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Advanced methodology for assessing distribution characteristics of Paris equation coefficients to improve fatigue life prediction

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ABSTRACT This paper offers a methodology for coping with information loss following consolidation of data on fatigue crack propagation rates derived from different experiments. It is customary, both in the literature and in standardization, to consolidate results of several experiments conducted under similar conditions, using identical materials. This reduces the ability to implement a probabilistic fracture mechanics approach in order to reliably calculate the distribution of the number of cycles needed to reach a critical value (CV; onset of instability or failure). Such reliable calculation requires, among other things, an estimation of the distribution characteristics of the crack progression curves coefficients represented by models such as Paris or NASGRO, and an estimation of joint distributions of equation coefficients representing such models. Consolidated data reduce the ability to estimate these required distribution characteristics. This work suggests an analytical approach that uses consolidated data, but enables the information to be treated as if it were possible to attribute the data to the various experimental specimens from which they were obtained. Consequently, information required for the evaluation of the distribution of the number of cycles needed to reach a CV can be obtained.

The proposed approach is generic and can be applied in additional scientific fields that can benefit from separation of data obtained from different experiments.

Keywords analytical approach; damage tolerance; fatigue; Paris equation; probabilistic fracture mechanics, reliability; risk.

NOMENCLATURE

A = Antilog of the 'independent' coefficient in the linear Paris equation
a = Crack length
af = Critical crack length
ai = Initial crack length
da/dN = Fatigue crack propagation rate
COV = Covariance
CV = Critical value
Kc = Fracture toughness
ΔK = Stress intensity factor range
Nf = Number of cycles to failure
P = Slope coefficient of the Paris curve
SD = Standard deviation
V = Variance
α = Shape coefficient
ε = Error
σmax = Maximum stress
σr = Stress amplitude

INTRODUCTION

In this paper, we offer a solution to the problem of information loss due to consolidated data results of crack

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propagation experiments. Consolidating data prevents the ability to distinguish between results obtained from different specimens. Standardization documents and the literature frequently use such consolidated data. Consolidated data significantly reduce the reliability of any evaluation of the distribution of number of cycles required to reach a critical value (CV).

The term *critical value* means a predetermined value such as catastrophic failure, a certain crack length that is critical for the said application or a region of instability in crack propagation rate and so on.

In the framework of development processes, risk assessment, reliability prediction and the planning of maintenance policies that are based on a damage tolerance approach, organizations from all areas of industry frequently apply a fracture mechanistic approach. According to such an approach, it is sometimes required to estimate the distribution of the number of fatigue cycles required to reach a CV.

To predict fatigue failure according to a fracture mechanistic approach, curves are used that describe fatigue crack propagation rate versus stress intensity factor range, as depicted in Fig. 1.

As seen in Fig. 1, there are three distinct regions: Region I describes the initial crack propagation stage, Region II describes the monotonous linear propagation of the crack and Region III describes the instability of crack propagation until fracture.[1]

For implementing a fracture mechanistic approach, models are used to estimate the number of fatigue cycles

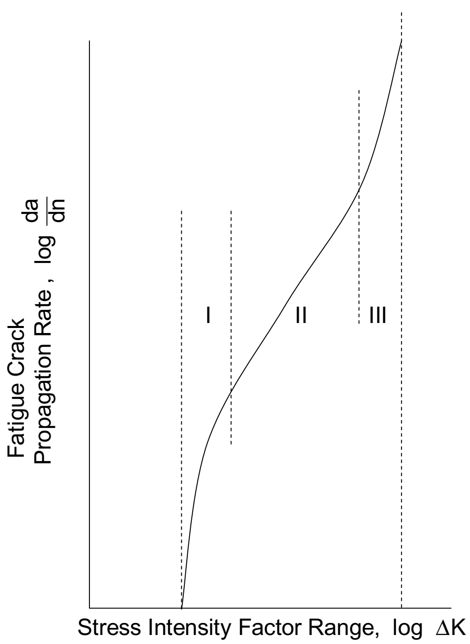

Fig. 1 The three regions of crack propagation.

required to reach a CV. Such models fully or partially describe the curve presented in Fig. 1 in mathematical terms. NASGRO[2] and Paris[3] are examples of such models. For instance, Eq. 1 describes the model according to the Paris approach:

$$\frac{da}{dN} = A(\Delta K)^P , \qquad (1)$$

where $\frac{da}{dN}$ is the increment of fatigue crack propagation per cycle, ΔK is the cyclic range of the stress intensity factor, P is the slope coefficient and A is the independent coefficient.

Calculations based solely on nominal values are insufficient, and the significance of the scatter must be addressed. When the scatter of crack propagation rates in different materials is compared, considerable variance is observed. Figure 2, for instance, taken from MIL-HDBK-5J[4], illustrates significant scatter in fatigue crack propagation rates for two different aluminium alloys (2124-T851 and 7050-T7451).

In some materials, significant scatter was seen in observations of experiments that describe the propagation rates of cracks obtained from several specimens made of the same material and tested under similar conditions.

If experimental data for each sample (of identical material tested under similar conditions) are available, regression can be performed on the results. Such regressions, with respect to the mathematical models, enable evaluation of the curve equation coefficients relevant for each specimen, defining a distinct curve for each specimen, which is similar to the curve depicted in Fig. 1. Crack growth rate is estimated for the type of material and test conditions, by referring to several curves obtained from different experiments.

From curve coefficients so obtained, it is possible to estimate the distribution characteristics of the relevant curve coefficients (Paris, NASGRO, etc.) and the joint distribution of the said coefficients for the type of material examined.

It is of great importance to take into consideration the joint distribution of the coefficients. Feng[5] for instance, writes that 'the pair of Paris parameters is mutually related, and therefore can not be analysed independently', and Annis[6] claims that 'parameters estimates are jointly distributed… It is how regression model parameters naturally behave… So any realistic simulation must sample from correlated join density'. In the article, Annis points out that failure to take the relationship between the Paris coefficients into consideration can cause a significant mistake, which might run to hundreds of percentage points of error. It is evident that failure to consider the joint distribution might lead to significant error in estimating the distribution of the number of fatigue cycles required to reach a CV.

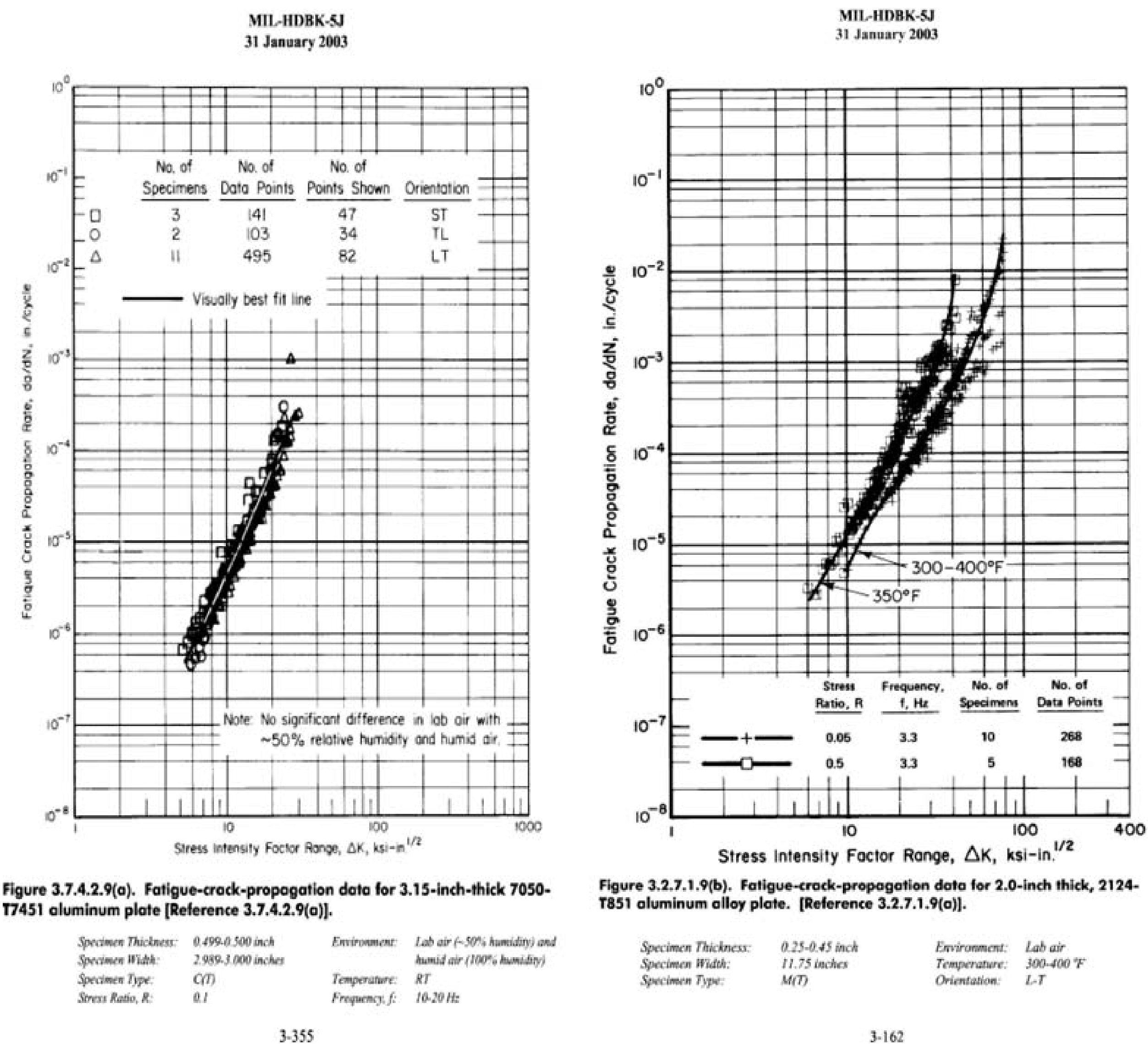

Fig. 2 Examples of broad and narrow distributions.

It is often the case, in some of the common data sources that apply a fracture mechanistic approach, such as MIL-HNDBK 5J,[4] NASGRO[2] and AFGROW,[7] that although results of crack propagation experiments are available in the literature, it is impossible to distinguish between results obtained from different specimens. This considerably affects the ability to distinguish between scatter in crack propagation rates that stems from errors and scatter that stems from physical variability, such as large variability among specimens. Such loss of information due to consolidating data detrimentally affects the ability to estimate the joint distribution relevant to the curve coefficients, which in turn considerably affects the reliability of simulations run in order to characterize the distribution of the number of cycles needed to reach a CV.

This paper presents results of work aimed at improving the ability to cope with the above-mentioned difficulties, which stem from the practice of data consolidation. In this paper, a new approach is presented that helps estimate fatigue crack propagation scatter even when it is impossible to distinguish between test results obtained from various specimens.

In the framework of this paper, the proposed approach was implemented utilizing the Paris equation. The paper describes the proposed model in detail and explains the method of estimating the distribution characteristics of the Paris coefficients and their joint distribution, even in cases (common in the literature and standardization) in which consolidated observations and results of crack propagation experiments are presented.

In doing so, the stage is set for the estimation of the distribution of the number of cycles to CV and common analyses can be executed by applying risk, reliability[8] and damage tolerance[9] theories.

Such analyses have both safety and economical implications for the definition of maintenance-oriented policies and for hazard/risk management policies.

Assessing the distribution of the number of cycles to failure is part of the body of work in the field of probabilistic fracture mechanics.[10,11]

DISTRIBUTION OF PARIS CURVES

This paper offers a methodology that enables the estimation of the distribution of the number of fatigue cycles required to reach a CV even when the data analysed is consolidated.

To demonstrate the suggested approach and for the sake of simplicity, we treat only the linear region of the curve (Fig. 1, Region II), although we acknowledge that the earliest stage of crack growth (Fig. 1, Region I) must also be examined. Because data are derived from experiments that cover the entire lifespan of the cracking samples – from initial crack growth up to catastrophic fracture – the affect of the variability of the initial crack growth region on the variability of the linear region is manifested in the raw data.

Applying the model over the linear region enables to estimate the distribution of the number of fatigue cycles expected until the crack grows from an initial crack (*a*0) to its final critical length (*af*), when these values are in the linear region or when the number of cycles in Region III is negligible compared to the number of cycles in the examined linear region. Such estimations are frequently called for when implementing a damage tolerance approach.[9]

The linear region of Fig. 1 is commonly described and analysed using the Paris approach.[3]

The concept described in this paper is generic and the model can be adjusted to fit more complex, nonlinear equations, such as the NASGRO equation.[2]

The proposed methodology suggests the adoption of an analytical approach whereby the scatter of the linear regions described by Paris curves obtained from each of the various experiments is employed, rather than executing a conventional single regression with respect to the entire set of data obtained from various experiments.[12] Figure 3 describes such a scatter of lines (on a log–log scale).

The proposed approach is based on the fact that, according to the common method, Paris curve data are based on consolidated experimental results from several specimens. In general, specimens may vary with respect to both their metallurgical and geometrical properties as well as the variance in experiment conditions.

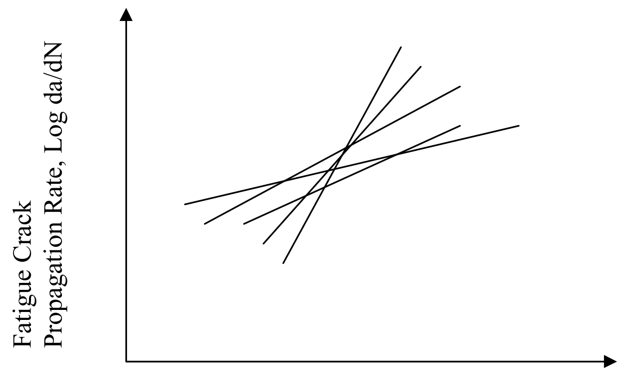

Fig. 3 Schematic scatter of lines (on a log–log scale) representing the linear zones for the different specimens.

As mentioned in the Introduction, the Paris equation is traditionally described as follows:

$$\frac{da}{dN} = A(\Delta K)^P . \tag{1}$$

We refer to the scatter of the two coefficients in the Paris equation, that is, the slope coefficient, P, and the independent coefficient, A.

Taking logarithms of both sides yields the following linear equation, which describes the Paris linear region on a log–log scale (Region II in Fig. 1):

$$Log\frac{da}{dN} = Log(A) + P \cdot Log(\Delta K). \tag{2}$$

When transitioning from a single Paris equation to the distribution of Paris curves, the calculation continues as follows. For a specific value of ΔK, the variance of the linear lines can be formulated as

$$V\left[Log\frac{da}{dN}\right] = V[Log(A)] + [Log(\Delta K)]^{2*}V(P) + 2^*[Log(\Delta K)]^*COV(Log(A), P) + V(\varepsilon), \tag{3}$$

where V is the variance, COV is the covariance and ϵ is the measurement error. Equation 3 describes a parabolic relationship between V[Log(*da/dN*)] and Log(ΔK), where V[Log(A)], $V(P)$, COV(Log(A),P) and $V(\varepsilon)$ constitute the coefficients of the parabolic equation.

This mathematical operation enables the coefficients described in Eq. 3 to be obtained, namely V[Log(A)], $V(P)$ and COV(Log(A),P), which give an indication as to the scatter of the Paris curve coefficients and the joint distribution between the said coefficients. Determination of these values facilitates the ability to estimate the distribution of the number of cycles to CV. By extracting the above coefficients from Eq. 3, the stage is set for estimating the desired data, even when the data are consolidated.

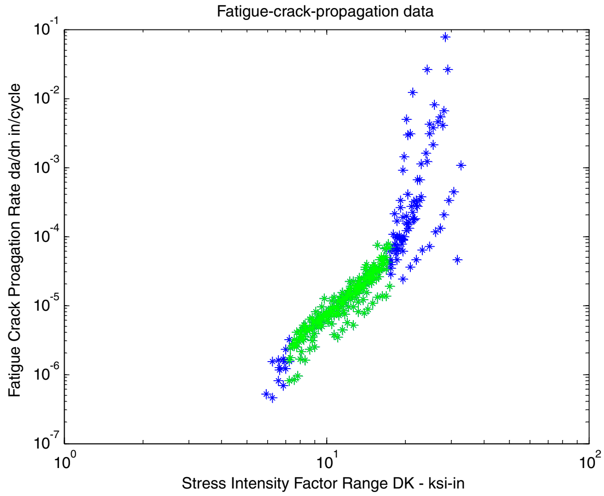

Fig. 4 Consolidated data obtained from experiments on specimens of aluminium alloy 2124-T851.

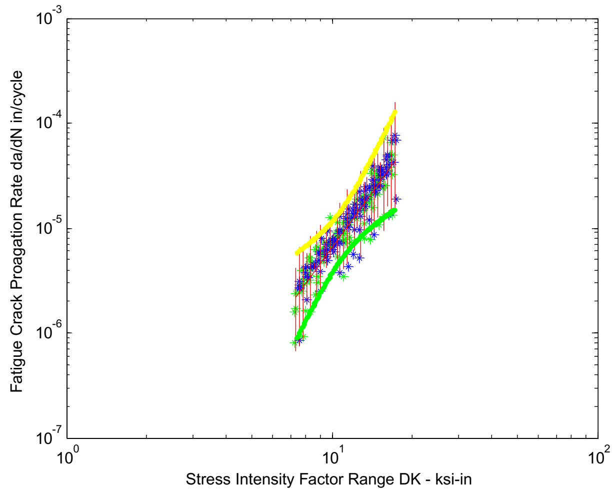

Fig. 5 Implementing the analytical approach to crack propagation data obtained from experiments with aluminium alloy 2124-T851. Note the division of the ΔK axis into small intervals and indication of 2 *SD* range for each interval.

The following procedure is used to extract the variances of P, Log(A) and their covariance COV(Log(A),P) from Eq. 3:

- Consolidated crack propagation data are plotted.

The consolidated data available in the literature and standardization documents are plotted as a graph with axes as shown in Fig. 1. As mentioned above, the data are obtained from different specimens, made of the same material and tested under similar conditions.

Figure 4 presents consolidated data for aluminium alloy 2124-T851. Note that it is impossible from the consolidated data presented, to relate the data to any of the specimens from which they were obtained.

- Observations located at the extremities are neglected.

In order to ensure that the observations that constitute the basis for calculation are taken only from the linear region of the curve, experimental observations at the extremities of the curve are neglected. Figure 4 describes the linear region data (Region II in green) relative to the entire data set.

- The horizontal axis (ΔK) is divided into intervals.

The horizontal axis (ΔK) is divided into intervals. In order to increase the probability that each interval will include observation points that originate from experiments with different specimens (i.e. independent data points), we minimize the size of the intervals on the horizontal axis (ΔK) to ensure that each interval includes at least two observations. It is obvious that the crack length and the resulting stress intensity increase with the progression of the experiment. Thus, the smaller the interval (ΔK) that includes observations with various values of crack propagation rate, the lower the probability that the observations will belong to the same specimen.

Figure 5 shows the division of the linear region into small intervals over the vertical axis (ΔK). Red lines are located in the middle of each interval. Observations are marked in blue and green alternately, whereby same-coloured observations belong to the same interval.

- V[Log(da/dN)] is calculated.

For each interval on the horizontal axis (ΔK), the variance of the observations with respect to the vertical axis is calculated. The length of the red lines located in the middle of the interval (Fig. 5) represents the range of two standard deviations based on a calculation of the V[Log(da/dN)] for each interval.

- Linear regression is executed.

Linear regression along the ΔK axis of the calculated results, V[Log(da/dN)], is performed in order to estimate the parabolic equation (Eq. 3) coefficients. Such linear regression enables the extraction of the following values: V(log(A)), V(P) and COV(Log(A),P).

This procedure completes the description of the extraction of values from Eq. 3. It is noted that although our analysis is based on consolidated data, we have nonetheless succeeded in extracting the distribution characteristic of the Paris curve coefficients and their joint distribution.

Figure 5 presents the parabolic curves that surround the observations (yellow and green). Coefficients of the parabolic curves (Eq. 3) were extracted for consolidated experiment data of crack propagation rate in aluminium alloy 2124-T851.

Substituting the coefficients into Eq. 3 results in the variance equation $V[\text{Log}(da/dN)]$ for the aluminium examined. The parabolic curves (Fig. 5) describe the range of two standard deviations calculated based on Eq. 3.

ESTIMATING THE DISTRIBUTION OF THE NUMBER OF CYCLES TO FAILURE BASED ON CONSOLIDATED DATA

After extracting the relevant coefficients from Eq. 3, we use the said coefficients to estimate the distribution of the number of cycles to failure, even when using consolidated data.

In order to calculate the number of cycles to failure, Nf, the equation describing the Paris curve (Eq. 1) must be integrated. An example of such integration using a simple computation was given by Dieter,[13] who described the evolution of fatigue cracking under specific conditions.

The critical crack length, af, at which catastrophic failure will occur (life termination) can be calculated from the following equation:

$$af = \left(\frac{1}{\pi}\right)\left(\frac{K_c}{6_{\max}\,\alpha}\right)^2. \tag{4}$$

Following Dieter's example, the total number of cycles (Nf) required to grow the crack to critical size, and thus cause failure, is given by:

$$Nf = \frac{(af)^{-\frac{p}{2}+1} - (ai)^{-\frac{p}{2}+1}}{\left(-\frac{p}{2}+1\right)\ A\,\sigma r^p\,\pi^{\frac{p}{2}}\,\alpha^p}. \tag{5}$$

In order to reliably predict the distribution of the number of cycles to failure, Nf, based on consolidated data, it is insufficient to calculate a specific (discrete) value of Nf. Rather, the scatter must be considered as well and a simulation performed (such as Monte Carlo simulation) based on Eq. 5, which includes the Paris curve coefficients (A, P). To implement the simulation and based on the values extracted from Eq. 3, the distribution characteristic of the Paris equation coefficients (A, P) are used and the joint distribution between them is taken into consideration.

Equation 5 indicates that in order to compute the number of cycles to failure, values of A must be used rather than values of Log(A) extracted from Eq. 3. Thus, log(A) must be converted into a value of A. Note, that the conversion of Log(A) to A is not trivial, but rather involves the conversion of the variance of A and the covariance, $V[\log(A)]$ and $COV[\log(A),P]$, respectively.

To convert the values of log(A), extracted from Eq. 3 in previous stages, into a value of A, as required in Eq. 5, the following procedure is executed:

- New simulations are performed based on the following calculated distribution data derived from Eq. 3: $V(P)$, $V(\log(A))$, $COV(\log(A),P)$ and mean values P and log(A). These simulations are used to calculate additional Paris curves that fit the above-mentioned distribution characteristics.

In the simulation, we use the conventional assumption that log(A) and P are normally distributed.[6]

- For each line obtained from the simulations, the log(A) coefficient is converted to A.

Thus, vectors A and P are obtained according to the Paris curve scatter, and the joint distribution between the two Paris equation coefficients is taken into consideration. Based on these vectors, Eq. 5 can be applied a great number of times and thus, the distribution of the number of fatigue cycles required to reach failure, Nf, can be estimated. In other words, a method was presented that enables estimation of the distribution of the number of cycles required to reach a CV (failure) despite the existence of consolidated data.

ADDITIONAL INSIGHTS AND LIMITATIONS

- The concept described in this paper is generic. The model described can be adjusted to fit more complex, nonlinear, equations such as the NASGRO equation.[2]
- The illustration we presented of the implementation of the methodology deals with the linear region of the Paris equation. It can be applied when using a damage tolerance approach, which requires an estimation of the distribution of number of cycles to reach CV when the crack already exists and its further growth takes place in the linear region of the Paris equation. This approach can be helpful, for example, in the planning of maintenance activities.[14]
- We assume that the scatter in the crack propagation experiment results, which stems from measurement errors and from the experiment itself, is significantly less than the 'natural' scatter that results from the variance in the properties of the materials of the different specimens.

Note that even without this last assumption, the joint distribution and variance of slope can still be estimated.

- Due to difficulties in obtaining raw data from crack propagation rate experiments, it is often the case that researchers who apply a fracture mechanics approach, must work with graphical data that include consolidated experiment results (without detailed raw data). Such a graph is presented in Fig. 2[4] and in the ASM Handbook.[15] The proposed methodology can be applied in such cases as well. The parameters required to simulate the distribution of the number of cycles required to reach a CV, can be initially estimated by dividing the horizontal axis, ΔK, into small intervals and then, for each interval, $V[\mathrm{Log}(da/dN)]$ is estimated from the amplitude and scatter of observations within each interval (with respect to the vertical axis). The subsequent calculation steps are identical to those presented above. Naturally, the method can be applied in cases in which the graphs are not overly 'cluttered'; that is, in cases in which variability can be estimated along the vertical axis for intervals defined on the horizontal axis.

- The analytical approach presented in this paper is valid for other applications and additional scientific domains that require distinction between results obtained from different specimens or sources. The proposed approach may be applied when consolidated data are present and when a mathematical model exists (such as the Paris equation, Eq. 2) that describes the regularity of an examined phenomenon. When information is sought on regularity derived from the model in specific cases (for instance, aluminium alloy 2124-T851), experiments are performed on specimens of those specific materials. The coefficients of the equation described in the model are extracted from the experimental observations. Thus, information can be gathered that indicates the regularity of the said phenomenon in specific cases. Naturally, many areas of research meet the criteria mentioned, such as creep models[1] and models that address the delamination rate in composite materials.[16]

Consolidated data are common in the literature and standardization documentation for several reasons. First, over the years researchers were not meticulous about saving results from the specific experiments they conducted on different specimens and sufficed with the consolidation of data, performing a single regression on the entire set of consolidated data. Another instance in which consolidated data may exist is when it is not possible to attribute the object from which experimental observations were obtained to the experimental observations themselves. In other areas of science there are instances in which, throughout the entire duration of the experiment, experimental observations obtained from an object cannot be attributed to the object subject of the experiment. This situation may arise, for example, in cases in which an experiment is performed on cells or particles, which are difficult to identify and track during the experiment itself. In such cases, observations are obtained, each of which originates from a different object. The approach offered in this paper enables extraction of the relevant curve coefficients and their joint distribution in such cases as well. Thus, in appropriate cases, it is possible to estimate the scatter involved in reaching a CV.

SUMMARY

The proposed analytical approach enables calculation of the distribution of Paris equation coefficients (A, P) as well as the joint distribution between them, even when available data are consolidated. These values are required to estimate the distribution of the number of cycles required to reach a CV, which is essential, for example, in the fields of reliability, risk assessment and damage tolerance.

The proposed model is based on the realization that consolidated data on fatigue crack propagation rates (vs. ΔK) can be attributed to several individual curves (Paris curves) that originate from different specimens.

The concept described in this paper is generic and the model can be adjusted to fit more complex, nonlinear, equations such as the NASGRO equation.

The proposed approach constitutes a generic solution in cases in which distinction between results obtained from different experiments provides further statistical insights. As such it can be employed in various scientific disciplines.

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