placed on the state at node $i$. For the stencil shown in Figure 5.1 the resulting weighting matrices $M$ are given by:

$$M_0 = \begin{bmatrix} -2\nu_n\nu_c & 0 \\ 0 & -2\nu_n\nu_c \end{bmatrix}$$  \hspace{1cm} (5.8a)
The terms proportional to $\nu_c$ contribute a simple five-point smoothing, the others have an advective effect. Utilizing this stencil, we can study issues concerning both accuracy and convergence in order to predict viable choices for the two parameters, $\nu_n$ and $\nu_c$, existing in the update scheme.

### 5.1 Accuracy

Performing a Taylor series expansion about $U_0$ shows second-order accuracy is maintained in the steady state for all choices of parameters. It is interesting to note
that, if we analyze the discretization in the limit of infinite dissipation, i.e., \( \nu_c \to \infty \), the Cauchy-Riemann system is given by:

\[
\begin{align*}
  u_t &= \omega \Delta u \\
  v_t &= \omega \Delta v,
\end{align*}
\]

where \( \omega \) is proportional to the product \( \nu_n \nu_c \). This product remains finite, under the stability constraint of \( \nu_n \nu_c < \frac{1}{2} \), derived in the following section. This system, now parabolic in time, shows the limit of the Lax-Wendroff discretization as a parabolic iteration. This system is easily discretized with a Galerkin FEM for second-order spatial accuracy and is compared in convergence properties with the standard Lax-Wendroff implementation, \( \nu_c = O(1) \), in the next section. However, because the analysis was performed on a regular grid, it was found that when the method was utilized on irregular grids, accuracy was lost with increasing dissipation, i.e., increasing \( \nu_c \). It is unclear to the author how to implement the method to preserve the accuracy on irregular grids for large damping coefficients. Therefore, we are limited to choosing \( \nu_c = O(1) \) for irregular grids, where it appears second-order accuracy is preserved even when utilized for Euler problems, (see Results in Chapter VII).

Other issues of importance for Lax-Wendroff methods include the use of extra non-linear damping mechanisms to ensure damping in all directions as well as the preservation of monotonicity constraints. Because we apply this method only to the elliptic-based subsystem, which has no real characteristics in the steady state, i.e., there exists no real direction such that

\[
(\mathbf{A} \frac{\partial}{\partial x} + \mathbf{B} \frac{\partial}{\partial y}) = 0,
\]

damping of errors will occur in all directions. This reduces the need for additional artificial damping necessary when Lax-Wendroff is used to approximate steady sys-
tems with a hyperbolic component. However, because the elliptic-based system is utilized on the subcritical side of discontinuities, we must have enough dissipation present to prevent post-shock oscillations.

In one dimension, the Lax-Wendroff method is monotone for \( \nu_c \geq 1 \), and thus we search for an equivalent criteria in two dimensions. The standard criterion for monotonicity of systems is given by the condition that all coefficient matrices, \( M \), have positive eigenvalues. In analyzing the coefficient matrices given in Equation (5.8a)-(5.8d), the corner point weights, \( M_2 \) and \( M_5 \), have eigenvalues,

\[
\lambda_{M_2, M_5} = \pm \frac{\nu_n}{6\sqrt{2}}
\]

which are clearly negative for all parameters excluding the limit of \( \nu_c \to \infty \). In this limit the nodal CFL must be zero, due to the stability constraint shown in the following section. This gives zero eigenvalues for the matrix weights at the corner points and also produces positive eigenvalues for all of the other matrix weights provided that \( \nu_n \nu_c \leq 1/2 \). Therefore, the only extension of the one-dimensional monotonicity constraint to higher dimensions, is the limit of a parabolic iteration of the second-order system. Therefore, if other values for the Courant numbers are chosen, it does appear that additional artificial dissipation would be necessary. It is needed to provide sufficient damping for complete elimination of the oscillations about discontinuities resulting from the non-positive property of the scheme. This is not addressed in this work.

5.2 Convergence

Continuing with the same sample grid shown in Figure 5.1, we now address the issues of stability and convergence rate, including error elimination via convection and damping. To determine the domain of parameters, \( \nu_n \) and \( \nu_c \), for stability, and
also to predict damping capabilities of the method, a Von Neumann stability analysis is performed, assuming periodic boundary conditions. The amplification matrix, $G$, where

$$U^{n+1} = GU^n,$$  \hspace{1cm} (5.10)

describes the evolution of the discrete approximation from time level $n$ to time level $n + 1$. The spectral radius of the amplification matrix, for the grid considered, is determined to be:

$$\rho^2 (G) = [1 - \nu_n \nu_c (2 - \cos k_x h - \cos k_y h)]^2 + \frac{\nu_n^2}{9} [\sin k_x h - 2 \sin k_y h - \sin (k_x h + k_y h)]^2 + \frac{\nu_n^2}{9} [\sin k_y h - 2 \sin k_x h - \sin (k_x h + k_y h)]^2.$$  

Requiring the spectral radius to be less than one over the entire frequency domain defines the stability region. First analyzing the spectral radius at corner points, i.e. \((0,0), (0,\pi), (\pi,\pi)\) and \((\pi,0)\) frequencies, we determine the following bound:

$$\nu_n \nu_c \leq 1/2.$$  \hspace{1cm} (5.11)

Because the lowest frequencies, \((0,0)\), give $\rho(G) = 1$ for all parameters, we also require that the spectral radius be a decreasing function with increasing frequency near this point to ensure that the stability bound is not exceeded. This provides the other bound of

$$\nu_n \leq \nu_c.$$  \hspace{1cm} (5.12)

In substituting these limits into the spectral radius calculation it is found that all other frequencies remain stable, proving the above bounds are sufficient for ensuring stability.
If we first assume error elimination via damping properties alone, we find for a given \( \nu_c \), the smallest amplification rate, \( \rho(G) = 1 - \frac{1}{2} \pi^2 h^2 (1 - 1/(2 \nu_c^2)) + O(h^3) \), is obtained for \( \nu_n = 1/(2 \nu_c) \). This is found by equalizing the (low,low) and the (high,high) frequencies for a finite grid of size \( h = 1/M \), i.e. requiring

\[
\rho \left( \frac{\pi}{M}, \frac{\pi}{M} \right) = \rho \left( \frac{\pi(M-1)}{M}, \frac{\pi(M-1)}{M} \right).
\]

The optimization of this over possible values of \( \nu_c \) gives the limit \( \nu_c \to \infty \), with \( \nu_n \to 0 \) but \( \nu_c \nu_n = 1/2 \). This is in fact equivalent to a Jacobi iteration scheme with a finite-element discretization of the remaining Laplacians, predicting convergence in \( O(M^2) \) iterations for grids consisting of \( M \times M \) vertices. Therefore, this predicts \( O(M^2) \) iterations till convergence for the Lax-Wendroff scheme, with optimal performance for the limiting Laplacian discretizations, if we consider damping alone.

However, numerical tests confirmed the importance of advecting the errors. Test cases were performed on the structured standard grid described previously. For the Lax-Wendroff scheme, \( \nu_c = \nu_n = \frac{1}{\sqrt{2}} \) was chosen to maximize advection of errors, as it provides the maximum advection rate within the stability bounds. Also, characteristic boundary conditions were easily imposed, as the grid boundaries were aligned with the axes, thus reducing wave reflections. These conditions are found by reducing the system to one dimension in the directions normal to the boundaries:

**x direction:**
\[
\begin{align*}
u_t + u_x &= 0 \quad \Rightarrow \quad \text{prescribe } u \text{ on } x = 0 \\
v_t - v_x &= 0 \quad \Rightarrow \quad \text{prescribe } v \text{ on } x = 1
\end{align*}
\]

**y direction:**
\[
\begin{align*}
u_t + v_y &= 0 \quad \Rightarrow \quad (u + v)_t + (u + v)_y = 0 \quad \Rightarrow \quad \text{prescribe } u + v \text{ on } y = 0 \\
v_t + u_y &= 0 \quad (u - v)_t - (u - v)_y = 0 \quad \Rightarrow \quad \text{prescribe } u - v \text{ on } y = 1
\end{align*}
\]

as shown in Figure 5.2.
Two iterative methods are compared with the Lax-Wendroff scheme for the parabolic system: Point Jacobi and Gauss-Seidel. By utilizing a Gauss-Seidel relaxation the work per iteration is greater than it is for the Point Jacobi or Lax-Wendroff methods. This is because the Gauss-Seidel relaxation sweeps across the grid updating the solution at vertices in the process, and utilizing the new updated values to compute the solutions at the remaining mesh points. Therefore, all triangles about the current vertex must be visited to evaluate the new state at that vertex. To compute the new solution at all mesh points, each of the triangles in the domain will thus be visited three times, to compute the solution at each of its three vertices. Also, because of the way the Gauss-Seidel relaxation must be implemented, the desirable property of the compactness of the algorithm, present for both the Point-Jacobi and Lax-Wendroff methods, is lost. No longer can the numerical scheme be implemented as a single loop over triangles, where each triangle is analyzed independently making the algorithm very efficient for parallel implementation. These issues would need to be addressed when evaluating the true advantage of the speed-up associated with this relaxation.

For the test cases, both the Point Jacobi and Gauss-Seidel methods are applied...
with their corresponding optimal relaxation parameters [11],

\[ \omega_{opt}^{PJ} = 1.0 \]
\[ \omega_{opt}^{GS} = 2 \left( 1 - \frac{\pi}{M} \right), \]

for an \( M \times M \) grid.

The example case analyzed has an exact solution given by:

\[
\begin{align*}
    u &= 4x^3 - 12xy^2, \\
    v &= 4y^3 - 12yx^2.
\end{align*}
\]  \hfill (5.13)

The initial condition consists of random numbers in the range \((-16, 8)\), which includes initial maxima of twice the solution maxima. The boundary conditions for the Lax-Wendroff scheme are the characteristic conditions described previously. For the parabolic systems, both \( u \) and \( v \) are given on all boundaries.

For a \( 21 \times 21 \) grid the residual histories are given in Figure 5.3. Clearly the advection of errors dominates the convergence, as the Lax-Wendroff scheme converges
much quicker than the Jacobi relaxation which was predicted to be superior in the analysis of damping rates only. The fastest convergence is achieved by the optimized Gauss-Seidel relaxation, however, it is repeated that this destroys the compactness of the scheme.

Next, a study to predict convergence rate as a function of the grid size is performed on grids of size $M = 21, 41, 81, 161, 321$. The $\ln(M)$ vs. $\ln(\text{iterations})$ is shown in Figure 5.4. As predicted by theory the Jacobi scheme converges in $O(M^2)$ iterations, while the optimized Gauss-Seidel converges in $O(M)$ iterations. The Lax-

![Figure 5.4: Convergence rates vs. grid size for model Cauchy-Riemann problem.](image)

Wendroff method also shows a convergence of $O(M)$ iterations, emphasizing the importance of the ability to advect errors.

For single-grid calculations, as opposed to multi-grid, the advantage of the standard Lax-Wendroff technique is that fast convergence due to advection is realized, without the necessity of optimizing the relaxation parameter for the Gauss-Seidel method, which is not trivial for arbitrary grids. With non-optimal coefficients, the method can quickly deteriorate to $O(M^2)$ behavior. However, the extension to Euler
for arbitrary grids also poses the non-trivial problem of developing non-reflecting boundary conditions. It is thus concluded that a compromise between damping and advection is needed, given the current status of boundary conditions. Also, the flexibility of the Lax-Wendroff scheme makes it appealing, not to mention the ease of combining it with the hyperbolic iterative schemes for the advective terms existing in the full system.

Another viable option for improving convergence performance is multi-grid acceleration, where damping rates in the high frequency range, i.e., \( k_x h \geq \pi / 2 \) \( \cup \) \( k_y h \geq \pi / 2 \) govern the efficiency. For a given \( \nu_c \), optimizing over this domain in the limit of decreasing mesh size, we find the lowest amplification rate,

\[
\rho(G) = \frac{1 + 9\nu_c^2}{15\nu_c^2 - 1} \quad \text{for} \quad \nu_n = \frac{6\nu_c}{15\nu_c^2 - 1}.
\]

Again the minimization over all \( \nu_c \), giving \( \rho(G) \to .6 \), yields a Jacobi iteration method for the remaining parabolic equations, with \( \nu_c \to \infty \), but \( \nu_n, \nu_c \to .4 \). This again leads to the loss of convection capabilities that proved very useful provided that boundary conditions could be effectively imposed.

Finally, we choose parameters for the Lax-Wendroff scheme that compromise between damping and convection properties, due to the eventual problem of boundary reflections in full Euler calculations. Assuming a finite damping coefficient, e.g. \( \nu_c = 1 \), high frequency optimization gives \( \nu_n = .43 \) and \( \rho(G) = .71 \). For this case the amplification rates over the entire frequency spectrum are shown in Figure 5.5.

For multi-grid considerations where the block containing the (low,low) frequencies is omitted, maximum damping occurs for equalization of spectral radii for the \( \{ \pi, \pi \} \) and \( \{ \pi / 2, \pi \} \) frequencies.

Although \( \rho = .714 \) is higher than values usually sought for elliptic solvers, we
have now recovered the ability to advect errors as well. In comparison, a Gauss-Seidel relaxation optimized for multi-grid enhancement provides a high frequency spectral radius of $\rho = 0.447$ [11]. But again, for the Gauss-Seidel method, the loss of compactness needs to be addressed in terms of overall best efficiency, precluding the benefit of enhancing efficiency via parallel implementation. For a complete study of multi-grid methods for this model system, as well as the extension to Euler, see Müller [16].

To summarize, a standard Lax-Wendroff iterative method for converging the model Cauchy-Riemann system is shown to provide second-order solutions in an efficient manner, by utilizing both damping and advection of errors. Unfortunately, to date no efficient characteristic boundary conditions exist that limit reflections in full Euler calculations on arbitrary domains. With the incorporation of the discretization discussed here into the full Euler system, shown in Chapter III, numerical experiments show that indeed a compromise between advection and damping provides the
best convergence times.

The other issue of importance addressed here is monotonicity. As implemented, the Lax-Wendroff method does not provide the damping necessary to prohibit oscillations on the subsonic side of discontinuities. Therefore, additional non-linear dissipation must be incorporated if this discretization is utilized for the subsonic acoustical system contained in the Euler system. This is left to future studies. However, in the test cases shown in the Results (Chapter III), with only weak shocks present, no noticeable oscillations occur.
CHAPTER VI

NUMERICAL ALGORITHM

The basic concept of reducing the Euler Equations into subproblems, discussed in Chapter III, defined three distinct approaches. The first method of reducing the system utilized discrete sets of simple waves to model the local flow field; the second used fewer simple waves plus steady solution components; the third decomposed the equations into elliptic-based and hyperbolic systems determined from the steady equations. In the preceding chapters, discretizations were proposed for the two different types of resulting operators. However, these reduced system were discussed in terms of a non-conservative formulation of the equations; some details concerning the implementation of these ideas in a conservative numerical code need to be addressed.

An appropriate linearization is needed in order to maintain conservation for the wave modeling schemes as well as the Hyperbolic/Elliptic splitting. There are other issues concerning conservation for each type of equation decomposition which are also discussed. Also, for the Hyperbolic/Elliptic splitting, the current practice of merging the discretizations of the elliptic-based and hyperbolic acoustic systems between the supersonic and subsonic regimes is shown. Other important issues within a discrete scheme include the boundary conditions and the method of calculating local time-step constraints for marching to steady state; these are also addressed in this chapter.
6.1 Conservation

To analyze the discretizations of Euler equations in quasi-linear form:

\[ U_t + AU_x + BU_y = 0 \]  \hspace{1cm} (6.1)

we need an average state at which the Jacobian matrices A and B are to be evaluated, in order to maintain a conservative flux balance. This is of course analogous to the one-dimensional case, treated in Chapter II, with the standard Roe linearization given by the parameter vector,

\[ Z = \sqrt{\rho} \begin{bmatrix} 1 \\ u \\ v \\ H \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}. \]  \hspace{1cm} (6.2)

Considering a one-dimensional transition between two states, \( U_1 \) and \( U_2 \), the conservation property

\[ \Delta F = A \Delta U, \]  \hspace{1cm} (6.3)

is satisfied for \( \bar{A} = A(Z) \), where \( Z = \frac{1}{2}(Z_1 + Z_2) \).

In multidimensional cases, specifically in two dimensions, there is a natural extension of the Roe linearization [4]. For each cell we need the fluctuation given by:

\[ \Phi = \int \int \left( \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dA, \]  \hspace{1cm} (6.4)

with a conservative linearization providing cell average gradients such that:

\[ \bar{F}_x = \bar{A} \bar{U}_x, \quad \bar{G}_y = \bar{B} \bar{U}_y. \]  \hspace{1cm} (6.5)

If we linearize the two-dimensional system with respect to the parameter vector \( Z \),

\[ \int \int (F_x + G_y) dA = \int \int (A_ZZ_x + B_ZZ_y) dA, \]  \hspace{1cm} (6.6)
where $A_Z$ and $B_Z$ are the Jacobians of the flux functions $F$ and $G$ with respect to $Z$, we find that these Jacobians:

\[
A_Z = M_{FZ} = \begin{bmatrix}
z_2 & z_1 & 0 & 0 
\frac{z_1-1}{\gamma} z_4 & \frac{z_2+1}{\gamma} z_2 & -\frac{z_2-1}{\gamma} z_3 & \frac{z_2-1}{\gamma} z_1
0 & z_3 & z_2 & 0 
0 & z_4 & 0 & z_2
\end{bmatrix}
\] (6.7)

and

\[
B_Z = M_{GZ} = \begin{bmatrix}
z_3 & 0 & z_1 & 0 
0 & z_3 & z_2 & 0 
\frac{z_3-1}{\gamma} z_4 & \frac{z_3-1}{\gamma} z_2 & \frac{z_2+1}{\gamma} z_3 & \frac{z_2-1}{\gamma} z_1
0 & 0 & z_4 & z_3
\end{bmatrix}
\] , (6.8)

are linear functions of the parameter vector. If we assume linear variation in $Z$ over mesh elements, defining constant gradients $\hat{Z}_x$ and $\hat{Z}_y$ over the cell, the average gradients of the flux functions, $\hat{F}_x$ and $\hat{G}_y$ are easily calculated:

\[
\hat{F}_x = \iint A_Z dA \hat{Z}_x = \overline{A_Z} \hat{Z}_x = A_Z(\overline{Z}) \hat{Z}_x
\] (6.9)

and

\[
\hat{G}_y = \iint B_Z dA \hat{Z}_y = \overline{B_Z} \hat{Z}_y = B_Z(\overline{Z}) \hat{Z}_y.
\] (6.10)

Therefore, the average state at which $\overline{A_Z}$ and $\overline{B_Z}$ are calculated for conservation is the average state of $Z$ in a cell, given by the assumption of linear variation as:

\[
\overline{Z} = \iint Z dA = \frac{1}{3}(Z^1 + Z^2 + Z^3),
\] (6.11)

where $Z_i$ denotes the parameter vector evaluated at node $i$. From this average state of $Z$ we can also obtain the consistent average state of the primitive variables in each
The corresponding average spatial gradients of the primitive variables are thus calculated by:

$$\hat{\nabla}_x = M_{VZ} \hat{Z}_x, \quad \hat{\nabla}_y = M_{VZ} \hat{Z}_y,$$

(6.13)

where $M_{VZ}$ is the transformation matrix from the parameter vector to primitive variables,

$$M_{VZ} = \frac{\partial V}{\partial \bar{Z}} = \frac{\partial V}{\partial \bar{Z}}(\bar{Z}),$$

(6.14)

which is also calculated at the Roe average state. This gives the following for the gradients of the primitive variables:

$$\hat{\nabla}V = \begin{bmatrix}
\hat{\nabla}_x \rho \\
\hat{\nabla}_x u \\
\hat{\nabla}_x v \\
\hat{\nabla}_x p
\end{bmatrix} = \begin{bmatrix}
2\sqrt{\bar{\rho}}\bar{\nabla}z_1 \\
\left( -\bar{u}\bar{\nabla}z_1 + \bar{\nabla}z_2 \right) / \sqrt{\bar{\rho}} \\
\left( -\bar{v}\bar{\nabla}z_1 + \bar{\nabla}z_3 \right) / \sqrt{\bar{\rho}} \\
\frac{\bar{z}_1}{\gamma} - \bar{u}\bar{\nabla}z_2 - \bar{v}\bar{\nabla}z_3 + \bar{\nabla}z_4
\end{bmatrix}.$$  

(6.15)

This linearization seems to be the simplest way of converting conservative residuals to the forms needed by the decomposition, and it guarantees conservation however large the local gradients are. However, it is to be expected that any linearization in the presence of large gradients will have some inconsistency, as this one does. If we consider an isolated shear flow where pressure and density are constant, in most instances there will be a numerical pressure gradient calculated from the above formulation due to the assumption of linear variation in the velocity and total enthalpy.
fields, rather than the pressure field, (see Appendix A for example). Other alternatives such as assuming linear variation in the primitive state vector as opposed to the Roe parameter vector have been considered, but pose problems in the calculation of the Jacobian matrices, as they are not linear functions of the primitive vector. Therefore, despite the discrepancy of spurious pressure gradients, the Roe linearization does provide the average state that yields a conservative formulation.

This linearization is necessary for all of the schemes discussed in this work. However, other issues concerning conservation which differ between the Wave Model Schemes and the Hyperbolic/Elliptic Decomposition now are also addressed.

### 6.1.1 Wave Modeling Conservation Issues

In the discrete wave models given in Chapter III, the local primitive spatial gradients were modeled as a set of planar waves:

\[
\begin{align*}
\hat{V}_x &= \sum_k \alpha_k^k \cos \theta^k r^k, \\
\hat{V}_y &= \sum_k \alpha_k^k \sin \theta^k r^k,
\end{align*}
\]  

with the corresponding time derivative given by:

\[
V_t = -\sum_k \alpha_k^k \lambda_m^k r^k.
\]  

Because the scheme does not produce vanishing cell fluctuations, but rather provides a method of converging accumulated signals at nodes, the distribution within each cell must be of the conservative fluctuation \( \Phi_U = -U_t S_r \). The linearization provides the appropriate primitive spatial gradients and average state from which the wave strengths, orientations and corresponding eigenvectors are determined. A simple transformation of variables from the primitive to conservative state will then produce the appropriate time gradients of the conserved quantities:

\[
U_t = M_{UV} V_t = -M_{UV} \sum_k \alpha_k^k \lambda_m^k r^k = -\sum_k \alpha_k^k \lambda_m^k M_{UV} r^k,
\]
where $M_{UV}$ is the Jacobian of the conservative state with respect to the primitive state,

$$M_{UV} = \frac{\partial U}{\partial V}. \quad (6.19)$$

Essentially what we recover is the transformation of the eigenvectors $r^k$ into their conservative counterparts

$$r^k_{U} = M_{UV} r^k, \quad (6.20)$$

and the conservative time change for each wave,

$$U^k_t = -\alpha^k \lambda^k_m r^k_{U}. \quad (6.21)$$

This provides a fluctuation, $\Phi^k_{U}$, for each wave,

$$\Phi^k_{U} = -\alpha^k \lambda^k_m r^k_{U} S_{r}. \quad (6.22)$$

which is distributed among the vertices of the triangle via the multidimensional upwinding schemes discussed in Chapter IV.

### 6.1.2 Hyperbolic/Elliptic Splitting: Conservation Issues

The method of decoupling the preconditioned Euler system into hyperbolic and elliptic-based subsystems resulted in a set of “characteristic equations.” Although the analysis and underlying framework of the scheme is based on this characteristic form, conservation is easily enforced in the final numerical algorithm.

Given the preconditioned characteristic form,

$$W_t + (PA)_{W} W_x + (PB)_{W} W_y = 0, \quad (6.23)$$

we decompose the system into elliptic-based and hyperbolic components by splitting $$(PA)_{W}$$ and $$(PB)_{W}$$ into their diagonal and block diagonal components representing
the advective and elliptic-based operators respectively, i.e.

$$(PA)_W = (PA)_W^H + (PA)_W^E, \quad (PB)_W = (PB)_W^H + (PB)_W^E.$$  \hspace{1cm} (6.24)$$

Appropriate discretizations for the different operators are then utilized to distribute the preconditioned fluctuation, \(\{\Phi\}_W = -W_i S_i. \) These distribution matrices, \(A_i^E\) and \(A_i^H\), representing the proportion of the elliptic and hyperbolic fluctuations sent to each of the vertices, determined from the corresponding discretization method, i.e.

\[
\{\Phi\}_W^i = A_i^E \left( (PA)_W^E W_x + (PB)_W^E W_y \right) + A_i^H \left( (PA)_W^H W_x + (PB)_W^H W_y \right),
\]

\hspace{1cm} (6.25)

can be combined into one distribution matrix \(A_i\) representing the proportion of the total fluctuation sent to each vertex such that:

\[
\{\Phi\}_W^i = A_i \{\Phi\}_W, \quad \sum_i A_i = I.
\]

\hspace{1cm} (6.26)

The preconditioned fluctuation, \(\{\Phi\}_W^\tau\) in a triangle \(\tau\) is thus given by:

\[
\{\Phi\}_W^\tau = -\sum_i A_i^\tau W_i S_i^\tau = \sum_i A_i^\tau \left( (PA)_W W_x + (PB)_W W_y \right) S_i^\tau.
\]

\hspace{1cm} (6.27)

In looping over all cells and accumulating signals at nodes, an explicit integration scheme evolves the solution in time by

\[
W_i(t + \Delta t) = W_i(t) - \frac{\Delta t}{S_i} \sum_\tau A_i^\tau \{\Phi\}_W^\tau.
\]

\hspace{1cm} (6.29)

It is clear that the above formulation will produce incorrect jump relations when capturing discontinuities, because converged nodal solutions will balance \(\Delta (PW)\) rather than \(\Delta U\). As before, this discrepancy is a consequence of the fact that, when
steady state is reached, cell fluctuations do not converge to zero, but rather a balance at vertices is obtained.

Therefore, to establish conservation, the non-preconditioned fluctuation in conserved variables needs to be distributed in each cell and balanced at vertices for a conservative steady state. Again considering a single triangle, the preconditioned characteristic fluctuations are transformed into conservation form via:

$$\Phi_u = -U_iS^r = M_{uw}P^{-1}\{P\Phi\}_W,$$

where $M_{uw}$ is the Jacobian of the conserved state $U$ w.r.t. the characteristic state $W$, i.e.

$$M_{uw} = \frac{\partial U}{\partial W}. \quad \text{(6.31)}$$

After accumulating the change to the conservative state at vertices, the preconditioner can be utilized in its original context to speed up convergence. The evolution of the conservative state vector $U$ is given by:

$$U_i(t + \Delta t) = U_i(t) - \frac{\Delta t}{S_i} P_i \sum_{\tau} M_{uw}P_{\tau}^{-1} A_{Li}^{\tau} \{P\Phi\}_{W}^\tau,$$

where $P_i$ is the preconditioner evaluated at the old nodal state $U_i(t)$.

Both conservative and non-conservative formulations of the presented decoupled method have been implemented and tested in numerical experiments. While the non-conservative formulation is very robust in obtaining steady state solutions, the conservative formulation suffers from stability issues within the stagnation region. A paper by Darmofal and Schmid [2] gives a detailed look at stability issues arising from the preconditioned system, based on eigenvector degeneracy in the limit of $M \rightarrow 0$. This degeneracy appears to manifest itself more readily in the conservative scheme.
6.2 Merging of Acoustic Discretization Through the Sonic Line

In Chapter III, the smooth transition through the sonic line, of the analytical acoustic system in the Hyperbolic/Elliptic decomposition, was shown. We now discuss the steps taken to ensure that the discrete representations of the acoustics in the subsonic and supersonic regimes, also smoothly merge in the sonic limit. In the current state of the work, with a Lax-Wendroff solver utilized in subsonic flows and the limited N Scheme utilized for the characteristics in supersonic flow, a slightly ad-hoc fix is needed to merge the two discretizations. Some specifics of this merging are discussed, although it is noted that many other alternatives are possible.

The method utilized for the results shown in this work consists of choosing a region about the sonic line, which we denote as the sonic region, where both the supersonic and subsonic methods are employed simultaneously. We define the sonic region by:

\[ \sqrt{M^2 - 1} = \beta \leq 0.1. \]

The discretization methods proposed in this thesis are upwind methods along the steady characteristic directions in the supersonic region and a Lax-Wendroff method for the subsonic system. However, when we enter the sonic region, from the supersonic side, i.e. \( M > 1 \) such that \( \beta = 0.1 \), the use of the characteristic directions for upwinding is phased out while the Lax-Wendroff method is phased in. Eventually, when the other limit of the sonic-region is reached, i.e. \( M < 1 \) such that again \( \beta = 0.1 \), the use of the characteristic directions has been completely phased out and the Lax-Wendroff method is utilized for the full system. A cubic blending function, \( \psi \), is defined between the two endpoints of the sonic region. This function scales the proportion of the acoustic residual discretized with the Lax-Wendroff method,
and therefore goes from a value of zero at the supersonic limit of the sonic region to one at the subsonic limit. Also, a zero derivative in $\psi$, with respect to $\beta$ at both endpoints is enforced. Therefore, the proportion of the acoustic residual discretized by the upwind scheme is equal to $(1 - \psi)$.

An important issue in the sonic region is that both the supersonic and subsonic discretizations have vanishing dissipation in the streamline direction. This can allow for such non-physical phenomena as entropy-violating expansion shocks. Thus, the streamwise dissipation is frozen. For the advection operators, this is done by freezing the characteristic directions as those defined at the sonic region limit. Therefore, the direction of the characteristics, relative to the streamline, remains at

$$\tan \mu = \pm \frac{1}{\beta_{fr}}, \quad \beta_{fr} = 0.1.$$  

This prevents them from becoming completely perpendicular to the streamline. For the system solver, the streamwise dissipation terms are simply frozen at their values given at the sonic region limit on the subsonic side. This prevents the elliptical domain of influence from completely flattening out, but rather maintains its streamwise range given for $\beta = 0.1$.

6.3 Time-Step Calculation

The evolution of the solution to steady state, given in the previous section, is achieved by marching forward in time with a standard local forward-Euler time integration. Therefore, at each vertex $i$ in the domain, a local maximum time-step is needed. Because the system we are discretizing is hyperbolic in time, the maximum time-step per iteration is limited by the time it takes the fastest wave to traverse one computational cell. We thus define the nodal time-step, $\Delta t^i$ as the minimum
cell time-step $\Delta t^r$ over all cells, $\tau$, for which $i$ is a vertex, i.e.

$$\Delta t^i = CFL \min_{\tau \in \tau} \Delta t^r,$$  \hspace{1cm} (6.32)

where the $CFL$ number is the global Courant number or scaling of the time-step for stability.

In each cell, thus, we need the time it takes the fastest local wave, with speed $\lambda_{max}$, to traverse the cell length $h$,

$$\Delta t^r = \frac{h}{\lambda_{max}},$$  \hspace{1cm} (6.33)

where $h$ is defined as the length of the shortest side of the mesh simplex. In the wave-model methods, this maximum wave speed is given by an acoustic wave moving in the direction of flow,

$$\lambda_{max} = |q| + a.$$  \hspace{1cm} (6.34)

For the Hyperbolic/Elliptic Decomposition, where the time-dependent system is preconditioned, the speeds at which the waves travel are altered in order to optimize the condition number of the system and thus maximize convergence performance. In this work, all wave speeds are scaled to the local flow velocity $q$ and therefore

$$\lambda_{max} = |q|.$$  \hspace{1cm} (6.35)

In stagnation regions, where the flow speed goes to zero, a limit must be placed to bound the time-step away from infinity. We utilize a fraction of the freestream velocity $q_{\infty}$ in order to obtain proper scaling for all non-dimensional forms of the system, providing a modified maximum speed

$$\lambda_{max} = \max \left( |q|, 10^{-3} q_{\infty} \right).$$  \hspace{1cm} (6.36)

This bound on the wave speed proved sufficient for the cases tested and presented in this work. It is noted that the non-conservative formulation of the scheme must
be utilized in flows with stagnation points in order to prevent instability during the initial transients.

6.4 Boundary Conditions

The last area of importance in the actual implementation of the fluctuation splitting methods discussed in this work is that of boundary conditions. Because we must solve problems on finite domains, boundary conditions are necessary in order to define information which is sent from the outside world to the finite domain being modeled. The two basic categories of boundaries discussed and utilized are solid walls and inflow/outflow boundaries.

6.4.1 Solid Walls

We utilize a slip boundary condition along solid surfaces which constrains the flow to be tangent to the wall. This is easily enforced in a strong way for cell vertex schemes since the nodes lie on the solid surface. It simply reduces to forcing flow tangency on the wall at every iteration, by initializing the flow field to this state and not allowing the interior scheme to make changes to the normal component of velocity, i.e., assuming the direction of flow is known at boundary vertices.

To determine the normal to a curved surface at each vertex we used the average of the scaled normals to the two faces of which that vertex is a member, see Figure 6.1. These normals are scaled to the length of their corresponding face. Therefore, the direction of tangency at vertex $i$, $\vec{t}_i$, is given as

$$\vec{t}_i = \vec{x}_{i+1} - \vec{x}_{i-1},$$

with $\vec{n}_i$ defined as perpendicular to $\vec{t}_i$.

Another alternative is to use a weak boundary condition to drive the final solution
Figure 6.1: Determination of normal vector at vertex on a solid wall surface.

To tangency along solid surfaces, by enforcing no flow through faces which lie on a solid surface. This is done by modifying the fluctuation in cells with a face on a wall. Essentially the total fluctuation can be interpreted as a sum of fluxes normal to the three faces of the triangular cell, see Figure 6.2, such that

$$\Phi = F_n^1 + F_n^2 + F_n^3. \quad (6.38)$$

These normal fluxes are calculated under the same assumption as interior cells, i.e., linear variation of the Roe parameter vector. To enforce the boundary condition, the normal flux through the edge on the wall, $F_n^3$, is found by knowing that there is no flow through the edge and therefore the only flux through the face is due to the pressure force at the wall. This alters the fluctuation in the cell, which is in turn distributed to the vertices via the desired scheme, i.e., Wave Model or Hyperbolic/Elliptic Decomposition, in the same manner as for all interior cells.

In numerical experiments the weak condition appeared to suffer from a spurious mode where velocity vectors would not approach tangency, but rather an oscillatory mode where the velocity averaged over each face had zero normal component. Most likely, a more sophisticated approach, which not only modifies the cell fluctuation but also the distribution scheme to account for the solid wall, is needed. In the
Figure 6.2: Fluctuation in triangle due to normal fluxes through faces.

test cases displayed in this thesis, the strong condition was utilized because of its robustness, and effect on convergence and accuracy. It is noted that other solid boundary conditions have been developed, such as a vanishing-ghost-cell approach [18]. However, in a comparison with the strong condition this approach proved to provide equivalent solutions with a slight slow-down in convergence rate.

6.4.2 Inflow/Outflow

At inflow and outflow boundaries, characteristic conditions are applied depending on whether the local flow conditions are supersonic or subsonic. The numerical boundary conditions are determined by those physical quantities given outside of the computational domain, which must be specified at the boundary. The four independent scenarios are discussed.

- **Supersonic Inflow**
All four characteristics run into the domain from the exterior and therefore all variables are specified.

- **Supersonic Outflow**

No incoming characteristics from the exterior domain and therefore no boundary conditions are specified. The interior scheme alone determines the exit conditions.

- **Subsonic Inflow**

In this case, the two hyperbolic characteristics are incoming and therefore the entropy and enthalpy at the inlet must be specified. The elliptic or acoustic portion of the flow represents one incoming and one outgoing characteristic, as discussed for the model Cauchy-Riemann system in Chapter V. As in the model system, assuming a one-dimensional analysis along the incoming streamline, we see that the direction of the flow, $\theta$, is carried in from the exterior and therefore must be specified at the inflow boundary, while the pressure is determined from the interior scheme.

For internal flow problems, specifically the channel flow test case analyzed in the Results (Chapter VII), specifying the flow direction to be constant across the inlet cut does not physically model true conditions. Although this is standard practice it violates the true physics if it is assumed that a straight channel of infinite length is upstream of the inlet cut.

For the NACA 0012 cases also shown in the Results, the flow direction given by the boundary condition is found from a vortex correction applied at the far field.
• Subsonic Outflow

For subsonic outflow conditions the two hyperbolic characteristics come from the interior domain and therefore the interior scheme will determine the exit conditions for enthalpy and entropy. The acoustic or elliptic system will again have one incoming and one outgoing characteristic. However, now the exit pressure is determined from the incoming wave and therefore must be specified by a boundary condition and the direction of flow through the exit plane is determined by the interior scheme.
CHAPTER VII

RESULTS

The fluctuation-splitting schemes based on the various methods of reducing the Euler equations presented in Chapter III are now tested in a variety of flow regimes. In all cases, the hyperbolic advective operators for both the simple waves utilized in the wave models, as well as the hyperbolic characteristic equations found in the Hyperbolic/Elliptic Splitting Method, are discretized using the N Scheme coupled with the minmod limiter. This produces nearly second-order, monotone, solutions as shown for scalar advection operators in Chapter IV. The elliptic-based subsystem contained within the H/E Splitting is solved using the Lax-Wendroff scheme, as discussed in Chapter V, utilizing a cell CFL of 1.2. All cases are initialized with freestream conditions; in the case of the H/E Splitting for flow fields containing a stagnation region, the flow field is initially updated using a non-conservative formulation of the method, in view of the stability issues discussed in Chapter VI.

7.1 Comparative Results

Two test cases are analyzed in order to compare the three main methods of reducing the Euler system of equations to recognizable subproblems: Roe’s Wave Model D, the Steady/Unsteady Splitting and the Hyperbolic/Elliptic Decomposition
method. All cases analyzed were run on unstructured triangular grids generated with a frontal Delaunay method developed by J.-D. Müller [15].

7.1.1 Channel Flow

The first comparison of methods is performed on subsonic flow through a channel with 10% cosine shaped bumps on both the upper and lower walls, the grid consisting of 495 total nodes is shown in Figure 7.1. The inflow Mach number is given as $M_\infty = 0.5$, where the theoretical result is a subsonic, isentropic, symmetric solution about the bump. The boundary conditions are: the upper and lower surfaces are solid walls, and characteristic conditions are applied at the subsonic inlet and outlet.

![Figure 7.1: 10% cosine bump channel consisting of 40 points on the upper and lower surfaces with a total of 495 nodes.](image)

The pressure contours for the three splitting methods are shown in Figure 7.2, with the corresponding entropy contours shown in Figure 7.3. Clearly the six-wave model produces the most dissipative result. The asymmetry in pressure about the bump is the most pronounced and corresponds to the most entropy production. The Steady/Unsteady Splitting is less dissipative as seen in the pressure field and the decrease in entropy production along the walls, however, the profiles are very rough due to the problem with this method that wave directions depend upon small, oscillating residuals. The best result, seen by the near symmetry in the pressure field
as well as the least entropy production is for the Hyperbolic/Elliptic Decomposition Method. This decomposition method produces a very smooth solution field even on this coarse grid, as opposed to the rough solutions produced by the other two methods. It is also the only method to converge completely, as seen in the residual history plot shown in Figure 7.4. The six-wave model converges about one and a half orders before entering some type of limit cycle, although at this point the solution field has stabilized. The Steady/Unsteady Splitting has the most trouble in obtaining a solution, as the residual barely drops one order in magnitude. It does not appear to enter into a limit cycle; the solution field continues to change, and no consistent pattern is ever observed.

7.1.2 Subcritical Flow Over NACA0012

The standard subcritical test case of flow over a NACA 0012 given by a freestream Mach number, \( M_\infty = 0.63 \), at a 2° angle of attack is performed to again compare the three methods from the previous test case. The grid consists of 131 nodes on the body, with a far field located at 30 chords, giving a total of 2133 nodes. The grid is shown in Figure 7.5. This is a fairly coarse grid, relative to the standard grids used in workshops which usually contain at least twice as many points on the body. The boundary conditions imposed are a solid wall tangency condition on the surface of the airfoil; with the freestream values plus a vortex correction imposed at the far field boundary.

The convergence history of the calculated lift coefficient is shown in Figure 7.6. Again, the problem with the Steady/Unsteady Splitting is clearly seen, as the lift coefficient does not converge to a given value, but enters some random oscillatory mode. For the six-wave model, the lift coefficient converges to a constant value,
Figure 7.2: Pressure contours (41 contours, $P_{\text{min}} = .55$, $\Delta P = .005$) in cosine bump channel, $M_{\infty} = 0.5$, (a) Roe Six Wave Model D (b) Steady/Unsteady Splitting and (c) Hyperbolic/Elliptic Decomposition.
Figure 7.3: Entropy contours (51 contours, $s_{\text{min}} = .01$, $\Delta s = .0004$) in cosine bump channel, $M_\infty = 0.5$, (a) Roe Six Wave Model D (b) Steady Unsteady Splitting and (c) Hyperbolic Elliptic Decomposition.
Figure 7.4: Residual History of $L_2$ norm of the $u$ momentum for the cosine bump test cases.

Figure 7.5: Detail of NACA 0012 grid.
albeit much too low. Unlike the channel flow, where the residual was shown to enter a limit cycle, in this case the residual still hangs up after only dropping three orders but enters an oscillatory pattern with smaller amplitude allowing the solution to converge to nearly steady conditions. In Table 7.1 the results of the three decomposition methods are given, including the lift and drag coefficients as well as the maximum spurious entropy production, occurring at the leading edge of the airfoil for all cases. Also included in the table are the accepted values from the AGARD [22] test data. Again, the unnecessary dissipation of the six-wave model is seen by the low lift coefficient of only approximately half of the accepted value, as well as the high drag coefficient 50 times higher than that produced by the Hyperbolic/Elliptic Decomposition solution. Also, the maximum entropy production is nearly 40 times that of the H/E Decomposition, as shown in the plot of the entropy production along the body
for the three methods shown in Figure 7.7. The Steady/Unsteady Splitting, while showing a marked improvement over the six-wave model, still produces nearly three times the entropy of the H/E Decomposition, and of course fails to reach a converged solution. Clearly, the H/E Decomposition produces the best results, showing good agreement with the accepted values. Also included in Table 7.1 are values computed for the H/E Decomposition on a grid with twice the resolution, i.e. 261 body nodes with a total of 11,372. The solution is nearly the same as for the coarse grid, giving the same lift coefficient with a decrease in drag (now zero to four decimal places). Also, a decrease in maximum entropy production is seen.

The Mach contours for the solution obtained with the H/E Splitting on the coarse grid (131 body nodes) are given in Figure 7.8, including a blow-up of the stagnation region in Figure 7.9. Both plots show the smooth solution obtained with this method, and the ability to maintain the quality of the solution even in the stagnation region, where the flow is changing rapidly.

### 7.2 Hyperbolic/Elliptic Splitting

Because of the superiority of the Hyperbolic/Elliptic Decomposition Method over the other methods discussed in this work, various other flows over a NACA 0012 were
Figure 7.7: Spurious entropy production along body for subcritical flow over NACA0012, $M_\infty = 0.63$, $\alpha = 2^\circ$.

Figure 7.8: Mach Contours for flow over NACA 0012 at $M_\infty = 0.63$, $\alpha = 2^\circ$. 
computed with the H/E Decomposition in order to demonstrate the versatility and robustness of the method. Several solutions, including subcritical flow, low speed flow, transonic and supersonic cases, are presented on the same grid shown in the previous section, Figure 7.5.

### 7.2.1 Low-Mach-Number Flow Over NACA 0012

In order to demonstrate the accuracy of the scheme even in the incompressible limit, two low-speed NACA 0012 airfoil cases at zero angle of attack are analyzed. Mach contour plots for a freestream Mach number of .1 and .01 are shown in Figures 7.10 and 7.11 respectively. The solutions appear self-similar, by a scaling factor given by the freestream Mach number, as predicted by incompressible theory. A comparison of the pressure coefficient along the body for both cases with that given by a panel method code [12] is shown in Figure 7.13. Clearly the solutions produced
with the compressible code are nearly equivalent to the solution in the incompressible limit. The numerical dissipation is very low giving coefficients of drag for both cases at 0.0005, with maximum entropy generation on the order of $10^{-5}$. The convergence histories (Figure 7.12), as well as the solutions, are identical, confirming the theoretical analysis of this form of preconditioning.

Figure 7.10: Low-speed flow over NACA 0012 at zero angle of attack, $M_\infty = 0.1$; Mach contours.

For comparison, a solution of the $M_\infty = 0.1$ flow over the airfoil computed with a Lax-Wendroff method used to solve the full Euler system is presented; a Mach contour plot is shown in Figure 7.14. This computation is performed without the use of preconditioning matrices, which are known to preserve the accuracy of the numerical scheme even in low speed flows [40]. It can be seen from the contour plot that the solution computed with the full Lax-Wendroff method is degraded for this nearly incompressible flow field, with excessive numerical dissipation generated about
Figure 7.11: Low-speed flow over NACA 0012 at zero angle of attack, $M_\infty = 0.01$; Mach contours.

Figure 7.12: Normalized Residual History for $M_\infty = 0.1$ and $M_\infty = 0.01$ cases.
Figure 7.13: $C_p$ along airfoil for $M_{\infty} = 0.1$, $M_{\infty} = 0.01$ and the result of a panel method.

the body. Also shown is a plot of the spurious entropy production along the airfoil, Figure 7.15, found in the computations from both the H/E Decomposition and the full Lax-Wendroff method. The Lax-Wendroff solution results in a drag coefficient of 0.069 compared with 0.0005 for the H/E splitting, as well as entropy production at the leading edge of nearly 35 times the magnitude of that produced by the H/E method.

7.2.2 Subsonic Flow Case

The spurious entropy generation along a NACA 0012 for a subcritical case with a freestream Mach number of $M_{\infty} = 0.5$ at no angle of attack is shown in Figure 7.16, for both the H/E Decomposition and for a standard Lax-Wendroff solver. Again, the entropy near the leading edge of the airfoil is greater for the standard Lax-Wendroff scheme, by approximately a factor of 20. The decrease in entropy
Figure 7.14: Low-Speed-Flow over NACA 0012 at zero angle of attack, $M_\infty = 0.1$ computed with a Lax-Wendroff method for the full Euler system: Mach contours.

Figure 7.15: Entropy production along body for $M_\infty = 0.1$ case showing H/E Decomposition compared with basic Lax-Wendroff solver.
produced by separating the system and only utilizing the Lax-Wendroff scheme for the elliptic-based subproblem, is not as pronounced as in the low-speed airfoil case discussed previously, although still significant. The better results produced by the H/E splitting method for the low-speed case are most likely due to the increase in accuracy in these flow regimes from the use of preconditioning matrices, as found in their original development for convergence acceleration. This test case, however, demonstrates that the increase in accuracy found in the low-speed case was not due to preconditioning alone, but to the splitting of the system as well. If it had been, we would expect to find little gain in the present test.

Figure 7.16: Entropy production along NACA 0012 for subsonic flow case with $M_\infty = 0.5$ at $0^\circ$.

7.2.3 Transonic Flow Over NACA 0012

A standard transonic case of $M_\infty = 0.85$ at a $1^\circ$ angle of attack was also computed, Mach contours are shown in Figure 7.17, and, pressure coefficient along the body is
shown in Figure 7.18. The lift and drag coefficients computed as 0.384 and 0.059, are plotted versus solutions found in a GAMM workshop [7], shown in Figure 7.19. The dotted circle represents what were considered good solutions from the workshop. Considering the coarseness of our grid, with approximately half the number of body nodes, our solution falls very near the spectrum of accepted good solutions. In the solution plots, no noticeable oscillations occur on the subsonic sides of the shocks, even though monotonicity is not guaranteed for the Lax-Wendroff method at the chosen Courant numbers.

Figure 7.17: Transonic flow over NACA 0012, $M_\infty = 0.85, \alpha = 1^\circ$; Mach contours.

### 7.2.4 Supersonic NACA 0012

A standard supersonic case of $M_\infty = 1.2$ at a $0^\circ$ angle of attack was also computed. Mach contours are shown in Figure 7.20 and the Mach number upstream of shock and along the body are shown in Figure 7.21. Again, the solution displays the
Figure 7.18: Pressure coefficient along body for flow over transonic NACA 0012, $M_\infty = 0.85$, $\alpha = 1^\circ$.

Figure 7.19: $C_l$ vs. $C_d$ for solutions from GAMM workshop for transonic NACA 0012, $M_\infty = 0.85$, $\alpha = 1^\circ$. The dashed circle indicates the region of acceptably accurate solutions.
ability of the method to capture discontinuities sharply, considering the coarseness of the grid. The stagnation region is clearly defined, with an expansion occurring along the surface of the airfoil. The sonic expansion is smoothly produced with the numerical scheme, demonstrating the smoothness of the merging of the subsonic and supersonic discretizations of the acoustical system.

Figure 7.20: Supersonic NACA 0012, $M_\infty = 1.2$, $\alpha = 0^\circ$ Mach contours.

### 7.2.5 Low Speed Flow over NACA 0012 at High Angle of Attack

Low speed flow over a NACA 0012 with $M_\infty = 0.05$ at a $25^\circ$ angle of attack is computed to demonstrate the method for a case in which an isolated compressible region ($M_{\text{max}} = 0.29$) is contained in a nearly incompressible flow field. The pressure contours (Figure 7.22) display the isolated compressible region in the expansion around the leading edge. The computation gives a $C_d = 0.038$ and $C_l = 2.8$ (for comparison $2\pi \alpha = 2.74$, and a panel method code produced $C_l = 2.87$.) The Mach
Figure 7.21: Mach number upstream and along body for supersonic flow over NACA 0012, $M_{\infty} = 1.2$, $\alpha = 0^\circ$.

Mach number and entropy generation along the body (Figures 7.23 and 7.24) show the ability of the method to capture the expansion with low numerical error considering the coarseness of the grid, giving a maximum entropy growth on the order of $10^{-4}$. Also shown is a comparison of the pressure coefficient along the airfoil for the H/E splitting with a panel method solution [12] in Figure 7.25. This shows the ability of the compressible code to accurately predict a mostly incompressible flow.
Figure 7.22: NACA 0012, $M_\infty = 0.05$, $\alpha = 25^\circ$; pressure contours and some streamlines.

Figure 7.23: NACA 0012, $M_\infty = 0.05$, $\alpha = 25^\circ$ Mach number along body.
Figure 7.24: NACA 0012, $M_\infty = 0.05$, $\alpha = 25^\circ$; entropy generation on body.

Figure 7.25: NACA 0012, $M_\infty = 0.05$, $\alpha = 25^\circ$; $C_p$ along body for H/E Splitting compared with a panel method solution.
CHAPTER VIII

CONCLUSIONS

We have compared three different approaches in the development of a truly multi-dimensional algorithm for the Euler equations. The wave-modeling ideas provided a reduction of the system to a set of planar waves governed by scalar advection operators. This ideology dominated the research effort for many years, mainly because it was a direct extension of the successful splitting method in one dimension. While numerous models were developed and tested, none met with universal success. Contrary to this philosophy, a radically new approach in the Hyperbolic/Elliptic splitting was recently advocated, following original ideas of Ta’asan [35]. We presented a decomposition of the linearized steady system into hyperbolic and elliptic operators. With the use of preconditioning matrices, this decoupling of the steady system was preserved even in the iterative scheme. In supersonic regimes, the decomposition resulted in a set of four decoupled purely hyperbolic or advective operators, representing the well-known characteristics of the steady equations. This provides four clearly defined directions along which upwind-biased discretizations can be made, without the need for choosing sets of waves as in the wave model schemes. In subsonic flows, two advective operators remain, plus an elliptic-based coupled system, which is isolated from the hyperbolic operators in the linear form. Again, the hyper-
bolic operators provide distinct directions to bias the discretizations along. However, the elliptic-based system represents isotropic spreading of information and therefore we propose central discretizations. Most importantly, the need to choose sets of waves to model the local flow, of the infinitely many that exist, has been eliminated.

The decoupled system presented in this work provides an alternative to the form proposed by Ta'asan [35]. He proposes a canonical form with direct coupling between the elliptic and hyperbolic operators even in the linearized system. Our system, when applied to non-linear problems, will only have coupling due to the solution-dependent coefficients. It is unclear yet whether this indirect coupling, as opposed to the direct coupling of Ta’asan, poses any advantage regarding accuracy or convergence rates. It seems that in supersonic regimes our complete decoupling should provide the optimal scheme, as it essentially reduces to pure characteristic theory applied in a conservative way. It is this physical implication of the complete decoupling in supersonic flows that leads us to continue to advocate our formulation, with the natural extension into the subsonic regime preserving the decoupling.

In order to solve the system based on the decomposed forms, we first analyzed the resulting reduced operators, both hyperbolic and elliptic. For the advective operators, scalar analysis shown in Chapter IV provided a nearly second-order, monotone, compact spatial discretization based on the N Scheme limited using minmod. This is the chosen method of upwinding for the Euler results shown in this work. While test cases for weaker or even compressive limiters such as harmonic or superbee showed a slight increase in spatial accuracy, their main advantage is in their ability to minimize the spreading of shear layers. Unfortunately, a price must be paid for the improved discontinuity capturing in that local positivity constraints can be met, only on sufficiently regular grids. Therefore, if a compressive limiter is desired for the capturing of
contact surfaces, the grid regularity conditions must be met within the vicinity of the discontinuity. It is not yet clear whether the added benefit of sharper discontinuities will outweigh this added cost.

While the discretization of the hyperbolic operators seems clear the optimal scheme for the elliptic-based system has not yet been found. For this system, where there is no preferential direction of propagation, we desire a central discretization, such that errors are effectively damped in all directions. By adding hyperbolic time terms to the spatially elliptic system, as is done in this work, producing the Cauchy-Riemann equations augmented with time terms shown in Chapter V, this coupled system is easily solved using the same basic framework as for the advective operators. Because of the flexibility and compactness of utilizing a Lax-Wendroff method for time marching, it is employed for the Euler results shown in this work. The analysis of the model system showed this does provide a second-order scheme, however, contrary to one-dimensional analysis, in two dimensions there is no choice of finite coefficients that guarantees oscillation-free solutions, and therefore non-linear limiting will eventually need to be incorporated if this method is to be utilized for problems with stronger discontinuities.

Besides accuracy, another area of importance concerning the elliptic-based system is the rate at which solutions converge. The scheme allows for the expulsion of errors via damping and advection through boundaries. The two parameters, nodal and cell Courant numbers, can be adjusted to favor one over the other. Although the analysis of the model Cauchy-Riemann system provides no definitive answers in choosing the variable coefficients, with current problems of reflections at boundaries it seems clear that a compromise between damping and advection is needed.

Following the success of the chosen discretizations for the model problems repre-
senting the reduced systems of the various Euler decompositions, these were utilized in the full Euler code to evaluate the accuracy of the decompositions. The original wave-modeling ideas, consisting of six waves representing all local flow disturbances, in space as well as time, had too much dissipation, producing results similar to first-order dimension-split solvers. This loss of accuracy, attributed to the presence of unsteady waves even in the absence of time residuals, was alleviated by allowing steady patterns to account for part of the spatial gradients present within a local simplex. The Steady/Unsteady splitting did greatly improve the accuracy of the method, providing solutions with very low entropy generation along bodies. However, probably due to the erratic dependence of wave orientations on small cell residuals, convergence proved impossible to achieve in all test cases analyzed. As shown in the results, for flows over a NACA 0012 airfoil, even the pressure coefficient failed to stabilize.

While little success was realized with the wave modeling approach, the Hyperbolic/Elliptic decoupling produced accurate solutions with numerical dissipation lower than that obtained with the wave models. Clearly the elimination of advection operators to govern acoustic wave propagation in subsonic regimes where the steady system is elliptic, is the advantage over previous multi-dimensional models.

In Chapter VII, results from the Hyperbolic/Elliptic decomposition, for a range of flow conditions about a NACA 0012 airfoil, were shown. For the standard AGARD subcritical case, the computed solution provided lift and drag coefficients in good agreement with the accepted values from a potential code. Also shown are the results of a transonic flow case from a GAMM Workshop [7]. Although our solution was computed on a grid with half, and in some case one fourth, the refinement of those contained in the workshop, results close to the range of accepted values were
achieved.

In order to demonstrate the ability of the compressible code to remain accurate even in low-speed flow cases, several computations were presented. Each provided solutions with very low spurious entropy production and resulted in small drag coefficients. These solutions compared very well with those given by an incompressible panel method, showing that the correct incompressible limit is found. Also, one of the low-speed solutions of the preconditioned Hyperbolic/Elliptic decomposition was compared with that produced by a standard Lax-Wendroff scheme used for the full Euler equations. For this case, with \( M_\infty = 0.1 \), the excessive dissipation that manifests itself within standard compressible codes in low speed cases, is seen in the Lax-Wendroff solution. The decomposition method produced a drag coefficient less than one percent of that given by the Lax-Wendroff solution.

While some of the improvement in accuracy of the decomposition method over a Lax-Wendroff solver is due to preconditioning, some is also due to the splitting of the equations into hyperbolic and elliptic-based components. To display this, a compressible flow over a NACA 0012 airfoil was calculated with both methods. Again the decomposition method resulted in a more accurate solution, where the full Lax-Wendroff method produced twenty times more spurious entropy along the airfoil.

While the accuracy of our method for subcritical flows appears to compare well with results shown by Ta’asan, the convergence rates utilizing a multi-grid method implemented by Müller [16] are well below the multi-grid performance achieved by Ta’asan [36]. In our decomposition, complete decoupling of the linear system, even in the iterative scheme, would allow the individual operators, both elliptic and hyperbolic, to converge independently, providing convergence rates predicted in the
analysis of the model systems. However, for conservative Euler calculations coupling
does occur via the coefficients, and studies by Müller [16] show that convergence is
affected by this, reducing performance. This subject still requires further study. It is
noted that completely different discretization methods are utilized by Ta’asan versus
our method. Before a clear judgment can be made between the different canonical
forms, each must be implemented utilizing the same discretization and iterative
techniques.

To summarize, although there remain issues of major exploration within the
method presented in this work, preliminary results show great promise, encouraging
a belief that this path of breaking the system of equations into canonical forms is the
key to multi-dimensional flow simulation. Accurate solutions were presented for a
variety of flow situations, including nearly incompressible flow fields, displaying the
versatility of the method. This is a desirable quality especially when considering very
low-speed airfoil problems at high angles of attack. The issue of the loss of robustness
of the conservative implementation, most likely a consequence of the degeneracy
described by Darmofal and Schmid [2], needs to be addressed in continuing work.

8.1 Future Work

It seems that the main problem in the existing method is in solving the elliptic
sub-system found in the steady equations. The current method of converting this
acoustic system into a hyperbolic “pseudo” time-dependent system and employing
a Lax-Wendroff method, does not appear to be the optimal choice. We discuss an
alternate path in the discretization and solving of this system that could be followed
in future work. Also, the extension of the Hyperbolic/Elliptic decomposition to three
dimensions is discussed.
8.1.1 Discretization of the elliptic system

While great strides were taken to isolate the subsonic acoustical system from the hyperbolic characteristics, the iteration on the elliptic-based system, in this thesis, essentially resorts to what we know best: a hyperbolic time system. While we advocate the use of non-biased discretizations, we utilize a Lax-Wendroff method for converging the system creating symmetric stencils but with biased weights. Physically the elliptic system represents the isotropic distribution of information and thus should require symmetric weights providing for symmetric solutions. Also, with the difficulty in developing non-reflective boundary conditions for real flow problems, the damping properties for this system are most likely the key in optimizing convergence.

The damping rates for the Lax-Wendroff method, even when incorporating multi-grid techniques by optimizing over high frequencies, predict convergence rates worse than those achieved for the Cauchy-Riemann system by Brandt [1]. This suggests an alternate approach to the elliptic-based system is necessary to achieve acceptable convergence rates as well as more accurate (or symmetric) solutions. Some preliminary ideas on where the effort should focus are discussed here.

We return to the model elliptic system given by the Cauchy-Riemann equations:

\[
\begin{bmatrix}
\partial_x & \partial_y \\
\partial_y & -\partial_x \\
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
\end{bmatrix} = \mathcal{L}U = 0.
\] (8.1)

In reapplying the operator \(\mathcal{L}\),

\[
\mathcal{L}(\mathcal{L}U) = 0,
\] (8.2)

we find that this \(2 \times 2\) elliptic system decouples into

\[
\begin{bmatrix}
\partial_{xx} + \partial_{yy} & 0 \\
0 & \partial_{xx} + \partial_{yy} \\
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
\end{bmatrix} = 0.
\] (8.3)
If we augment this system with time terms,

$$ U_t + \nabla^2 U = 0, \quad (8.4) $$

we now have a parabolic system in time, as opposed to the original hyperbolic time system. This follows work of Brandt [1] in which optimal convergence is achieved for solving Cauchy-Riemann in a parabolic “time-marching” sense.

These equations can be discretized by a standard Galerkin Finite-Element Method, providing the desired symmetric stencils, as expected for an elliptic system. Either a Jacobi or Gauss-Seidel relaxation, can be used to solve the system, and when coupled with a standard multi-grid scheme should produce the fastest solutions. Because the Gauss-Seidel method violates the local compactness of the algorithm, a detailed analysis into the overall cost versus a compact Jacobi iteration will need to be performed. In this analysis, both the added work per iteration and the loss of parallel enhancement for the Gauss-Seidel relaxation, needs to be taken into account.

Unlike utilizing a hyperbolic time marching method, where one numerical boundary condition is needed per boundary, for the parabolic system both equations need boundary conditions on each boundary. However, the information needed at each boundary does not change, in that one condition is given by a Dirichlet condition for the known variable and the other is a Neumann condition coming from the first-order equations on the boundary. For example, if $u$ is given on a boundary aligned with the $x$ axis, the corresponding Neumann condition for $v$, in this case the derivative of $v$ in the $x$ direction, is found from the first order system:

$$ -v_x + u_y = 0 \quad \Rightarrow \quad v_x = u_y. \quad (8.5) $$

So the physical coupling of the variables, $u$ and $v$, remains, but is only enforced in the numerical scheme by the boundary conditions.
In the Euler equations, it is the pressure field and the local flow angle, $\theta$, which are contained in the elliptic system. By converting the system to its second-order, or parabolic, form, the natural boundary conditions following those expressed for the model system are surprisingly similar to familiar conditions utilized in existing numerical codes. For a solid wall, the tangency condition places a Dirichlet condition on $\theta$ as the flow direction is defined. From the first-order system we obtain a Neumann condition for pressure given by:

$$\frac{\partial p}{\partial n} = -\rho q^2 \frac{\partial \theta}{\partial s} = \frac{\rho q^2}{R}$$

(8.6)

where $R$ is the radius of curvature of the body.

Unfortunately the implementation of these ideas into a full Euler code is not trivial. The simultaneous solving of this parabolic system with the hyperbolic marching schemes used for the advective operators is not clear, especially how to do so in a conservative way. Essentially, the preconditioning matrix which preserves these desired forms in the unsteady equations will now contain differential operators, converting the first-order acoustic system into second-order parabolic equations, providing a complete decoupling of all characteristic variables (in the linearized sense). Of course, nonlinear coupling between systems occurs through the coefficients, as well as coupling at boundaries between the parabolic operators. It is highly probable that this coupling will reduce the convergence rates predicted by the linear model problems. However, it may produce a much more efficient method, and slightly more accurate solutions due to the symmetry in the weights for the acoustic system, than that presented in this work.

The other area of concern in the existing method is the robustness of the algorithm in stagnation regions. Due to the degeneracy of the acoustic eigenvectors of the
preconditioned system in this limit, unbounded spurious transient growth can occur. However, in the proposed system, where the acoustic field is now converged via parabolic operators, this unbounded transient growth should no longer be an issue.

8.1.2 Extension to three dimensions

The extension of the Hyperbolic/Elliptic decomposition to three dimensions is not straightforward. First of all, complete decoupling of the linearized system, as in the two-dimensional system, is not possible. We find that in the three-dimensional system, the two streamline characteristic equations for entropy and enthalpy still decouple, and for these two decoupled advective operators, the upwind schemes presented for two-dimensional scalar advection in this work have a natural extension to three dimensions, as shown by Deconinck, Struijs et. al. [6]. However, the other three equations are not further reducible when considering first-order operators.

Unfortunately, the analysis of the three remaining coupled equations is not easily extendible from the two-dimensional methods. This system of equations, in streamline coordinates, is given by:

\[ \beta^2 p_s + \rho q^2 (\theta_{n_1} + \phi_{n_2}) = 0 \]
\[ \rho q^2 \theta_s + p_{n_1} = 0 \]
\[ \rho q^2 \phi_s + p_{n_2} = 0, \]

where \( s \) is defined as the local streamline direction, with \( n_1 \) and \( n_2 \) perpendicular to \( s \), forming a right-handed Cartesian coordinate system. We denote the local flow angles by \( \theta \) and \( \phi \), both necessary in defining the direction of flow in three dimensions. Again \( \beta = \sqrt{M^2 - 1} \) provides the distinction between supersonic and subsonic regimes.

In supersonic flows, of the three real characteristics of the coupled system, one is
defined in the flow direction. The other two define the Mach cone generated about
the streamline. An ordinary differential equation along the streamline can only be
written if differential operators are used. This results in the advection of helicity
(the component of vorticity in the direction of flow) \( \psi \), where

\[
\psi = \mathbf{q} \cdot \mathbf{\omega} = q \left( \frac{\partial \phi}{\partial n_1} - \frac{\partial \theta}{\partial n_2} \right)
\]

along the streamline. However, the other equations representing the Mach cone
remain coupled. The resulting advection of helicity and the system representing the
Mach cone are not direct extensions of model systems found in the two-dimensional
analysis. Therefore, the proper discretizations are not straightforward.

In subsonic regimes, the system of three equations now contains one real eigen-
value plus two complex ones. If we allow the use of first-order differential operators,
the system can be written as second-order, or Laplacian, operators in the following
form:

\[
\begin{align*}
-\beta^2 p_{ss} + p_{n_1 n_1} + p_{n_2 n_2} &= 0 \\
q \left( -\beta^2 \theta_{ss} + \theta_{n_1 n_1} + \theta_{n_2 n_2} \right) &= -\psi_{n_2} \\
q \left( -\beta^2 \phi_{ss} + \phi_{n_1 n_1} + \phi_{n_2 n_2} \right) &= \psi_{n_1}.
\end{align*}
\]

Unlike in two dimensions, two of the resulting stream-wise biased Laplacian equations
have source terms which are given by gradients in helicity. As in the supersonic case,
this is not a direct extension of what was recovered in the two-dimensional analysis.
Again, research into the development of proper discretization techniques for this
system must be performed.

Although reductions of the full system are thus attainable in three dimensions,
the discretizations of the resulting sub-systems are not known. In this thesis, it was
shown that the jump from a successful model based on physics in one dimension, to two dimensions, was not trivial. It is the belief of the author that the extension of the successful two-dimensional ideas presented in this work, to three dimensions, is just as hard. However, the improvement over existing methods should prove to be even greater in the three dimensional case, and be well worth the effort.
APPENDIX A

Inconsistency due to Linearization

The spatial gradients of the primitive state, determined from the parameter vector

\[ \mathbf{Z} = \begin{bmatrix} \sqrt{\rho} \\ \sqrt{\rho u} \\ \sqrt{\rho v} \\ \sqrt{\rho H} \end{bmatrix}, \quad (A.1) \]

are given by:

\[ \nabla \mathbf{V} = \begin{Bmatrix} \nabla \rho \\ \nabla u \\ \nabla v \\ \nabla p \end{Bmatrix} = \begin{Bmatrix} 2\sqrt{\rho} \nabla z_1 \\ \frac{-\nabla z_1 + \nabla z_2}{\sqrt{\rho}} \\ \frac{-\nabla z_1 + \nabla z_3}{\sqrt{\rho}} \\ \frac{2-1}{\gamma} \sqrt{\rho} \left( \frac{\nabla z_1 - \nabla z_2 - \nabla z_3 + \nabla z_4}{\sqrt{\rho}} \right) \end{Bmatrix}, \quad (A.2) \]

where

\[ \mathbf{V} = \begin{bmatrix} \bar{\rho} \\ \bar{u} \\ \bar{v} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \bar{z}_1^2 \\ \bar{z}_2/\bar{z}_1 \\ \bar{z}_3/\bar{z}_1 \\ \frac{2-1}{\gamma} \left( \bar{z}_4 - \frac{1}{2} (\bar{z}_2^2 + \bar{z}_3^2) \right) \end{bmatrix}, \quad (A.3) \]

and

\[ \bar{H} = \bar{z}_4/\bar{z}_1. \quad (A.4) \]
In the assumption of linear variation of the parameter vector, the above formulations provide gradients and average values for the primitive state. The inconsistency in this linearization is seen when analyzing a cell which crosses a shear layer through which density and pressure are constant. The following example demonstrates this problem.

In Figure A.1 a shear layer crossing a triangle is shown. The shear layer is defined

![Diagram of shear layer crossing a triangle](image)

Figure A.1: Sample case of shear crossing a triangle.

with velocities given by,

\[ \vec{q}_1 = \beta \vec{q}_2 = \beta \vec{q}_3 = (u, 0), \quad (A.5) \]

with \( \beta \) equal to some constant representing the strength of the shear. The density and pressure are constant throughout and therefore,

\[ \rho_1 = \rho_2 = \rho_3 = \rho^* \quad \text{and} \quad p_1 = p_2 = p_3 = p^*. \quad (A.6) \]
The stagnation enthalpy, given by \( H = \frac{\gamma}{\gamma - 1} \rho + \frac{1}{2} q^2 \) is thus

\[
H_1 = \frac{\gamma}{\gamma - 1} \rho^* + \frac{1}{2} a^2, \quad H_2 = H_3 = \frac{\gamma}{\gamma - 1} \rho^* + \frac{1}{2} \beta^2 a^2. \tag{A.7}
\]

This defines the parameter vector at each vertex, given by:

\[
Z_1 = \begin{pmatrix}
\sqrt{\rho^*} \\
\sqrt{\rho^*} u \\
0 \\
\sqrt{\rho^*} \left( \frac{\gamma - 1}{\gamma - 1} \rho^* + \frac{1}{2} a^2 \right)
\end{pmatrix}, \quad Z_2 = Z_3 = \begin{pmatrix}
\sqrt{\rho^*} \\
\sqrt{\rho^*} \beta u \\
0 \\
\sqrt{\rho^*} \left( \frac{\gamma - 1}{\gamma - 1} \rho^* + \frac{1}{2} \beta^2 a^2 \right)
\end{pmatrix}, \tag{A.8}
\]

and thus,

\[
Z = \begin{pmatrix}
\frac{\sqrt{\rho^*}}{3} \\
\frac{(1 + 2\beta)}{3} \sqrt{\rho^*} u \\
0 \\
\sqrt{\rho^*} \left( \frac{\gamma - 1}{\gamma - 1} \rho^* + \frac{1 + 2\beta^2}{6} a^2 \right)
\end{pmatrix} \quad \text{and} \quad \hat{Z}_y = \begin{pmatrix}
0 \\
\frac{1 - \beta}{h} \sqrt{\rho^*} u \\
0 \\
\frac{1 - \beta^2}{2h} \sqrt{\rho^*} a^2
\end{pmatrix}. \tag{A.9}
\]

The gradients in the \( x \) direction are zero. Substituting Equation A.9 into Equations A.2 and A.3 gives the average primitive state and gradients for the cell as:

\[
\nabla_v = \begin{pmatrix}
\rho^* \\
\frac{(1 + 2\beta)}{3} u \\
0 \\
\rho^* + \frac{\gamma - 1}{\gamma} \frac{(1 - \beta)^2}{9} \rho^* a^2
\end{pmatrix} \quad \text{and} \quad \hat{\nabla}_y = \begin{pmatrix}
0 \\
\frac{1 - \beta}{h} u \\
0 \\
\frac{\gamma - 1}{\gamma} \frac{(1 - \beta)^2}{6h} \rho^* a^2
\end{pmatrix}. \tag{A.10}
\]

Clearly a spurious pressure gradient exists which is proportional to \((\Delta u)^2\), where \(\Delta u\) is the jump in the velocity across the shear. It is noted that the average pressure state found in the cell is not the constant pressure given at the vertices, but is altered by a factor due to the linearization also proportional to \((\Delta u)^2\). Existence of the spurious pressure gradient, not balanced by any gradient of \(v\), implies that the decomposition
will predict a pair of acoustic waves traveling vertically. These give an additional numerical dissipation that prevent the code from preserving this shear flow as an exact solution. It is likely that inconsistencies of this order are inevitable features of any discontinuity-capturing technique.
BIBLIOGRAPHY
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ABSTRACT

MULTI-DIMENSIONAL FLUCTUATION SPLITTING SCHEMES FOR THE EULER EQUATIONS ON UNSTRUCTURED GRIDS

by
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Schemes for approximating steady solutions to the two dimensional inviscid gas-dynamic equations on unstructured triangular grids are presented. This family of cell-vertex schemes, known as fluctuation-splitting or distribution methods, incorporates solutions to multi-dimensional subproblems as building blocks, and is developed in hopes of providing more accurate solutions than standard methods, which solve multi-dimensional problems utilizing one-dimensional physics.

Three different methods of decomposing the Euler equations into recognizable subproblems are presented. The first decomposition utilizes simple planar waves to model the local flow, where each wave is governed by an advection operator. The second decomposition utilizes fewer planar waves, while utilizing steady solution components. The last decomposition is based on the recognition of different mechanisms of information propagation present in the steady system of equations. It is
shown that the steady Euler equations can be decoupled into hyperbolic and elliptic components. In supersonic regimes, this yields four advective operators representing the steady characteristics. In subsonic regimes, besides the two hyperbolic components, an elliptic system of two coupled equations representing the acoustical field is recovered. Preconditioning matrices are then employed to preserve this decoupling even in the transient system.

Next, methods are presented for discretizing the resulting reduced operators, both hyperbolic and elliptic, with analysis and results for model systems. For advective operators, a nearly second-order, compact, monotone, upwind distribution scheme for scalar problems is presented. For the elliptic-based system, without a direction of propagation, central discretizations advanced with a Lax-Wendroff scheme are utilized. Following the success of the discretizations in solving the model problems, they were utilized for full Euler calculations, with results presented. While the wave-model schemes did not meet with universal success, results for the hyperbolic/elliptic decoupling method display accurate solutions for a variety of flows including nearly incompressible cases.