Testing $H_0: \beta_j^* = \bar{\beta}_j$

Recall under linear model w/ random features,

$$\sqrt{n}(\hat{\beta}_n - \beta^*) \xrightarrow{d} N(0, \underbrace{\Sigma_x^{-1}\Sigma_g\Sigma_x^{-1}}_{\substack{\shortparallel \\ \mathrm{Avar}[\hat{\beta}_n]}})$$

$\Sigma_x \triangleq E[x_i x_i^T]$

$\Sigma_g \triangleq E[(x_i\epsilon_i)(x_i\epsilon_i)^T]$

z-test:

test-statistic: $z_n \triangleq \dfrac{\sqrt{n}([\hat{\beta}_n]_j - \bar{\beta}_j)}{[\widehat{\mathrm{Avar}}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}}$, where $\widehat{\mathrm{Avar}}[\hat{\beta}_n]$ is any CONSISTENT estimator of $\mathrm{Avar}[\hat{\beta}_n]$

critical value: set $t$ so that $P_0(|z_n| > t) = \alpha$

Claim: the z-statistic $z_n = \dfrac{\sqrt{n}([\hat{\beta}_n]_j - \bar{\beta}_j)}{[\widehat{\mathrm{Avar}}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}}$ is $N(0,1)$ under the linear model w/ random features & $H_0$

Pf. We know from asymptotic properties of OLS that

$$\sqrt{n}([\hat{\beta}_n]_j - \bar{\beta}_j) \xrightarrow{d} N(0, [\mathrm{Avar}[\hat{\beta}_n]]_{j,j}) \text{ under } H_0$$

$$\frac{\sqrt{n}([\hat{\beta}_n]_j - \bar{\beta}_j)}{[\mathrm{Avar}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}} \xrightarrow{d} N(0,1)$$

$$z_n = \frac{\sqrt{n}([\hat{\beta}_n]_j - \bar{\beta}_j)}{[\widehat{\mathrm{Avar}}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}} = \underbrace{\frac{\sqrt{n}([\hat{\beta}_n]_j - \bar{\beta}_j)}{[\mathrm{Avar}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}}}_{\substack{\downarrow d \\ N(0,1)}} \boxed{\frac{[\mathrm{Avar}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}}{\underbrace{\boxed{[\widehat{\mathrm{Avar}}[\hat{\beta}_n]]_{j,j}^{\frac{1}{2}}}}_{\substack{\downarrow p \\ \mathrm{Avar}[\hat{\beta}_n]_{j,j}^{\frac{1}{2}}}}}} \xrightarrow{p} 1$$

$$\xrightarrow{d} N(0,1) \cdot 1 = N(0,1)$$

critical value: reject $H_0$ when $|z_n| > z_{\alpha/2}$, when $z_{\alpha/2} = \Phi^{-1}(1 - \alpha/2)$

$\Phi(t) = P(Z \le t)$, $Z$ is a $N(0,1)$ random variable

Formally, $P_0(|z_n| > z_{\alpha/2}) \xrightarrow[n\to\infty]{} P(|Z| > z_{\alpha/2})$ when $Z \sim N(0,1)$ (def of convergence in probability)

$= \alpha$

This is a result about type I error rate control:

$$P_0(\text{reject } H_0) \to \alpha$$

Alternative way of stating rejection rule: accept $H_0$ if

$$-z_{\alpha/2} \le Z_n \le z_{\alpha/2}$$

$$-z_{\alpha/2} \le \frac{\sqrt{n}\,([\hat\beta_n]_j - \bar\beta_j)}{\widehat{Avar}[\hat\beta_n]_{j,j}^{\frac12}} \le z_{\alpha/2}$$

$$-\frac{z_{\alpha/2}}{\sqrt n}\widehat{Avar}[\hat\beta_n]_{j,j}^{\frac12} \le [\hat\beta_n]_j - \bar\beta_j \le \frac{z_{\alpha/2}}{\sqrt n}\widehat{Avar}[\hat\beta_n]_{j,j}^{\frac12}$$

$$\bar\beta_j \le [\hat\beta_n]_j + \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12} \qquad [\hat\beta_n]_j - \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12} \le \bar\beta_j$$

$$\bar\beta_j \in \left[[\hat\beta_n]_j - \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12},\ [\hat\beta_n]_j + \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}\right]$$

Claim: $\left[[\hat\beta_n]_j - \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12},\ [\hat\beta_n]_j + \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}\right]$ is a $1-\alpha$-level confidence interval for $\beta_j^*$:

$$P\left(\beta_j^* \in \left[[\hat\beta_n]_j - \frac{z_{\alpha/2}}{\sqrt n}[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12},\ [\hat\beta_n]_j + \frac{z_{\alpha/2}}{\sqrt n}\widehat{Avar}[\hat\beta_n]_{j,j}^{\frac12}\right]\right)$$

$$= P\left(-z_{\alpha/2} \le \frac{\sqrt n\,([\hat\beta_n]_j - \beta_j^*)}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} \le z_{\alpha/2}\right)$$

Recall $\frac{\sqrt n\,([\hat\beta_n]_j - \beta_j^*)}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} \xrightarrow{d} N(0,1)$ so

$$P\left(-z_{\alpha/2} \le \frac{\sqrt n\,([\hat\beta_n]_j - \beta_j^*)}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} \le z_{\alpha/2}\right) \to P(-z_{\alpha/2} \le Z \le z_{\alpha/2}),\ \text{where } Z \sim N(0,1)$$

$$= 1-\alpha$$

Power of z-test:

$$H_1: \beta_j^* = \bar\beta_j + d$$

$$Z_n = \frac{\sqrt n\,([\hat\beta_n]_j - \bar\beta_j)}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} = \frac{\sqrt n\,([\hat\beta_n]_j - \overbrace{(\bar\beta_j + d)}^{\beta_j^* \text{ under } H_1})}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} + \boxed{\frac{\sqrt n\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}}}$$

$[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}$ $\quad [Avar[\beta_n]]_{j,j}^{\frac12}$ $\quad \dfrac{(\,[Avar[\beta_n]]_{j,j}^{\frac12}\,)}{\mu_n}$

$$\downarrow N(0,1)$$

$P_1(\text{reject } H_0) = P_1(|Z_n| > z_{\alpha/2})$ where $Z \sim N(0,1)$, $\mu_n$ is normalized

$\approx P(|Z + \mu_n| > z_{\alpha/2})$, effect size

$= P(Z + \mu_n > z_{\alpha/2}) + P(Z + \mu_n \le -z_{\alpha/2})$

$= P(Z > z_{\alpha/2} - \mu_n) + P(Z \le -z_{\alpha/2} + \mu_n)$

$$= \underbrace{1 - \Phi(z_{\alpha/2} - \mu_n)}_{\substack{\downarrow n\to\infty \\ 1 \text{ if } \mu_n > 0 \\ \downarrow n\to\infty \\ 0 \text{ if } \mu_n < 0}} + \underbrace{\Phi(-z_{\alpha/2} - \mu_n)}_{\substack{\downarrow n\to\infty \\ 0 \text{ if } \mu_n > 0 \\ \downarrow n\to\infty \\ 1 \text{ if } \mu_n < 0}}$$

$$P_1(\text{reject } H_0) \approx P(|Z + \mu_n| > z_{\alpha/2}) = P\left(\left|Z + \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}}\right| > z_{\alpha/2}\right)$$

**Sample size calculation:**

Fix a power level $\beta$, what is the required sample size to achieve this power?

$$\beta \approx P\left(\left|Z + \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}}\right| > z_{\alpha/2}\right) = P\left(Z + \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} > z_{\alpha/2}\right)$$

$$+ P\left(Z + \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} < -z_{\alpha/2}\right)$$

Assume $d > 0$

$$\beta \approx P\left(Z + \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}} > z_{\alpha/2}\right) = 1 - \Phi\left(z_{\alpha/2} - \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}}\right)$$

solve for $n$ to obtain

$$\Phi^{-1}(1-\beta) = z_{\alpha/2} - \frac{\sqrt{n}\, d}{[\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}}$$

$$\sqrt{n}\, d = [\widehat{Avar}[\hat\beta_n]]_{j,j}^{\frac12}\,(z_{\alpha/2} - \Phi^{-1}(1-\beta))$$

$$n = \frac{[\widehat{\mathrm{Avar}}[\hat{\beta}_n]]_{j,j} \left(z_{\alpha/2} - \Phi^{-1}(1-\beta)\right)^2}{\delta^2}$$