1. Introduction

Let \( X, Y \subset \mathbb{P}^2 \) be plane curves of degree \( d, e \) intersecting properly, i.e., if \( Z_1, \ldots, Z_r \) are all the irreducible components of \( X \cap Y \), then \( \text{codim } Z_i = \text{codim } X + \text{codim } Y, \forall i = 1, \ldots, r \). Bezout’s theorem tells us they intersect in exactly \( de \) points counting multiplicity. More precisely, we have the following:

**Theorem 1.1 (Bezout).** Let \( f, g \in k[x, y, z] \) be relatively prime homogeneous polynomials of degrees \( d \) and \( e \). The Hilbert polynomial of \( k[x, y, z]/(f, g) \) is the constant polynomial \( de \).

Moreover, if \( X, Y \) intersect transversely, then \( de \) is actually the number of geometric points of intersection. It’s natural to ask what the situation would be in higher dimensions. In fact, we can define a fundamental class \( [X] \) for each variety \( X \) and we have a higher dimensional analog of Bezout’s theorem:

**Theorem 1.2 ([1], Theorem 1.1).** Let \( A, B \subset X \) be subvarieties of a smooth variety \( X \) and they intersect properly. We can associate to each irreducible component \( C_i \) of \( A \cap B \) a positive integer \( m_i \) in such a way that

\[
[A][B] = \sum m_i [C_i].
\]

We’ll first try to give a description of the intersection of two subvarieties inside a complex projective variety from a homological point of view, and prove the above theorem in the case we are working over complex numbers. After that, we move on to define Chow ring for general quasi-projective varieties over any algebraically closed field \( k \). We will see that many of the ideas used in the complex projective case will in fact carry over to the general case, and with more algebra, we can make the same results work in the general case.
2. Borel-Moore homology

As a preliminary step, we define Borel-Moore homology in this section. We will mainly follow the treatment in [3]. This concept will be crucial in our definition of the fundamental class associated to a subvariety later. To capture the feature of a subvariety, we want to have some invariants relating a topological subspace to the ambient space. On the other hand, we want this invariant to be still determined by the subspace itself, i.e., not depending on the ambient space we choose. Keeping this intuition in mind, there are some obvious candidates for us, e.g., relative homology and cohomology. In fact, we use the latter to define the Borel-Moore homology as follows:

**Definition 2.1.** Let $X$ be a topological space that can be embedded as a closed subspace of $\mathbb{R}^n$. The $i$-th Borel-Moore homology group $\overline{H}_i(X)$ is defined to be the group $H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X)$.

**Remark.** 1) All the homology or cohomology groups here and below will be taken with $\mathbb{Z}$ coefficients.
2) The Borel-Moore homology group is well-defined in the sense that $\overline{H}_i(X)$ is independent of the choice of embedding of $X$ in the Euclidean space. We will not prove this fact, but this is saying these groups are defined intrinsically.

We will now develop some properties that we will use later.

**Proposition 2.2.** Let $X$ be a topological space that can be embedded as a closed subspace into $\mathbb{R}^n$.

(a) If $X$ can be embedded as a closed subspace into an oriented smooth manifold $M$ of dimension $m$, then

$$\overline{H}_i(X) \cong H^{n-i}(M, M \setminus X).$$

(b) $\overline{H}_i(X)$ is also independent of the embedding in (a).

(c) For an oriented compact $n$-dimensional smooth manifold $M$, the Borel-Moore homology groups and the usual singular homology groups agree.

(d) Assume $X$ can be embedded in some smooth manifold $M$ as above, and let $U$ be an open subset of $X$. Then we have a restriction map

$$\overline{H}_i(X) \to \overline{H}_i(U).$$

(e) Let $U \subseteq X$ as above, and let $Y = X \setminus U$. There is a long exact sequence

$$\cdots \to \overline{H}_i(Y) \to \overline{H}_i(X) \to \overline{H}_i(U) \to \overline{H}_{i-1}(Y) \to \cdots$$

(f) If $X$ is a disjoint union of open subsets $X_\alpha, \alpha \in I$, then $\overline{H}_i(X) = \bigoplus_{\alpha \in I} \overline{H}_i(X_\alpha)$.

**Proof.** (a) By Whitney embedding theorem, we know that $M$ can be embedded as a closed subset of some $\mathbb{R}^n$. We then have an exact sequence:

$$0 \to TM \to T\mathbb{R}^n \to \nu_M \to 0,$$

where $\nu_M$ denotes the normal bundle of $M$ in $\mathbb{R}^n$. We can choose a tubular neighborhood $U \subset \mathbb{R}^n$ of $M$ such that $U \cong \nu_M$ as vector bundles. In this setting, we have

$$H^i(M, M \setminus X) \cong H^{i+n-m}(\nu_M, \nu_M \setminus X) \cong H^{i+n-m}(U, U \setminus X).$$

Now by excision, \( H^{i+n-m}(U, U \setminus X) \cong H^{i-n-m}(\mathbb{R}^n, \mathbb{R}^n \setminus X) \). Hence,
\[
H^{m-i}(M, M \setminus X) \cong H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X) = \overline{H}_i(X).
\]

(b) Suppose \( X \) is embedded as a closed subset of two different manifolds \( M, N \) of dimensions \( m, n \) respectively. Then by (a) we have \( H^{m-i}(N, N \setminus X) \cong \overline{H}_i(X) \cong H^{m-i}(M, M \setminus X) \).

(c) \( M \) embeds into itself. So by (a) we have
\[
\overline{H}_i(M) \cong H^{n-i}(M, M \setminus M) = H^{n-i}(M) \cong H_i(M),
\]
where the last isomorphism is given by Poincaré duality.

d) Let \( M^o = M \setminus (X \setminus U) \). Then \( U \) is closed in \( M^o \). Again using (a) gives \( \overline{H}_i(X) \cong H^{n-i}(M, M \setminus X) \), and \( \overline{H}_i(U) \cong H^{n-i}(U, U \setminus X) \). Contravariance of cohomology gives the restriction map induced from the inclusion \( i : U \to X \):
\[
i^* : \overline{H}_i(X) \cong H^{n-i}(M, M \setminus X) \to H^{n-i}(U, U \setminus X) \cong \overline{H}_i(U).
\]

(e) Let \( M = \mathbb{R}^n \). For the triple \( M \setminus X \subset M \setminus Y \subset M \), we have the long exact sequence for cohomology:
\[
\cdots \to H^{n-i}(M, M \setminus Y) \to H^{n-i}(M, M \setminus X) \to H^{n-i}(M \setminus Y, M \setminus X) \to H^{n-i+1}(M, M \setminus Y) \to \cdots
\]
Using isomorphisms in (a) and (d), the above sequence is exactly the long exact sequence:
\[
\cdots \to \overline{H}_i(Y) \to \overline{H}_i(X) \to \overline{H}_i(U) \to \overline{H}_{i-1}(Y) \to \cdots.
\]

(f) This follows directly from the corresponding property for relative cohomology.

3. INTERSECTIONS OF SUBVARIETIES

In this section, we try to associate to every subvariety a fundamental class and use that to describe the intersection of two subvarieties as in [3]. From now on, \( X \) will be a projective smooth complex variety of dimension \( n \). Then \( X \) is actually a compact oriented \( 2n \)-dimensional real manifold.

**Lemma 3.1.** A smooth complex projective variety \( X \) of pure dimension \( n \) is a real \( 2n \)-dimensional smooth manifold.

**Proof.** Such \( X \) is given by \( X = Z(f_1, \ldots, f_l) \) for some polynomials \( f_1, \ldots, f_l \) on \( \mathbb{C}^N \). These polynomials are apparently holomorphic. Smoothness on \( X \) implies that the Jacobian has rank \( l \), and so \( X \) is a smooth submanifold of \( \mathbb{C}^N \), and thus a \( 2n \)-dimensional real manifold. \( \square \)

**Lemma 3.2.** Let \( A \) be an algebraic subset of a nonsingular algebraic variety, and let \( k \) be the dimension of \( A \). Then \( \overline{H}_i(A) = 0 \) for \( i > 2k \), and \( \overline{H}_{2k}(A) \) is a free abelian group with a generator for each \( k \)-dimensional irreducible component of \( V \).

**Proof.** We first deal with the case when \( A \) is smooth and purely \( k \) dimensional. In this case \( A \) is a real \( 2k \)-dimensional smooth manifold. Write \( A = A_1 \cup \cdots A_r \), where \( A_i \) are the irreducible components of \( A \). We claim that these are exactly the connected components of \( A \). In fact, every irreducible component is connected, and conversely, if \( x \in A_i \cap A_j, i \neq j \), then \( x \) must be a singular point, contradicting smoothness. Hence the union above is in
fact a disjoint union. By what we have in Proposition 2.2, we can thus conclude that \(\overline{\eta}_i(A) = 0, \forall i > 2k\), and \(\overline{\eta}_{2k}(A) = \bigoplus_{j=1}^r H_{2k}(A_j)\), with each \(H_{2k}(A_j) \cong \mathbb{Z}\). For general \(A\), we induct on the dimension \(k\). If \(k = 0\), then \(A\) is pure 0-dimensional, and this is done in the previous case. Suppose we have the desired conclusion for all algebraic subsets of dimension less than \(k\). Now write \(A = Z \cap (A \setminus Z)\) such that \(Z\) is closed, and \(A \setminus Z\) is pure \(k\)-dimensional. Then \(\dim Z < k\), and \(\overline{\eta}_i(Z) = 0, \forall i > 2k - 2\). The long exact sequence for \(A \setminus Z \subset A\) is:

\[
0 = \overline{\eta}_{2k}(Z) \to \overline{\eta}_{2k}(A) \to \overline{\eta}_{2k}(A \setminus Z) \to \overline{\eta}_{2k-1}(Z) = 0.
\]

Thus, \(\overline{\eta}_{2k}(A) \cong \overline{\eta}_{2k}(A \setminus Z)\), which by the special case we’ve done, is a free abelian group with a generator for each \(k\)-dimensional irreducible component of \(A\). All the higher dimensional homology groups of \(A\) vanish by looking at the long exact sequence. This finishes the proof of lemma. \(\square\)

It’s worth noting that \(\overline{\eta}_*\) are not like the usual singular homology in the sense that there is no pushforward for general maps between topological spaces, but we do have the following:

**Lemma 3.3.** Let \(f : X \to Y\) be a proper continuous maps of topological spaces (that admits closed embeddings into Euclidean spaces). Then there is a pushforward map \(f_* : \overline{\eta}_i(X) \to \overline{\eta}_i(Y)\), and that the pushforward maps are functorial.

In view of the above lemmas, we can make the following definition:

**Definition 3.4.** Let \(V\) be an irreducible subvariety of \(X\), and \(\dim V = k\). The fundamental class \([V]\) of \(V\) in \(X\) is the image in \(H^{2c}(X) := H^{2n-2k}(X)\) of the canonical generator of \(H_{2k}(V) = \mathbb{Z}\). The refined class \(\eta_V\) is the image of \([V]\) in \(H^{2c}(X, X \setminus V)\) via the isomorphism in Proposition 2.2(a):

\[
\overline{\eta}_{2k}(V) \cong H^{2n - 2k}(X, X \setminus V) =: H^{2c}(X, X \setminus V).
\]

Now we are ready to state and prove the following theorem concerning the intersection of two subvarieties of \(X\):

**Theorem 3.5.** Let \(V, W \subseteq X\) be subvarieties of \(X\), with \(a = \dim V, b = \dim W\). Write \(V \cap W = Z_1 \cup \cdots \cup Z_r\), where \(Z_i\) are the irreducible components of \(V \cap W\). If the intersection is proper, then we have a well-defined product (cup product) for elements in cohomology groups, i.e. :

\[
\eta_V \cdot \eta_W = m_1 \eta_{Z_1} + \cdots + m_r \eta_{Z_r},
\]

for some uniquely determined integers \(m_1, \cdots, m_r\). Further, if \(V, W\) intersect generically transversely, then

\[
\eta_V \cdot \eta_W = \sum_{i=1}^r \eta_{Z_i}.
\]

**Proof.** Note that \(\eta_V \in \overline{\eta}_{2a}(V) \cong H^{2n - 2a}(X, X \setminus V)\), and \(\eta_W \in \overline{\eta}_{2b}(W) \cong H^{2n - 2b}(X, X \setminus W)\), we have a well-defined cup product for elements in cohomology groups, i.e. :

\[
\cdots : H^{2n - 2a}(X, X \setminus V) \otimes H^{2n - 2b}(Y, Y \setminus W) \to H^{4n - 2a - 2b}(X, X \setminus (V \cap W)), \eta_V \otimes \eta_W \mapsto \eta_V \cdot \eta_W.
\]

Note also that \(H^{4n - 2a - 2b}(X, X \setminus (V \cap W)) \cong \overline{\eta}_{2a + 2b - 2n}(V \cap W)\). That the intersection is proper means that each \(Z_i\) has dimension \(a + b - n\). By Lemma 3.2, \(\overline{\eta}_{2a + 2b - 2n}(V \cap W)\)
is a free abelian group on $\eta_{Z_1}, \ldots, \eta_{Z_r}$. Hence there are unique integers $m_1, \ldots, m_r$ such that

$$\eta_V \cdot \eta_W = m_1 \eta_{Z_1} + \cdots + m_r \eta_{Z_r}.$$ 

To prove the last assertion in the theorem, we observe that if $X$ is a complex vector bundle of rank $c = n - a$ over $V$ and $V$ is embedded as the zero section in $X$, then $\eta_V$ by definition is the Thom class $\gamma_X$ of this vector bundle. Another observation is that we can restrict $X$ to its open subsets that meet $V$ and $W$: if $X^\circ$ is such an open subset, and we let $V^\circ := X^\circ \cap V$, then the restriction map $Z \cong H^{2n-2a}(X, X \setminus V) \to H^{2n-2a}(X^\circ, X^\circ \setminus V^\circ) \cong \mathbb{Z}$ sends $\eta_V$ to $\eta_{V^\circ}$, and the restriction map is actually an isomorphism. Now we can restrict $X$ to its open subset by choosing open subset $U$ around each $x \in V \cap W$ so that $U$ is biholomorphic to $\mathbb{C}^n$ and $V \cap U, W \cap U$ meet transversely in some $Z_i$. In this way $X$ restricts to a vector bundle over $V \cap W$ given by a direct sum of vector bundles over $Z_i$’s, and the Thom class of such a vector bundle is the product of Thom classes. This is exactly $\eta_V \cdot \eta_W = \eta_{Z_1} + \cdots \eta_{Z_r}$. 

□

Remark. The uniquely determined $m_i$’s in the theorem are called intersection multiplicities along $Z_i$. These in fact agree with the intersection multiplicities defined in Algebraic Geometry.

4. CHOW RING

If we do not restrict ourselves to complex projective varieties, we can in fact cook up everything we had so far in a similar spirit. From now on, let $k$ be an algebraically closed field of characteristic zero, and let $X$ be any smooth quasi-projective variety of dimension $n$.

Mimicking the construction of the usual homology groups, we define the group of cycles on $X$:

**Definition 4.1.** The group of cycles on $X$, denoted by $Z(X)$, is the free abelian group on smooth subvarieties of $X$. A cycle is some $Z = \sum_i n_i Y_i \in Z(X)$, where $Y_i$ are subvarieties of $X$. The cycle $Z$ is called effective if all $n_i \geq 0$.

It’s easy to see that $Z(X)$ is actually graded by dimensions $0 \leq k \leq n$ or codimensions $0 \leq c \leq n$:

$$Z(X) = \bigoplus_k Z_k(X) = \bigoplus_c Z^c(X).$$

To each subvariety $Y \subseteq X$, we have a canonical effective cycle associated to it. Write $Y = Y_1 \cap \cdots \cap Y_r$, where $Y_i$ are the irreducible components. Define the canonical cycle to be $\langle Y \rangle := \sum_{i=1}^r l_i Y_i$, where $l_i$ is the length of the local ring $O_{Y,Y_i} := \lim_{\substack{U \subset X \\ U \cap Y \neq \emptyset}} O(U)$.

In homology theory, we have a chain complex after defining cycles, and the homology groups are given by some quotient. Likewise, we will define the equivalence relation on $Z(X)$.
Definition 4.2. The rational equivalence group \( \text{Rat}(X) \) is the subgroup generated by
\[ \langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle, \]
where \( \Phi \subseteq \mathbb{P}^1 \times X \) is a subvariety not contained in any \( \{t\} \times X \).

One can think of this definition as some sort of analog of the notion of cobordism in the manifold theory: The equivalent cycles in some sense "bound" a subvariety of one dimensional higher. In Algebraic Geometry, one object having the same flavor as "boundary" would be divisors. So one may expect there's some way of defining this equivalence via divisors, and there is. Here is an equivalent definition.

Definition 4.3. Let \( \text{Rat}_k(X) \) be the group generated by \( r \in \text{Div}(X) \), where \( r \in \text{Frac}(W^*) \) for some \( (k+1) \)-dimensional subvariety \( W \subseteq X \). Define \( \text{Rat}(X) = \bigoplus_k \text{Rat}_k(X) \).

We'll use either definition depending on different situations. Now we define the Chow group to be
\[ A(X) := \mathbb{Z}(X)/\text{Rat}(X). \]
For subvariety \( Y \subseteq X \), we define the fundamental class \( [Y] \) to be the class of \( \langle Y \rangle \) in \( A(X) \). With these definitions, we can state an analog of Theorem 3.5:

Theorem 4.4 ([1], Theorem 1.5). Let \( X \) be a smooth quasi-projective variety. Then there exists a unique product structure on \( A(X) \) such that if two subvarieties \( V, W \subseteq X \) intersect generically transversely, then
\[ [V] \cdot [W] = [V \cap W]. \]
Further, if \( V, W \) intersect properly, then for each irreducible components \( Z \) of \( V \cap W \) there is a unique positive integer \( m_Z(V, W) \), called the intersection multiplicity of \( V \) and \( W \) along \( Z \), such that
\[ [V][W] = \sum m_Z(V, W)[Z] \in A(X), \]
and \( m_Z(V, W) = 1 \) if and only if \( V \) and \( W \) intersect transversely at a generic point of \( Z \).

Remark. This makes \( A(X) \) into an associative commutative ring graded by codimensions
\[ A(X) = \bigoplus_{c=0}^{\dim X} A^c(X). \]
With this theorem, we are finally able to call \( A(X) \) the Chow ring.

We will see a proof of this theorem in the later section. Here is a basic example:

Example/Proposition 4.5 ([1], Proposition 1.13). \( A(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n] \).

Proof. We'll show that there is no other generator other than \( [\mathbb{A}^n] \), i.e., if \( Y \subseteq \mathbb{A}^n \) is any proper subvariety of \( \mathbb{A}^n \), then \( Y \) is rationally equivalent to 0. So we need to find some subvariety of \( \mathbb{P}^1 \times \mathbb{A}^n \) with one end being \( Y \) and the other end being \( \emptyset \). More precisely, we assume without loss of generality that \( 0 \notin Y \). Let
\[ U = \{(t, tz) \in (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n : z \in Y\} = Z(\{f(z)t : f(z) = 0 \text{ on } Y\}). \]
Let \( W \) be the closure of \( U \) in \( \mathbb{P}^1 \times \mathbb{A}^n \). Let \( \pi : W \rightarrow \mathbb{P}^1 \) be the projection map. Then for each \( t \in \mathbb{A}^1 \setminus \{0\} \), the fiber in \( W \) is \( tY \), in particular, when \( t = 1 \in \mathbb{A}^1 \setminus \{0\} \), the fiber in
W is exactly Y. We claim that the fiber of W above \( \infty \in \mathbb{P}^1 \) is empty. In fact, choose a polynomial \( g \) such that \( g \) vanishes on all of Y but \( g(0) = c \) for some \( c \neq 0 \). The function \( G(t, z) = g(\frac{z}{t}) \) on \( \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^n \) thus extends to a regular function on \( \mathbb{P}^1 \times \mathbb{A}^n \) taking constant value \( c \) on \( \infty \times \mathbb{A}^n \). Hence \( \pi^{-1}(\infty) = \emptyset \) and Y is rationally equivalent to 0. This is what we wanted to show. \( \square \)

**Remark.** This example matches our topological intuition. Suppose \( k = \mathbb{C} \), and we have \( \mathbb{C}\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}\mathbb{P}^{n-1} \). In this case, the Borel-Moore homology groups for \( \mathbb{P}^n \) are just the singular homology groups. By looking at the long exact sequence for Borel-Moore homology groups, we have \( \overline{H}_{2n}(\mathbb{C}^n) \cong H_{2n}(\mathbb{C}\mathbb{P}^n) = \mathbb{Z} \).

### 5. Comparing Borel-Moore homology groups with Chow rings

Let \( X \) be a smooth quasi-projective variety. Although defining product structure on \( A(X) \) is really hard, it still has roots in the topological ideas we already have. In this section, we’ll try to state and prove some properties that homology groups and Chow groups have in common, and define the product structure for Chow rings when \( X \) is a smooth variety. We begin by establishing a result similar to Lemma 3.2.

**Proposition 5.1** (\([1]\), Proposition 1.8). If \( X \) is irreducible of dimension \( k \), then \( A_k(X) \cong \mathbb{Z} \), and is generated by \([X]\). More generally, if \( X = X_1 \cup X_2 \cup \cdots \cup X_r \), where \( X_i \)'s are the irreducible components of \( X \), then the classes \([X_i]\) generate a free abelian subgroup of rank \( r \) in \( A(X) \).

**Proof.** The first assertion follows basically by definition. \([X_i]\)'s are clearly among the generators of \( A(X) \). Conversely \( \text{Rat}(X) \) is generated by subvarieties in \( \mathbb{P}^1 \times X \), each of which has to be contained in some \( \mathbb{P}^1 \times X_i \). \( \square \)

For Borel-Moore homology groups, \( \overline{H}_* \) is not covariant in general. Despite that, we do have proper pushforward in the case when we have proper maps. This is also true for Chow groups.

**Proposition 5.2** (\([3]\)). If \( f : X \to Y \) is a morphism between smooth complex projective varieties, then there exists a Zariski open subset \( U \) of \( f(V) \) such that \( V \to f(V) \) determines a finite-sheeted cover from \( f^{-1}(U) \cap V \) to \( U \), and we have

(a) if \( \dim f(V) < \dim V \), then \( f_*([V]) = 0 \).

(b) \( f_*[V] = d[f(V)] \) if \( V \) has degree \( d \) over \( f(V) \).

**Proof.** By Lemma 3.3 We have a commutative diagram

\[
\begin{array}{ccc}
\overline{H}_{2k}(V) & \xrightarrow{i_*} & \overline{H}_{2k}(X) \\
\downarrow{f_*} & & \downarrow{f_*} \\
\overline{H}_{2k}(f(V)) & \xrightarrow{i_*} & \overline{H}_{2k}(Y) \\
\end{array}
\]

where \( k = \dim V, n = \dim X, m = \dim Y \). In the case when \( \dim f(V) < k \), then \( f_*[V] = 0 \) by Lemma 3.2. In the other case, pick \( U \) open in \( f(V) \) such that \( f^{-1}(U) \cap V \to U \) is a \( d \)-sheeted cover. Replace \( U \) by a small open ball in the smooth locus of \( f(V) \). Then \( f^{-1}(U) \cap V = \bigcup_{j=1}^d U_j \), where each \( U_j \) is homeomorphic to \( U \). Then we have the following commutative diagram:
\[ \mathcal{H}_{2k}(V) \longrightarrow \mathcal{H}_{2k}(f^{-1}(U) \cap V) = \bigoplus_{j=1}^{d} \mathcal{H}_{2k}(U_j) \]

\[ f_* \downarrow \quad \quad \quad \quad \downarrow f_* \]

\[ \mathcal{H}_{2k}(f(V)) \longrightarrow \mathcal{H}_{2k}(U) \]

from which we see that \([V]\) is sent to \(d[f(V)]\) in the map \(\mathcal{H}_{2k}(f(V)) \to \mathcal{H}_{2k}(U)\). \(\square\)

**Proposition 5.2’** ([1], Theorem 1.20; [2], Theorem 1.4). Let \(f : X \to Y\) be a proper map of quasi-projective varieties, and let \(V \subseteq X\) be a subvariety. Then there exists \(f_* : Z_k(X) \to Z_k(Y)\) such that

(a) If \(\dim f(V) < \dim Y\), then \(f_*\langle V \rangle = 0\);

(b) If \(\dim f(V) = \dim Y\), and \(f|_V\) has degree \(d\), i.e., \(d = [\text{Frac}(V) : \text{Frac}(f(V))]\), then \(f_*\langle V \rangle = d\langle f(V) \rangle\);

(c) \(f_*\) extends linearly to all cycles on \(X\).

Further, \(f_*\) on cycle groups induces \(f_* : A_k(X) \to A_k(Y), \forall k\).

**Sketch of Proof.** By the remark after proof of Theorem 1.11 in 2.5.2 in [4], \(W = f(V)\) is a closed subvariety of \(Y\). This induces a finite map \(\text{Frac}(W) \to \text{Frac}(V)\) if \(\dim W = \dim V\). Define \(d\) to be \([\text{Frac}(V) : \text{Frac}(W)]\) if \(\dim V = \dim W\), and 0 if \(\dim W < \dim V\). Define \(f_*\langle V \rangle = d\langle f(V) \rangle\). It’s clear that \(f_*\) extends linearly to a group homomorphism \(f_* : Z_k(X) \to Z_k(Y)\). The functoriality of \(f_*\) follows immediately from the multiplicativity of degrees of finite field extensions.

It thus remains to show that if \(Z\) is a \(k\)-cycle on \(X\) rationally equivalent to 0, then \(f_*\langle Z \rangle\) is also rationally equivalent to 0 on \(Y\). This can be reduced to the case when \(Z = (r)\) where \(r\) is a rational function on some subvariety of \(X\). Further replace \(X\) by this subvariety and \(Y\) by \(f(X)\), the theorem follows from the following lemma, which we will not prove.

**Lemma 5.3** ([2], Proposition 1.4). Let \(f : X \to Y\) be a proper surjective map of quasi-projective varieties, and \(r \in \text{Frac}(X)^\times\). Then

(a’) \(f_*\langle (r) \rangle = 0\) if \(\dim Y < \dim X\);

(b’) \(f_*\langle (r) \rangle = [N(r)]\) if \(\dim Y = \dim X\), where \(N(r)\) is the norm of \(r\) of the field extension \(\text{Frac}(X)/\text{Frac}(Y)\). \(\square\)

This above Lemma is already sufficient to prove the Bezout’s theorem. The original proof we had was using hilbert polynomials. Now with Lemma 5.3 we give another proof of Bezout’s theorem:

**Alternate proof of Theorem 1.1**. We claim that we can reduce the case when \(Z(f)\) and \(Z(g)\) are disjoint unions of \(d\) and \(e\) lines respectively, in which case the intersection is just the set of \(de\) points, and the theorem follows. Let \(g'\) be another homogeneous polynomial of degree \(e\) relatively prime to \(f\). Then \(\frac{g'}{g}\) defines a rational function on \(Z(f)\). Let \(\pi : Z(f) \to Z(f,g)\) be the projection. The previous lemma tells us that \(\pi_*[\frac{g'}{g}] = 0\). This
is exactly saying
\[ \sum_{p \in Z(f,g)} v_p(g) = \sum_{p \in Z(f,g')} v_p(g'), \]
\[ \text{i.e., the number of points in the intersection counted with multiplicities agree when we replace } g \text{ by } g'. \]
So we can take \( g' = L^e \) for some line \( L \). We can do the same thing for \( f \) and reduce to the simplest case we had at the beginning of the proof. \( \square \)

For smooth real manifolds, Borel-Moore homology groups behave like cohomology groups. Thus we should expect to have pullbacks, and we do have.

**Proposition 5.4** (3). Let \( f : X \to Y \) be a map of smooth complex projective varieties. Let \( V \subseteq Y \) be an irreducible subvariety of codimension \( c \) such that \( W = f^{-1}(V) \) is an irreducible subvariety of \( X \) of codimension \( c \) and the following holds: there exists a neighborhood \( U \) of a smooth point of \( V \) on which \( V \cap U \) is defined by polynomials \( h_1, \ldots, h_c \) such that \( W \cap f^{-1}(U) \) is the submanifold of \( f^{-1}(U) \) defined by \( h_1 \circ f, \ldots, h_c \circ f \). Then the pullback \( f^* : H^{2c}(Y, Y \setminus V) \to H^{2c}(X, X \setminus W) \) has the property that \( f^*(\eta_V) = \eta_W \), where \( \eta_V, \eta_W \) are the refined classes defined in Definition 3.4.

**Proposition 5.4'** (2). Let \( f : X \to Y \) be a map of smooth quasi-projective varieties. Then there is a unique map of groups \( f^* : A^*(Y) \to A^*(X) \) such that if \( V \subseteq Y \) is a subvariety generically transverse to \( f \), i.e. \( f^{-1}(V) \) is generically reduced, and \( \text{codim}_Y f^{-1}(V) = \text{dim}_X f^{-1}(V) \), then \( f^*(\langle V \rangle) = \langle f^{-1}(V) \rangle \).

In greater generality, we have a pullback whenever we have flat maps.

**Proposition 5.5.** Let \( f : X \to Y \) be a flat map of schemes, i.e., \( f \) locally given by \( \text{Spec} \ A \to \text{Spec} \ B \) with \( A \) a flat \( B \) module. Then we have a map \( f^* : A_k(Y) \to A_{k+n}(X) \) such that for any smooth subscheme \( V \subseteq Y \) of pure dimension \( k \), \( f^*(\langle V \rangle) = \langle f^{-1}(V) \rangle \).

Now for smooth \( n \)-dimensional variety \( X \) over the ground field, we can define the intersection product in the Chow ring \( A(X) \). Let \( V, W \) be two subvarieties of \( X \) of dimensions \( k, l \) respectively, and let \( \Delta : X \to X \times X \) be the diagonal embedding. On the group of cycles, we have a map
\[ Z_k(X) \times Z_l(X) \to Z_{k+l}(X \times X), \]
sending \( (\langle V \rangle, \langle W \rangle) \) to \( \langle V \times W \rangle \), and extend linearly to a map \( Z(X) \times Z(X) \to Z(X \times X) \). If this map induces a map \( A(X) \otimes A(X) \to A(X \times X) \), then by composing with the pullback \( \Delta^* : A(X \times X) \to A(X) \), we get a product map \( A(X) \times A(X) \to A(X) \) sending \( (\langle V \rangle, \langle W \rangle) \) to \( \langle V \cap W \rangle \). In fact, the map given on cycles indeed induces a map on the Chow ring. To see this, we need the following:

**Theorem 5.6** (2). Let \( \alpha, \beta \) be two cycles on \( X \).

(a) If \( \alpha \) or \( \beta \) is rationally equivalent to \( 0 \), then \( \alpha \times \beta \) is also rationally equivalent to \( 0 \).

(b) If \( f, g \) are flat(resp. proper), then \( f \times g \) is also flat(resp. proper). Moreover, \( (f \times g)_*(\alpha \times \beta) = f_*(\alpha) \times f_*(\beta) \), \( (f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times f^*(\beta) \).

Remark. The previous argument together with the theorem gives a proof of Theorem 4.4.
References


