A FOUNDATION FOR STRATEGIC AGENDA VOTING*

JOSE APESTEGUIA†, MIGUEL A. BALLESTER‡, AND YUSUF CAN MASATLI OGLU§

Abstract. We offer complete characterizations of the equilibrium outcomes of two prominent agenda voting institutions, that are widely used in the democratic world: the amendment, also known as the Anglo-American procedure, and the successive, or equivalently the Euro-Latin procedure. Our axiomatic approach allows a proper understanding of these voting institutions, and allow comparisons between them, and with other voting procedures.

Keywords: Strategic Voting, Agendas, Committees, Institutions, Axioms.

JEL classification numbers: C72; D02; D71; D72.

1. Introduction

A proper understanding of the democratic institutions that are being used in practice is a prime concern in the social sciences. In this paper we focus on two prominent voting procedures that are widely used in parliamentary, legislative, and committee decision-making, world-wide: the amendment and the successive agenda procedures. While the amendment procedure is extensively used in the Anglo-American world, like in the US Congress, the successive procedure is in place in many European countries as well as the European parliament. In a nutshell, in this paper we offer for the first time two foundations of these two procedures that enhance our understanding of these key voting institutions.

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Both, the amendment and the successive procedures, are voting institutions used to collectively select an alternative among a set of alternatives. They represent natural extensions of simple majority voting to cases where there are more than two alternatives. In both cases, the alternatives are ordered, forming an agenda, and are considered sequentially, taking at each step in the sequence binary decisions using majority voting. In the particular case of the amendment procedure, two alternatives are jointly considered at each step, and the binary choice consists in deciding by majority voting which alternative is eliminated, and hence which alternative is confronted with the next one in the agenda. For the sake of illustration consider three alternatives, ordered as \((a, b, c)\). Then in this case \(a\) is voted against \(b\), and the winner against \(c\). The winner of this last confrontation is declared elected. See Figure 1(a) for a graphical representation.

In the successive procedure, an alternative is considered at each step in the sequence, and the binary choice made by majority voting is whether to select it, or to reject it, and in case of rejection, to consider the next alternative in the sequence. For instance, in the above example, voters must in fact decide between accepting \(a\) or rejecting it, in which case they confront the problem of selecting an alternative from \(\{b, c\}\). If they prefer \(a\) to the alternatives in \(\{b, c\}\) then the voting ends. Otherwise they compare the next alternative in the sequence, which is \(b\), with the remaining one, alternative \(c\). Figure 1(b) provides a graphical representation.

\[
\begin{array}{ccc}
\begin{array}{c}
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\text{a} \\
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\text{a}
\end{array} & \quad & \begin{array}{c}
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\end{array}
\end{array}
\]

(a) Amendment Procedure   (b) Successive Procedure

**Figure 1.** Structure of Amendment Procedure and Successive Procedure with the agenda \((a, b, c)\).
It turns out that versions of these voting procedures are extensively used in committee decision-making, as well as in parliamentary institutions world-wide. Moreover, there seems to be a geographical concentration of the type of agenda voting institution in use. While the amendment procedure is prevalent in the anglo-american world, european countries adopt the successive procedure. See Table 1 for an illustration.\footnote{Black (1948) represents the first formal treatment of agenda voting institutions. Riker (1958) and Farquharson (1969) study the agenda voting procedures actually used in the US Congress. See also Shepsle and Weingast (1982), Ordeshook and Schwartz (1987), and Schwartz (2008). See Rasch (2000) for a treatment of the European case.}

<table>
<thead>
<tr>
<th>Amendment</th>
<th>Successive</th>
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<tbody>
<tr>
<td>Anglo-American</td>
<td>Euro-Latin</td>
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<tr>
<td>USA</td>
<td>Austria, Belgium, Czech Republic, Denmark</td>
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<tr>
<td>Canada</td>
<td>France, Germany, Greece, Hungary, Iceland</td>
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<tr>
<td>UK</td>
<td>Ireland, Italy, Luxembourg, Netherlands</td>
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<td>Sweden</td>
<td>Norway, Poland, Portugal</td>
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<td>Finland</td>
<td>Slovakia, Slovenia, Spain</td>
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<td>Switzerland</td>
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\textbf{Table 1.} Agenda voting procedures by country.

Given then the practical importance of these voting procedures, it is not surprising that they have been subject to substantial theoretical and empirical research, that has lead to the clarification of important aspects. Specifically, there is now a good understanding of the nature of the elected outcomes when voters either vote strategically or naively (see Farquharson, 1969; Miller, 1977, 1980; McKelvey and Niemi, 1978; Plott and Levine, 1978; Moulin, 1979, 1986; Shepsle and Weingast, 1984; Banks, 1985; Eckel and Holt, 1989; and Bag, Sabourian and Winter 2009). The role of the agenda on the final elected outcome has also been the subject of intense research (see McKelvey, 1981; Shepsle and Weingast, 1982; Ferejohn, Morris Fiorina and McKelvey 1987; Dutta, Jackson and Le Breton, 2002; and Bernheim, Rangel and Rayo, 2006). Finally, there are papers that study which voting procedure maximizes in expectation the well-being of society (see Moser, 2007; see also Apesteguia, Ballester and Ferrer, 2011). These questions are of prime importance for the understanding of these voting institutions.
Here we take a different approach, and offer, for the first time, complete characterizations of these voting institutions that uniquely identify them. We establish the sets of properties that when imposed on a decision rule give the same outcome that the one obtained with strategic voters in the corresponding voting procedure. We show that the two procedures are characterized by two systems of three properties each, sharing a common intuition. The two systems of three properties are formed by a (i) Condorcet type property, (ii) a property that guides the election in the presence of cycles generated by binary majority voting, and (iii) a consistency property imposing structure on the elections across related sets of alternatives. These characterizations allow a deep understanding of the properties satisfied by the voting institutions, and facilitates comparisons between them, and with other voting procedures. Furthermore, the identification of the characterizing properties of the voting institutions allows to evaluate their normative attractiveness, and stimulates the study of the consequences of relaxing or strengthening some of these properties.

The first two properties are both Condorcet type, but follow different directions. While in the case of the Euro-Latin procedure it is the classic Condorcet Consistency property, in the case of the Anglo-American procedure it is a Condorcet Loser Consistency property. That is, in the former case it is imposed that if there is an alternative that is preferred to every other alternative by a majority of voters, this is the alternative that should be selected. This is typically regarded as a fundamental property to be desired in any sensible voting procedure. In the Anglo-American case, on the other hand, the Condorcet property dictates that if there is an alternative that is ranked below every other alternative by a majority of voters, not only this alternative should not be elected, but it should not affect the final election either. This is clearly another desirable property.

The second type of properties discriminate between alternatives in the presence of cycles generated by majority voting. Cycles give rise to a difficult problem: what alternative in the cycle should be elected. This is a fundamental problem in political economy, that through this second set of properties, we show that the two voting institutions address it in a simple and natural way. The two voting procedures approach the problem from the same angle, that is by systematically identifying an alternative as especially prominent. However, like in the
above Condorcet case, they follow logically opposed directions. While the Euro-Latin procedure involves identifying a prioritarian alternative, one that is chosen whenever it is part of a triple of alternatives forming a cycle, the Anglo-American procedure identifies an antiprioritarian alternative, an alternative that is never chosen when it is part of a triple of alternatives forming a cycle, and that identifies the alternative to be selected: precisely that alternative that is preferred by a majority to the antiprioritarian alternative.

The third and last two properties, impose consistency requirements in the elections between related sets of alternatives. The properties impose, in both cases, structure on the elected outcomes so that these do not depend in capricious ways on the set of alternatives to vote upon.

The rest of the paper is organized as follows. Section 2 formally presents the environment, gives the definitions of the voting procedures, and introduces the equilibrium notion used thereafter. Section 3 is devoted to the characterization of the Euro-Latin procedure, while Section 4 does the same for the Anglo-American procedure. Finally, Section 5 discusses the nature of the properties used in the characterizations of the procedures, shows the independence of the axioms by way of the study of alternative voting institutions, and establishes the connection between our exercise and implementation theory. All the proofs are contained in the Appendix.

2. Basic Definitions

Let $X$ be a finite set of $m$ alternatives and let $n$ denote the number of voters. For convenience we assume that $n$ is odd. A decision problem is a pair $(P, A)$, where $P = (P_1, \ldots, P_n)$ is a profile of preferences, with each $P_i$ being a complete, transitive, and asymmetric binary relation on $X$, and $A \subseteq X$ is a set of alternatives to vote for. A decision rule $v$ assigns to each decision problem $(P, A)$ an outcome $v(P, A) \in A$.²

An agenda $\vec{X} = (x_1, \ldots, x_m)$ is an ordered list of all the elements in $X$. Given the agenda $\vec{X}$, the associated successive procedure or equivalently the associated

²Dutta, Jackson, and Le Breton (2001, 2002) also consider decision rules in the domain of all the subsets of $X$. 
Euro-Latin procedure, assigns to any set of alternatives $A$, the alternative that survives the following process. The first alternative in $A$ according to the agenda is voted for approval. If the alternative is approved by a majority of individuals, the process stops and this alternative is implemented. If the alternative is rejected, the second alternative of $A$ in the agenda is voted for approval. If the alternative is approved by a majority of individuals, the process stops and this alternative is implemented. Otherwise, the next alternative of $A$ in the agenda is considered, and the process is repeated. If the final alternative in the agenda is considered, it is approved without voting. Consider the extensive game representation of the Euro-Latin procedure restricted to the set of alternatives $A$ when voters have preferences $P$, and denote by $\gamma_{EL}(P, \vec{X}, A)$ its strategic form representation.

Given the agenda $\vec{X}$, the associated amendment procedure or equivalently the associated Anglo-American procedure assigns to any set of alternatives $A$ the alternative that survives the following process. The first pair of alternatives of $A$ in the agenda is voted upon with the one obtaining a majority of votes advancing to the next stage. There it is paired against the next alternative of $A$ in the agenda, and the process is repeated until the final alternative in the agenda is reached. Consider the extensive game representation of the Anglo-American procedure restricted to the set of alternatives $A$, and denote by $\gamma_{AA}(P, \vec{X}, A)$ its strategic form representation.

Clearly, both procedures are subject to strategic manipulation by sophisticated voters. In the Euro-Latin procedure voters may approve an early alternative in the agenda in order to avoid the selection of a latter one. In the Anglo-American procedure voters may pass an alternative only because it can defeat a posterior one, and not because it is preferred to the one with whom it is competing. The characterization of the equilibrium outcomes under the two procedures signifies a challenge that we address in this paper.

A setback with sophisticated behavior is that absurd outcomes may be the result of Nash equilibrium. For example, in a simple two alternatives setting, all players voting for the same alternative, independently of their preferences, is a Nash equilibrium. The refinement that discards this sort of behavior is the use of undominated strategies. Clearly, in the former two alternatives example, voting for the less preferred alternative is weakly dominated, and hence would
be eliminated. It is well-known that binary voting procedures like the one we study here, are dominance solvable. That is, the iterated elimination of weakly dominated strategies in the strategic form representation leads to a unique Nash equilibrium outcome (see Moulin 1979, McKelvey and Niemi 1978, and Austen-Smith and Banks 2005). The use of Nash equilibrium in undominated strategies is, therefore, the standard practice in voting settings like ours, and this is the one we adopt here. We denote by $UNE[\gamma_{EL}(P, \vec{X}, A)]$ and $UNE[\gamma_{AA}(P, \vec{X}, A)]$ the corresponding equilibria in undominated strategies of the Euro-Latin and Anglo-American procedures, respectively.

3. Characterization of the Euro-Latin Procedure

Given a decision problem $(P, A)$, a Condorcet winner is an alternative in $A$ such that, for any other alternative $A$, a majority of voters ranks the former above the latter. In other words, the Condorcet winner majority dominates all other alternatives in $A$. The properties of a Condorcet winner makes it highly desirable as the social outcome of any political problem. This leads us to consider the classical Condorcet Consistency property.

**Condorcet Consistency (CC).** The decision rule selects the Condorcet winner whenever this alternative exists.

It is well-known that Condorcet winners do not always exist. The simplest situation in which Condorcet winners fail to exist involves three alternatives $x, y$ and $z$, and $x$ majority dominating $y$, $y$ majority dominating $z$, and $z$ majority dominating $x$. We refer to the former three alternatives situation as a Condorcet cycle. The presence of a Condorcet cycle immediately raises the problem of selecting one alternative from the triple forming the cycle. A possible approach to this problem entails identifying an alternative that for certain reasons is given priority over the rest of alternatives. This may be due because it represents the status quo, for example. In any case, this alternative is always selected in every Condorcet cycle that includes it. More formally, we say that $a$ is prioritarian in $A$ if, for any preference profile $P$, and for any triple $T_A$ of alternatives in $A$ forming
a Condorcet cycle that involves alternative $a$, we have $v(P, T_A) = a$.

**Condorcet Priority (CP).** The decision rule admits a prioritarian alternative for any set of alternatives.

It is convenient to explore in more detail three simple implications of CP that contribute to a better understanding of the property. First, note that CP implies that for every $A$ there is a *unique* prioritarian alternative. This is easy to see. Suppose on the contrary that $a$ and $a'$ are prioritarian in $A$. Then, there exist a preference profile $P$ and alternative $x \in A$ such that $a, a'$ and $x$ form a Condorcet cycle. It follows that whatever the outcome from $v(P, \{a, a', x\})$ leads to a contradiction. Second, it is also immediate that if alternative $a$ is prioritarian in $A$, then $a$ is prioritarian in every $B \subseteq A$ with at least three alternatives. That is, there is a great deal of consistency across menus of options in the determination of prioritarian alternatives. Finally, if alternative $a$ is prioritarian in $A$ and $a$ is part of a Condorcet cycle $T_A$ in $A$, the direction of the cycle is immaterial for the outcome from $v(P, T_A)$.

Let us now consider the third and last characterizing property of the Euro-Latin procedure. Suppose that the collectivity of voters has to select an alternative from a set $A$. A natural process involves to divide the scrutiny of alternatives in $A$ in three stages. In stage 1 the collectivity of voters, say the committee, decides over a subset of $A$, say $B$, in stage 2 the committee decides over the remaining alternatives in $A$, say $C$, and in stage 3, the committee decides over the selection from $B$ and that from $C$. More formally, we say that a non-empty partition $(B, C)$ of $A$ constitutes a division of $A$ if, for any $D \subseteq A$ and for any preference profile $P$, $v(P, D) = v(P, \{v(P, D \cap B), v(P, D \cap C)\})$. Notice that implicit in the definition of a division, we have admitted the possibility that some alternatives in $A$ may be dropped in the process.

**Division Consistency (DC).** The decision rule admits a division for any set of alternatives.
We now show that the three properties described above completely and uniquely characterize the equilibrium outcome of the Euro-Latin procedure.

**Theorem 1.** A decision rule $v$ satisfies CC, CP and DC if and only if there exists an agenda $\vec{X}$ such that $v$ is the undominated Nash equilibrium outcome of the Euro-Latin procedure.

The intuition of the ‘if’ part is simple. The ‘only if’ part of the proof proceeds as follows. Condorcet Priority guarantees that there is a single prioritarian alternative in each set of alternatives. We use this to construct an agenda $\vec{X}$ placing later those alternatives that are prioritarian in bigger sets. We then show that the outcomes of the decision rule coincide with the result in the Anglo-American procedure with agenda $\vec{X}$, when voters vote sincerely, that is when voters truthfully reveal their actual preferences. In order to do so we prove that for any set of alternatives $A$ and a prioritarian alternative $h$ of $A$, the sets $A \setminus \{h\}$ and $\{h\}$ form a division of $A$. That is, we show that the prioritarian alternative of a set can always be separated from the set, without further consequences on the selection. We finally use the well-known relationship by which the result of sincere voting in an Anglo-American procedure with agenda $\vec{X}$ is exactly the Nash equilibrium in undominated strategies of the Euro-Latin procedure with the agenda following the opposite order to the one in $\vec{X}$ (see Miller 1977). Then, we conclude that for the Euro-Latin procedure the construction of the agenda places the prioritarian alternative in $X$, $x_1$, first in the ordered list, followed by the prioritarian alternative in $X \setminus \{x_1\}$, and so on. Intuitively, therefore, under sophisticated voting in the Euro-Latin procedure, the strongest alternatives, those that are selected in every Condorcet cycle, are considered first in the agenda.

4. **Characterization of the Anglo-American Procedure**

Given a decision problem $(P, A)$, a Condorcet loser is an alternative in $A$ such that, for any other alternative $A$, a majority of voters places the former below the latter. Notice that the Condorcet loser, whenever it exists, is unique. The Condorcet loser is a highly undesirable alternative and hence it is not only that selecting it as the outcome of a decision problem would be absurd, but also, it is natural that any sensible voting procedure be robust to the presence or absence of
such an alternative. This leads us to impose the following consistency property.\(^3\)

**Condorcet Loser Consistency (CLC).** \(v(P, A) = v(P, A \setminus \{a\})\) whenever \(a\) is a Condorcet loser in \((P, A)\).

In the presence of a Condorcet cycle on the triple \(\{x, y, z\}\), our first approach above identifies an element that is given priority. An alternative approach consists on identifying an alternative that is never prioritized in such situations. This may stimulate changes if the alternative is the default one, or avoid risky outcomes when decisions are unclear or controversial. More formally, we say that \(a\) is antiprioritarian in \(A\) if, for any preference profile \(P\), and for any triple \(T_A\) in \(A\) that forms a Condorcet cycle involving alternatives \(a\) and \(x\), we have \(v(P, T_A) = x\) if and only if \(v(P, \{x, a\}) = x\), with \(x \neq a\). That is, alternative \(a\) is antiprioritarian if (i) it is never selected when being part of a Condorcet cycle \(T_A\), and (ii) the selected alternative from \(T_A\) is precisely the one that is preferred to \(a\) by the majority. That is, the alternative \(a\) that is considered as antiprioritarian identifies in each case the alternative to be selected.

**Condorcet Antipriority (CA).** The decision rule admits an antiprioritarian alternative for any set of alternatives.

Like in the case of prioritarian alternatives, in every \(A\) there is a unique antiprioritarian alternative, and also if \(a \in B \subseteq A\) is antiprioritarian in \(A\), \(a\) is also antiprioritarian in \(B\), \(|B| \geq 3\). Unlike in the case of Condorcet Prioritiy, now the direction of the cycle is relevant for the outcome in a Condorcet cycle. Clearly, if \(a\) is prioritarian and majority preferred to \(x\), \(x\) majority preferred to \(y\), and \(y\) majority preferred to \(a\) the outcome from \(\{a, x, y\}\) is option \(y\). However, if the direction of the cycle is the opposite, then the outcome is \(x\).

Our final property establishes a consistency requirement across sets of alternatives. Having identified one antiprioritarian alternative in a set \(A\), one should be able to remove those alternatives dominated by the antiprioritarian one, since

\(^3\)Indeed, in the Discussion section below we show that Euro-Latin procedures satisfy this consistency property too.
one can argue that they are not natural candidates for election, and concentrate on the rest. The election in this subset must coincide with the election in the original set.

Elimination Consistency (EC). If \(a\) is an antiprioritarian alternative in \(A\) and \(v(P, \{a, y\}) = a\), then \(v(P, A) = v(P, A \setminus \{y\})\).

We now show that Condorcet Loser Consistency, Condorcet Antipriority and Elimination Consistency completely and uniquely characterize the equilibrium outcome of Anglo-American procedures.

**Theorem 2.** A decision rule \(v\) satisfies CLC, CA and EC if and only if there exists an agenda \(\vec{x}\) such that \(v\) is the undominated Nash outcome of the Anglo-American procedure.

We now provide the intuition for the ‘only if.’ We first construct the agenda \(\vec{x}\) as follows. Consider first the antiprioritarian alternative \(x_1\) in \(X\), and place it at the end of the agenda. Then, consider the antiprioritarian alternative \(x_2\) in \(X \setminus \{x_1\}\) and place it in the second to last position in the agenda, and so on. If we interpret antiprioritarian alternatives as the most controversial ones, this means that the Anglo-American procedure places these alternatives at the very end of the agenda.

The second step in the proof involves showing that for every \(P\) and \(A\), the election from a decision rule satisfying the properties is the limit \(x^*\) of a sequence of stepping stones in the agenda. This sequence is formed by: (i) the last alternative of \(A\) according to the agenda \(\vec{x}\), (ii) the last alternative of \(A\) that majority dominates the alternative in (i), (iii) the last alternative of \(A\) that majority dominates the alternatives in (i) and (ii), etc. The proof of this step uses the following idea. The last alternative is antiprioritarian, and thus, by EC and CLC, we can concentrate only on those alternatives that dominate it. A recursive argument concludes the proof. Finally, we show through an inductive argument, along the lines of Shepsle and Weingast (1984), that \(UNE[\gamma_{AA}(P, \vec{x}, A)]\) is exactly the limit of such sequence.
5. Discussion

5.1. Comments on the Axiomatic Structures of the Euro-Latin and Anglo-American Procedures. We have shown that while Euro-Latin procedures are fully characterized by Condorcet Consistency, Condorcet Priority and Division Consistency, Anglo-American procedures are fully characterized by Condorcet Loser Consistency, Condorcet Antipriority, and Elimination Consistency. It is illuminating to note the shared structure of the two systems of properties characterizing the two voting procedures.

First, both, Condorcet Consistency for Euro-Latin procedures and Condorcet Loser Consistency for Anglo-American procedures, follow the fundamental principle of Condorcet-type reasoning. While the former imposes selecting the alternative that majority dominates all others whenever it exists, the latter imposes that the outcome should not depend on the presence of the alternative that is majority dominated by all others, whenever it exists. Furthermore, given the desirability of the two properties, one may wonder whether Condorcet Loser Consistency is satisfied by Euro-Latin procedures and the same is true for Condorcet Consistency and Anglo-American procedures. The answers to both questions are clearly yes. In fact we can prove that we can replace Condorcet Consistency by Condorcet Loser Consistency in our characterization of the Euro-Latin procedure. When Division Consistency holds, then both properties are indeed equivalent.

**Proposition 1.** A decision rule \( v \) satisfying \( DC \), satisfies \( CC \) if and only if satisfies \( CLC \).

Second, properties Condorcet Priority in the Euro-Latin case and Condorcet Antipriority in the Anglo-American case, again follow the same type of logic, with opposite directions. Both properties apply in the presence of cycles, and while the former identifies an alternative that gains prevalence whenever it is present, the latter identifies an alternative as unchoosable. These two properties shape the order of the agenda. While in the Euro-Latin case the prioritarian alternatives come first in the agenda, in the Anglo-American case the antiprioritarian alternatives come last in the agenda.

Finally, Division Consistency and Elimination Consistency both impose structure on the outcomes selected by the voting procedures across sets of alternatives.
5.2. Other Voting Procedures and the Independence of the Axioms.

We now suggest other voting institutions than the Anglo-American and Euro-Latin ones, with the purpose of exploring the consequences of relaxing each one of the properties we have studied, one at a time. This illustrates the power of the axioms, that is, the structure they are imposing, and at the same time it shows the independence of the properties.

Consider first a decision rule that completely ignores the preferences of the voters, and follows some external order over the alternatives, say a virtue’s book. More concretely, the virtue’s book establishes a linear order over the set of alternatives and the decision rule selects in every decision problem the maximal alternative according to the book. The rule trivially satisfies Condorcet Priority, the prioritarian alternative in a set being the most virtuous alternative in the set. Also, given the maximizing nature of the rule, Division Consistency holds since any division of the set of alternatives is inessential for the final selection. However, since the virtue’s book completely disregards the views of individuals, it might be the case that all individuals in society prefer the less virtuous alternative in a set to the most virtuous one and yet the rule chooses the latter alternative, violating Condorcet Consistency.

Alternatively, a modified version of the virtue’s decision rule would aim to block the less virtuous alternative, and in so doing would attend the views of the individuals by selecting the alternative that presents the highest resistance to it. That is, the modified virtue’s book rule selects the alternative with the largest margin with respect to the less virtuous alternative (and, say, it respects the selection of the virtue’s book in case of a tie). The rule satisfies Condorcet Antipriority. To see this, note that the antiprioritarian alternative in a set is the less virtuous alternative. Then, in case of a cycle, it follows immediately that the alternative that dominates the antiprioritarian has a larger margin than the other alternative. The rule also satisfies Elimination Consistency since no alternative is dominated by the less virtuous one. However, the rule fails to satisfy Condorcet Loser Consistency since it may be the case that the Condorcet loser is the alternative with the largest margin to the less virtuous one, and hence it would be selected by the modified virtue’s decision rule.
More democratic versions of the former two rules would involve attending the views of the society when these are consistent with Condorcet notions. The Condorcet virtue’s book would select the Condorcet winner whenever this exists, and would follow the virtue’s book rule otherwise. This rule trivially satisfies Condorcet Consistency. It also satisfies Condorcet Priority since the virtue’s book is applied in the presence of cycles. However, it fails to satisfy Division Consistency. To see this, consider a three elements set, and following a similar analysis to the one adopted in the proof of Theorem 1, note that the only possible division involves the separation of the prioritarian alternative from the two others. It might be the case that there is a cycle involving the three alternatives and hence the prioritarian alternative is selected in the three elements set, and that the selected alternative among the non-prioritarian ones majority dominates the prioritarian, provoking the latter not to be selected in the three-stage process, violating Division Consistency.

The Condorcet modified virtue’s book would similarly discard Condorcet losers, applying successively the property of Condorcet Loser Consistency, whenever this is possible, and behaving like the modified virtue’s book when Condorcet losers do not exist. This, apart from obviously satisfying Condorcet Loser Consistency, satisfies Condorcet Antipriority since the modified virtue’s book is applied in the presence of cycles. However, it violates Elimination Consistency. Consider a set of alternatives \( \{a, y, z\} \) over which \( a \) is the antiprioritarian alternative and suppose that \( a \) majority dominates alternative \( y \), and \( y \) dominates \( z \). It might be the case that, by focussing on the largest margin against \( a \), the rule selects alternative \( y \), violating Elimination Consistency.

Finally, consider a binary game tree where (i) all alternatives appear once as terminal nodes of the tree and (ii) there exists at least one node with two non-terminal successors. Define, then, the generalized agenda rule as the rule that selects the undominated Nash equilibrium of the normal form game induced by the binary game. This rule obviously satisfies Condorcet Consistency and Condorcet Loser Consistency. Now, consider one of the special nodes in point

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4Some of the conclusions on the independence of the axioms require of at least four alternatives, such as the argument that follows.
(ii) and four alternatives that are successors of it, two from each of the two non-
terminal successors. It is immediate to see that for any subset of three alternatives
involving a Condorcet cycle, the selection of the rule is the alternative that stems
away alone from the two others. Consequently, there is no prioritarian alternative
in the set of four alternatives. Hence, Condorcet Priority does not hold in general.
A similar reasoning shows that Condorcet Antipriority is also violated. Finally,
the rule satisfies Division Consistency by the equilibrium nature of the rule and
Elimination Consistency since no antiprioritarian alternative exists.

5.3. **Connection to Implementation Theory.** We would like to note here
that our exercise can be related to classical implementation theory. In the lan-
guage of implementation, our exercise might be restated as follows. Decision
rules satisfying CC, CP and DC, or alternatively satisfying CLC, CA and EC
can be implementable via the undominated Nash solution. In particular, our
analysis shows that we can use simple mechanisms for the implementation, since
the former is implemented using the Euro-Latin procedure and the latter by the
Anglo-American procedure.

The main difference between our approach and the one typically used in im-
plementation theory is that, in our case, a decision rule is defined not only over
all possible preference profiles but also over all possible subsets of alternatives,
while implementation theory typically works over the grand set of alternatives
only.

**APPENDIX A. PROOFS**

**Proof of Theorem 1:** We first prove that the Nash equilibrium in undominated
strategies of an Euro-Latin procedure satisfies the properties. Clearly, CC is
immediate. To check CP, we just need to observe that the first alternative of
the set according to the agenda is prioritarian. The reason is that, if such an
alternative is involved in a Condorcet cycle, its rejection would imply the election
of the majoritarian winner alternative among the other two. Because of the
Condorcet cycle structure, that alternative is majority inferior to the first one.
Finally, any set of alternatives admits a division in which all the alternatives but

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5See Jackson (2001) for a survey on implementation theory.
the first one, according to the agenda, are separated from the first alternative. We now prove the converse statement using two lemmata.

**Lemma 1.** Given $A$ and $P$, for any division $(B, C)$ of $A$ and for any prioritarian alternative $h$ of $A$, with $h \in B$ and $|B| \geq 2$, the set $\{v(P, B \setminus \{h\}), v(P, C), h\}$ has a Condorcet winner.

**Proof of Lemma 1:** Suppose on the contrary that the set $\{v(P, B \setminus \{h\}), v(P, C), h\}$ has no Condorcet winner. Given that the set has three alternatives, they must form a Condorcet cycle. By hypothesis, $A$ has no Condorcet winner. Given that $(A \setminus \{h\})$ forms a division of $A$, by DC it follows that $v(P, (A \setminus \{h\}), v(P, C), h)) = v(P, (v(P, B \setminus \{h\}), h), v(P, C)))$. Hence, we must have both $v(P, (v(P, B \setminus \{h\}), h)) = h$ and $v(P, (v(P, C), h)) = h$. By CC, it can only be that $h$ is a Condorcet winner over the set $\{v(P, B \setminus \{h\}), v(P, C), h\}$, leading to an absurd. □

**Lemma 2.** Given $A$, for any prioritarian alternative $h$ of $A$, $(A \setminus \{h\}, \{h\})$ forms a division of $A$.

**Proof of Lemma 2:** We prove it by induction over the cardinality of $A$. The result is trivial if the cardinality of $A$ is equal to 2. Suppose that the result is true for any subset of cardinality lower or equal than $k$. Let $A$ with $|A| = k + 1$. Let $h$ be a prioritarian alternative of $A$. We have to check that $v(P, S \cup \{h\}) = v(P, \{v(P, S), h\})$ for every $S \subseteq A \setminus \{h\}$. We do it by induction over the cardinality of $S$. The result is trivial if the cardinality of $S$ is equal to 1. Suppose that the result is true for any subset of cardinality lower or equal than $r < k$ and let $S \subseteq A \setminus \{h\}$ with $|S| = r + 1$. By DC, $S \cup \{h\}$ admits a division $(B, C)$. Let without loss of generality $h \in B$. Then, in particular, it must be $v(P, S \cup \{h\}) = v(P, \{v(P, B), v(P, C)\}) = v(P, \{v(P, B \setminus \{h\}), v(P, C)\})$ where $\hat{B} = B \setminus \{h\}$. Since $C$ is nonempty, it must be $|\hat{B}| \leq r$ and hence, by induction, $v(P, \{\hat{B} \cup \{h\}\} = v(P, \{v(P, B), h\})$. As a consequence,

$$v(P, S \cup \{h\}) = v(P, \{v(P, B \setminus \{h\}), v(P, C)\}) = v(P, \{v(P, \hat{B}), h\}, v(P, C)))$$

Since $h$ is prioritarian in $A$, it is also prioritarian in $S \cup \{h\} \subseteq A$. Lemma 1 guarantees that the set $\{v(P, \hat{B}), v(P, C), h\}$ has a Condorcet winner, denoted by
Suppose the claim is true for any set of alternatives with cardinality lower or equal than \( p \). Then, the decision rule selects the majoritarian alternative, and hence the claim follows.

Next, we prove that \( v \) is the outcome from the Anglo-American voting over the agenda \( \vec{X} \) when voters vote sincerely, that is truthfully reporting their preferences. We prove it by induction on the cardinality of set \( A \). By CC, if \( |A| = 2 \), the decision rule selects the majoritarian alternative, and hence the claim follows. Suppose the claim is true for any set of alternatives with cardinality lower or equal than \( p \). Now we prove that the claim is true for any set of alternatives \( A \) with \( |A| = p + 1 \). Let \( x_m \) be the last element of \( A \) in the agenda. Given the induction hypothesis, \( v(P, A \setminus \{ x_m \}) \) is the outcome of sincere voting in the Anglo-American procedure \( \gamma_{AA}(P, \vec{X}, A \setminus \{ x_m \}) \). Clearly, by CC \( v(P, \{ v(P, A \setminus \{ x_m \}), x_m \}) \) is the outcome of sincere voting in the Anglo-American procedure restricted to this pair of alternatives. Given that \( x_m \) is the last element of \( A \), it is straightforward that \( v(P, \{ v(P, A \setminus \{ x_m \}), x_m \}) \) is also the outcome of sincere voting for the Anglo-American procedure \( \gamma_{AA}(P, \vec{X}, A) \). To conclude the claim, we only need to show that \( v(P, \{ v(P, A \setminus \{ x_m \}), x_m \}) = v(P, A) \). By construction, \( x_m \) is a prioritarian alternative of \( \{ x_1, x_2, \ldots, x_m \} \). Hence, by Lemma 2 we know that for every subset \( S \) of \( \{ x_1, x_2, \ldots, x_{m-1} \} \) we have \( v(P, S \cup \{ x_m \}) = v(P, \{ v(P, S), x_m \}) \). In particular, we know that \( v(P, \{ v(P, A \setminus \{ x_m \}), x_m \}) = v(P, A) \).

To conclude the proof, we use the well-known result on the equivalence between the outcomes of sincere voting in Anglo-American procedures with a given
Proof of Theorem 2: We first prove that the Nash equilibrium in undominated strategies of an Anglo-American procedure satisfies the properties. CLC is immediate. To check CA, we just need to observe that the last alternative of the set according to the agenda is antiprioritarian. Suppose there exists a Condorcet cycle involving alternatives $x, y$ and $a$, where $x$ majority dominates $y$, $y$ majority dominates the last alternative in the triple $a$ and $a$ majority dominates $x$. Since the outcomes from $\{x, a\}$ and $\{y, a\}$ are $a$ and $y$ respectively, and $y$ majority dominates $a$, the outcome of the first election, namely between $x$ and $y$, will be $y$, which will be confirmed in the final election. Hence, $a$ is antiprioritarian, as announced. Finally, we prove that the Nash equilibrium in undominated strategies of an Anglo-American procedure satisfies EC. The property is trivial for sets involving two alternatives. Notice that for any set $A$ with at least three alternatives, the antiprioritarian alternative must be the last alternative of $A$ in the agenda, say $a$. This follows immediately from our previous argument on CA. Let $y$ be majority dominated by $a$. Clearly, $y$ cannot be the selected alternative in $A$ since $y$ reaching the final election against $a$ would entail the election of $a$. Then, it follows that at equilibrium, the outcome from $A$ must be the same that the outcome from $A \setminus \{y\}$, as desired. We now prove the converse statement.

Let $v$ satisfy CLC, CA and EC. We first construct an agenda $\vec{X}$. By CA, there exists one alternative which is antiprioritarian in $X$. Denote it by $x_1$. Suppose we have defined $x_1, \ldots, x_k$. By CA, there exists one alternative which is antiprioritarian in $X \setminus \{x_1, \ldots, x_k\}$. Denote it by $x_{k+1}$. This process defines an ordered list of alternatives $x_1, \ldots, x_n$. We construct the agenda by setting $\vec{X} = (x_n, x_{n-1}, \ldots, x_1)$.

Now, given the agenda $\vec{X}$, we associate to every pair $(P, A)$ a sequence of alternatives in $A$, $t_1(P, A), t_2(P, A), \ldots$ as follows. $t_1(P, A)$ is the last alternative in $A$ according to the agenda $\vec{X}$. $t_2(P, A)$ is the last alternative in $A$, according
to the agenda $\vec{X}$, that majority dominates alternative $t_1(P, A)$, if it exists. Otherwise, $t_2(P, A) = t_1(P, A)$. Given $t_1(P, A), \ldots, t_s(P, A)$, denote by $t_{s+1}(P, A)$ the first alternative from $A$ in the agenda $\vec{X}$ that majority dominates all alternatives $t_1(P, A), \ldots, t_s(P, A)$, if it exists. Otherwise, $t_{s+1}(P, A) = t_s(P, A)$.

**Lemma 3.** Given $A$ and $P$, $v(P, A) = \lim t_i(P, A)$.

**Proof of Lemma 3:** Notice that by construction, $t_1(P, A)$ is antiprioritarian in $A = A_1$. Thus, by EC, $v(P, A_1) = v(P, A'_1)$, where $A'_1$ contains exactly the alternative $t_1(P, A)$ and the set of alternatives that are majority dominate by $t_1(P, A)$ in $A_1$. If $A'_1 = \{t_1(P, A)\}$, we are done. Otherwise, $t_1(P, A)$ is a Condorcet loser in $A'_1$ by construction. By CLC, $v(P, A'_1) = v(P, A_2)$ where $A_2 = A'_1 \setminus \{t_1(P, A)\}$. By construction, $t_2(P, A)$ is the last alternative in $A_2$ according to the agenda $\vec{X}$. Hence, $t_2(P, A)$ is antiprioritarian in $A_2$, and by EC, $v(P, A_2) = v(P, A'_2)$ where $A'_2$ is the set of alternatives that are majority dominated by $t_2(P, A)$ in $A_2$, and that contains $t_2(P, A)$. If $A'_2 = \{t_2(P, A)\}$, we are done. Otherwise, $t_2(P, A)$ is a Condorcet loser in $A'_2$ by construction. Then, again $t_3(P, A)$ is the last alternative in $A_3 = A'_2 \setminus \{t_2(P, A)\}$ according to the agenda and given the finiteness of $X$, the iteration of this process proves the lemma. □

**Lemma 4.** Given $A$ and $P$, $UNE[\gamma_{AA}(P, \vec{X}, A)] = \lim t_i(P, A)$.

**Proof of Lemma 4:** We prove this lemma by induction over the cardinality of $A$. If $A$ contains two alternatives, the claim follows immediately. Suppose that the claim is true for sets containing up to $k$ alternatives, and consider $A$ with $k + 1$ alternatives. Take the first and second alternatives in $A$ with respect to the agenda, say $x$ and $y$. If the alternative $x$ (resp. $y$) is voted off, then by the induction hypothesis, the outcome of the equilibrium in undominated strategies is the limit $l_1 = \lim t_i(P, A \setminus \{x\})$ (resp., $l_2 = \lim t_i(P, A \setminus \{y\})$). By construction, the outcome of the Nash equilibrium in undominated strategies in $A$ is the alternative that is majority preferred among these two alternatives, $l_1$ and $l_2$. We now show that this coincides with the $\lim t_i(P, A)$. We distinguish between the following cases:

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6Shepsle and Weingast (1984) proved an analogous result. We include Lemma 4 here for completeness.
• $l_1 \neq y$ and $l_2 \neq x$. Then $l_1 = l_2$ and this element is different from $x$ and $y$. By construction of $l_1$, $l_1$ is both majority preferred to $x$ and $y$. Hence, $l_1$ is the limit of the sequence associated to $A$.

• $l_1 = y$, $l_2 \neq x$. By construction, $y$ majority dominates $l_2$ and hence, $y$ is the outcome of the Nash equilibrium in undominated strategies in $A$. But clearly, $y$ belongs to the sequence in $A$, and since $x$ does not belong to the sequence in $A \setminus \{y\}$, it does does not belong to the sequence in $A$ either. Hence, $y$ is the limit of the sequence of $A$, as desired.

• $l_1 \neq y$, $l_2 = x$. By construction, $x$ majority dominates $l_1$ and hence, $x$ is the outcome of the Nash equilibrium in undominated strategies in $A$. But clearly, $y$ does not belong to the sequence in $A$ since $y$ does not belong to the sequence in $A \setminus \{x\}$. As a consequence, $x$ belongs to it, and it is the limit of the sequence in $A$.

• $l_1 = y$ and $l_2 = x$. The outcome of the Nash equilibrium in undominated strategies in $A$ is the alternative in $\{x, y\}$ that majority dominates the other. But notice that $y$ belongs to the sequence in $A$, and $x$ is in the sequence if and only if $x$ majority dominates $y$. Hence, the limit of the sequence is the alternative in $\{x, y\}$ that majority dominates the other, as desired. □

The two lemmata together conclude the proof. ■

**Proof of Proposition 1:** We first prove that a decision rule $v$ satisfying DC and CC also satisfies CLC. Consider $(P, A)$ and suppose that the Condorcet loser $cl(P, A)$ exists. We need to prove that $v(P, A) = v(P, A \setminus \{cl(P, A)\})$. We define iteratively a finite family of nested sets $A_i$ each one containing the Condorcet loser $cl(P, A)$. Let $A_1 = A$. Given the sets $A_1, A_2, \ldots, A_i$ containing $cl(P, A)$, by DC, there exist a division of $A_i$, $B_i$ (that without loss of generality we assume to contain $cl(P, A)$) and $C_i$. Define $A_{i+1}$ as $B_i$ whenever $B_i$ contains some alternative in addition to the Condorcet loser. Otherwise, the family stops at $A_i$.

Now, from the constructed family, we know that:

1. $v(P, A_i) = v(P, \{v(P, B_i), v(P, C_i)\})$
2. $v(P, A_i \setminus \{cl(P, A)\}) = v(P, \{v(P, B_i \setminus \{cl(P, A)\}), v(P, C_i)\})$. 

Therefore, $v(P, A) = v(P, A \setminus \{cl(P, A)\})$. □
Hence, \( v(P, A_i) = v(P, A_i \setminus \{cl(P, A)\}) \) if \( v(P, B_i) = v(P, B_i \setminus \{cl(P, A)\}) \). That is, \( v(P, A_i) = v(P, A_i \setminus \{cl(P, A)\}) \) if \( v(P, A_{i+1}) = v(P, A_{i+1} \setminus \{cl(P, A)\}) \). The conjunction of all these relationships leads to \( v(P, A) = v(P, A \setminus \{cl(P, A)\}) \) if \( v(P, A_k) = v(P, A_k \setminus \{cl(P, A)\}) \), where \( A_k \) is the last set in the family. In this case, \( B_k = \{cl(P, A)\} \) and given that \( cl(P, A) \) is majority dominated by all alternatives in \( A_k \setminus \{cl(P, A)\} \) we have that \( v(P, A_k) = v(P, \{v(P, B_k), v(P, C_k)\}) = v(P, \{cl(P, A), v(P, A_k \setminus \{cl(P, A)\})\}) = v(P, A_k \setminus \{cl(P, A)\}) \), as desired.

We now prove that a decision rule \( v \) satisfying DC and CLC also satisfies CC. We prove it by induction over the cardinality of the set of alternatives. Suppose that the set has two alternatives. Then, the Condorcet winner is selected by direct application of CLC. Now suppose that, if they exist, Condorcet winners are always selected for the corresponding sets, that are supposed to have cardinality smaller or equal than \( k \). Let \( (P, A) \) with cardinality of \( A \) equal to \( k + 1 \) and suppose that the Condorcet winner \( cw(P, A) \) exists. By DC, there exists a division \( (B, C) \) of \( A \). Suppose without loss of generality that \( cw(P, A) \in B \). Hence, \( v(P, A) = v(P, B \cup C) = v(P, \{v(P, B), v(P, C)\}) \). Since \( cw(P, A) = cw(P, B) = cw(P, v(P, C) \cup \{cw(P, A)\}) \), by the inductive hypothesis we know that \( v(P, \{v(P, B), v(P, C)\}) = v(P, \{cw(P, A), v(P, C)\}) = cw(P, A) \). Hence, \( v(P, A) = cw(P, A) \), and CC holds.
References


