Recap: $(X_n)_{n=1}^{\infty} \to X$

Def: (convergence in probability): $(X_n)_{n=1}^{\infty}$ converges in probability to X iff $P(\|X_n - X\|_2 > \epsilon) \to 0$ for any $\epsilon > 0$

This is written: $(X_n)_{n=1}^{\infty} \xrightarrow{p} X$

$\operatorname{plim}_{n\to\infty} X_n = X$

Note X_n & X have same dim, but they can be scalars or vectors

Def. (convergence in distribution): $(X_n)_{n=1}^{\infty}$ converges in distribution to X iff $F_n(t) \to F(t)$ for all t at which F is continuous.

F_n: CDF of X_n

F: CDF of X

Alt def (also works for random vectors) $(X_n)_{n=1}^{\infty}$ converges in distribution to X iff $E[f(X_n)] \xrightarrow{n\to\infty} E[f(X)]$ for all bounded & continuous functions real-valued functions f.

Ex: $Z_1, Z_2 \overset{ind}{\sim} N(0,1)$

$X_n \triangleq Z_1 + \frac{1}{n} Z_2$

Q1: Does X_n converge in probability to Z_1 ✓

Q2: " " " " dist to Z_1 ✓

Q3: " " " " probability to Z_2 ✗

Q4: " " " " dist to Z_2 ✓

Central limit theorem (CLT): Assume $(X_i)_{i=1}^{\infty}$ is a IID sequence of random variables s.t. $E[X_i] = \mu$ & $Var[X_i] = \Sigma$. Define $\bar{X}_n \triangleq \frac{1}{n}\sum_{i=1}^{n} X_i$. CLT says

$$\underline{\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z}, \text{ where } Z \sim N(0, \Sigma)$$

$\bar{X}_n - \mu \approx \frac{Z}{\sqrt{n}}$

$\bar{X}_n \approx \mu + \boxed{\frac{Z}{\sqrt{n}}}$

Aside: LLN: $\bar{X}_n \xrightarrow{p} \mu$

Define $\varepsilon_n \triangleq \bar{X}_n - \mu$

LLN is equivalently: $\bar{X}_n = \mu + \boxed{\varepsilon_n}$ where $\varepsilon_n \xrightarrow{p} 0$

Continuous mapping theorem (CMT): Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables (can be either random scalars or random vectors) s.t. $(X_n)_{n=1}^{\infty} \xrightarrow{d/p} X$. Then for any continuous function f, we have $(f(X_n))_{n=1}^{\infty} \xrightarrow{d/p} f(X)$.

1st important use case: If $(X_n)_{n=1}^{\infty} \xrightarrow{\boxed{p}} X$ and $(Y_n)_{n=1}^{\infty} \xrightarrow{\boxed{p}} Y$, then

$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} \xrightarrow{p} \begin{bmatrix} X \\ Y \end{bmatrix}$. This in urn implies:

1. $X_n + Y_n \xrightarrow{p} X + Y$
2. $X_n Y_n \xrightarrow{p} XY$

$\begin{bmatrix} X_n \\ Y_n \end{bmatrix}$ converges jointly to $\begin{bmatrix} X \\ Y \end{bmatrix}$, : need to check $\mathbb{P}\left(\left\| \begin{bmatrix} X_n \\ Y_n \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix} \right\|_2 > \epsilon \right) \to 0$

$$\mathbb{P}\left(\left\| \begin{bmatrix} X_n \\ Y_n \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix} \right\|_2 > \epsilon \right) = \mathbb{P}\left(\left\| \begin{bmatrix} X_n \\ Y_n \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix} \right\|_2^2 > \epsilon^2 \right)$$

$$\leq \mathbb{P}\left(\| X_n - X \|_2^2 > \epsilon^2 \right) + \mathbb{P}\left(\| Y_n - Y \|_2^2 > \epsilon^2 \right)$$

(because $\left\| \begin{bmatrix} X_n \\ Y_n \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix} \right\|_2^2 = \| X_n - X \|_2^2 + \| Y_n - Y \|_2^2$)

$\to 0 + 0$

$X_n + Y_n \xrightarrow{p} X + Y$

Define: $f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + b$

By CMT: $\underbrace{f\left(\begin{bmatrix} X_n \\ Y_n \end{bmatrix}\right)}_{= X_n + Y_n} \xrightarrow{p} \underbrace{f\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right)}_{= X + Y}$

$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \overset{d}{\sim} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \overset{d}{\sim} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right)$

NOT TRUE: If $(X_n)_{n=1}^{\infty} \xrightarrow{d} X$ and $(Y_n)_{n=1}^{\infty} \xrightarrow{d} Y$, then $\begin{bmatrix} X_n \\ Y_n \end{bmatrix} \xrightarrow{d} \begin{bmatrix} X \\ Y \end{bmatrix}$ NOT TRUE

Slutsky's lemma: If $(X_n)_{n=1}^{\infty} \xrightarrow{d} X$ and $(Y_n)_{n=1}^{\infty} \xrightarrow{d} Y$ AND Y is a constant, then $\begin{bmatrix} X_n \\ Y_n \end{bmatrix} \xrightarrow{d} \begin{bmatrix} X \\ Y \end{bmatrix}$

As a consequence: $X_n + Y_n \xrightarrow{d} X + Y$
(of CMT) $X_n Y_n \xrightarrow{d} XY$.

Delta method: If $(X_n)_{n=1}^{\infty}$ satisfies $\sqrt{n}(X_n - \underset{\text{constant}}{\boxed{X}}) \xrightarrow{d} Y$, then $(\ldots)^{\infty}$ ... satisfies $\sqrt{n}\left(f(X_n) - f(X) \right) \xrightarrow{d} \nabla f(X)^T Y$ for any

$(f(X_n))_{n \geq 1}$ satisfies ...

differentiable f.

$$\sqrt{n}\,(f(X_n) - f(x)) \approx \sqrt{n}\,(f'(x)\,(X_n - x))$$ (Taylor's thm)

$$= f'(x)\left[\sqrt{n}\,(X_n - x)\right]$$

$$f(\tilde{x}) \approx f(x) + f'(x)(\tilde{x} - x) + \ldots$$

$$\to f'(x) \cdot Y$$ (Slutsky's)

Multivariate Taylor's thm: $\underset{1\times 1}{f(\tilde{x})} = \underset{1\times 1}{f(x)} + \underset{1\times d}{\nabla f(x)^T}\,\underset{d\times 1}{(\tilde{x} - x)} + \frac{1}{2}\,\underset{1\times d}{(\tilde{x} - x)^T}\,\underset{d\times d}{\nabla^2 f(x)}\,\underset{d\times 1}{(\tilde{x} - x)}$

$f : \mathbb{R}^d \to \mathbb{R}$