Supplementary Appendix to "Bias in Dynamic Panel Models under Time Series Misspecification"

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1 Proof of Lemma A1

For $V_{i,t} = \sum_{j=0}^{\infty} A^j U_{i,t-j}$, we first derive the following lemma.

Lemma B1 *Under Assumptions E and S, for fixed T and t,*

$$\mathbb{E}\left(V_{i,t-1}\sum_{s=1}^{T} V'_{i,s-1}\right) = (I_p - A)^{-1}\left(I_p - A^t\right)\Gamma_0 + \left\{(I_p - A)^{-1}\left(I_p - A^{T-t}\right)A\Gamma_0\right\}'.$$

Proof of Lemma B1 We derive that

$$\begin{aligned}
\mathbb{E}\left(V_{i,t-1}\sum_{s=1}^{T} V'_{i,s-1}\right) &= \mathbb{E}\left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1}\right)\left(\sum_{s=1}^{T}\sum_{k=0}^{\infty} U'_{i,s-k-1}A'^k\right) \\
&= \sum_{s=1}^{t}\mathbb{E}\left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1}\sum_{k=0}^{\infty} U'_{i,s-k-1}A'^k\right) + \sum_{s=t+1}^{T}\mathbb{E}\left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1}\sum_{k=0}^{\infty} U'_{i,s-k-1}A'^k\right) \\
&= \sum_{s=1}^{t}\sum_{j=0}^{\infty} A^{j+t-s}\mathbb{E}\left(U_{i,s-j-1}U'_{i,s-j-1}\right)A'^j + \sum_{s=1}^{T-t}\sum_{j=0}^{\infty} A^j\mathbb{E}\left(U_{i,s-j-1}U'_{i,s-j-1}\right)A'^{j+s} \\
&= \sum_{j=1}^{t} A^{j-1}\Gamma_0 + \Gamma_0\sum_{j=1}^{T-t} A'^j \\
&= (I_p - A)^{-1}\left(I_p - A^t\right)\Gamma_0 + \left\{(I_p - A)^{-1}\left(I_p - A^{T-t}\right)A\Gamma_0\right\}'
\end{aligned}$$

since Γ_0 is symmetric. ■

Proof of Lemma A1 Since $V^0_{i,t-1} = V_{i,t-1} - V_{i,\cdot-1}$,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} V^0_{i,t-1}U^{0\prime}_{i,t} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(V_{i,t-1} - \frac{1}{T}\sum_{s=1}^{T} V_{i,s-1}\right)\left(U'_{i,t} - \frac{1}{T}\sum_{s=1}^{T} U'_{i,s}\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} V_{i,t-1}U'_{i,t} - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} V_{i,t-1}\sum_{s=1}^{T} U'_{i,s} \\
&= C_{11}(N,T) - C_{12}(N,T),
\end{aligned}$$

where

$$\operatorname*{plim}_{N\to\infty} C_{11}\left(N,T\right) = \mathbb{E}\sum_{t=1}^{T}\left(\sum_{j=0}^{\infty} A^{j} U_{i,t-j-1} U_{i,t}^{\prime}\right) = 0$$

and

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} C_{12}\left(N,T\right) &= \frac{1}{T}\mathbb{E}\sum_{t=1}^{T}\left(\sum_{j=0}^{\infty} A^{j} U_{i,t-j-1} \sum_{s=1}^{T} U_{i,s}^{\prime}\right) = \frac{1}{T}\sum_{t=2}^{T}\sum_{j=0}^{t-2} A^{j}\mathbb{E}\left(U_{i,t-j-1}U_{i,t-j-1}^{\prime}\right) \\
&= \sigma^{2}\left(I_p - A\right)^{-1}\left(I_p - \frac{1}{T}\left(I_p - A\right)^{-1}\left(I_p - A^{T}\right)\right) M,
\end{aligned}$$

with $M = e_p e_p^{\prime}$. Second, we have

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} V_{i,t-1}^{0} V_{i,t-1}^{0\prime} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(V_{i,t-1} - \frac{1}{T}\sum_{s=1}^{T} V_{i,s-1}\right)\left(V_{i,t-1}^{\prime} - \frac{1}{T}\sum_{s=1}^{T} V_{i,s-1}^{\prime}\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T} V_{i,t-1}V_{i,t-1}^{\prime} - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} V_{i,t-1}\left(\sum_{s=1}^{T} V_{i,s-1}^{\prime}\right) \\
&= C_{21}\left(N,T\right) - C_{22}\left(N,T\right),
\end{aligned}$$

where

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} C_{21}\left(N,T\right) &= \sum_{t=1}^{T}\mathbb{E}\left(\sum_{j=0}^{\infty} A^{j} U_{i,t-j-1}\right)\left(\sum_{k=0}^{\infty} U_{i,t-k-1}^{\prime} A^{\prime k}\right) \\
&= \sum_{t=1}^{T}\sum_{j=0}^{\infty} A^{j}\mathbb{E}\left(U_{i,t-j-1}U_{i,t-j-1}^{\prime}\right) A^{\prime j} = T\Gamma_0
\end{aligned}$$

and using Lemma B1

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} C_{22}\left(N,T\right) &= \frac{1}{T}\sum_{t=1}^{T}\left(I_p - A\right)^{-1}\left(I_p - A^{t}\right)\Gamma_0 + \frac{1}{T}\sum_{t=1}^{T}\left\{\left(I_p - A\right)^{-1}\left(I_p - A^{(T-t)}\right) A\Gamma_0\right\}^{\prime} \\
&= \left(I_p - A\right)^{-1}\left(I_p - \frac{1}{T}\left(I_p - A\right)^{-1}\left(I_p - A^{T}\right) A\right)\Gamma_0 \\
&\quad + \left\{\left(I_p - A\right)^{-1}\left(I_p - \frac{1}{T}\left(I_p - A\right)^{-1}\left(I_p - A^{T}\right)\right) A\Gamma_0\right\}^{\prime}. \ \blacksquare
\end{aligned}$$

2 Proof of Lemma A2

We let $\widehat{\Gamma}_j = (1/NT)\sum_{i=1}^{N}\sum_{t=1}^{T} V_{i,t-1}^{0} V_{i,t-1+j}^{0\prime}$ for $j = 0, 1$. The first derive the following lemmas.

Lemma B2 *For large T,*

$$G_{01} = (1/T)\,\Omega + O\left(T^{-2}\right),$$
$$G_{11} = (1/T)\,\Omega A' + O\left(T^{-2}\right),$$
$$G_{12} = \left(\sigma^2/T\right)(I_p - A)^{-1}\left(I_p + O\left(T^{-1}\right)\right)M = \left(\sigma^2/T\right)(I_p - A)^{-1} M + O\left(T^{-2}\right).$$

Proof of Lemma B2 Since Assumptions S and E imply that $H_T = O(T^{-1})$ and A is bounded,

$$\begin{aligned} G_{01}' &= (1/T)\left[(I_p - A)^{-1}\left(I_p + O\left(T^{-1}\right)\right)\Gamma_0 + \left\{(I_p - A)^{-1}\left(I_p + O\left(T^{-1}\right)\right)A\Gamma_0\right\}'\right]' \\ &= (1/T)\left[\Gamma_0\left(I_p - A'\right)^{-1} + (I_p - A)^{-1} A\Gamma_0\right] + O\left(T^{-2}\right) \\ &= (1/T)\left[\Delta + \Lambda'\right] + O\left(T^{-2}\right) \\ &= (1/T)\,\Omega + O\left(T^{-2}\right), \end{aligned}$$

where $\Delta = \sum_{j=0}^{\infty}\Gamma_j = \sum_{j=0}^{\infty}\Gamma_0 A'^j = \Gamma_0\left(I_p - A'\right)^{-1}$ and $\Lambda' = (\sum_{j=1}^{\infty}\Gamma_j)' = (\Gamma_1\left(I_p - A'\right)^{-1})' = (I_p - A)^{-1} A\Gamma_0$. Note that Ω is symmetric. Similarly, since $G_{11} = G_{01}\Gamma_0^{-1}\Gamma_1 = G_{01}A'$ and A is bounded, the remaining results readily follows. ■

Lemma B3 *Under Assumptions E, S, I and NT,*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T} vec\left(V_{i,t-1}U_{i,t}'\right) \to_d \mathcal{N}\left(0, \sigma^2\left(M \otimes \Gamma_0\right)\right) \quad \textit{as } N, T \to \infty \textit{ jointly.}$$

Proof of Lemma B3 We first check the generalized Lindeberg-Feller condition for joint (i.e., double indexed) asymptotic normality as in Theorem 2 of Phillips and Moon (1999). As discussed in Hahn and Kuersteiner (2002), a sufficient condition for the theorem to hold is that $\mathbb{E}\left[\left(1/\sqrt{T}\right)\sum_{t=1}^{T}\varphi' V_{i,t-1}u_{i,t}\right]^4$ is bounded uniformly in i and T for all nonrandom vector $\varphi \in \mathbb{R}^p$ such that $\varphi'\varphi = 1$. Recall that $U_{i,t} = eu_{i,t} = (u_{i,t}, 0, \cdots, 0)'$, and thus we only need to look at the moment condition of the first column of $V_{i,t-1}U_{i,t}'$ to check if this condition holds. We let $z_{i,t} = \varphi' V_{i,t-1}u_{i,t} = \sum_{j=0}^{\infty}\varphi' A^j U_{i,t-1-j}u_{i,t}$, then $\mathbb{E}z_{i,t} = 0$ and $\mathbb{E}z_{i,t}z_{i,s} = 0$ for $t \neq s$ since $u_{i,t}$ is *i.i.d.* It follows that $\mathbb{E}\left[\left(1/\sqrt{T}\right)\sum_{t=1}^{T} z_{i,t}\right]^4 = \left(1/T^2\right)\sum_{t=1}^{T}\mathbb{E}z_{i,t}^4 + \left(1/T^2\right)\sum_{t\neq s}\mathbb{E}z_{i,t}^2\mathbb{E}z_{i,s}^2$. We observe that

$$\begin{aligned} \mathbb{E}z_{i,t}^2 &= \mathbb{E}\left(\sum_{j=0}^{\infty}\varphi' A^j U_{i,t-1-j}u_{i,t}\right)\left(\sum_{k=0}^{\infty}\varphi' A^k U_{i,t-1-k}u_{i,t}\right)' \\ &= \left(\mathbb{E}u_{i,t}^2\right)\mathbb{E}\left(\varphi'\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A^j U_{i,t-1-j}U_{i,t-1-k}' A'^k\varphi\right) \\ &= \sigma^2\varphi'\mathbb{E}\left(\sum_{j=0}^{\infty} A^j U_{i,t-1-j}U_{i,t-1-j}' A'^j\right)\varphi = \sigma^2\varphi'\Gamma_0\varphi < \infty \end{aligned}$$

since $\mathbb{E}A^j U_{i,t-1-j}U'_{i,t-1-k}A'^k = 0$ if $j \neq k$; and Γ_0 if $j = k$. Moreover,

$$\mathbb{E}z_{i,t}^4 = \left(\mathbb{E}u_{i,t}^4\right)\mathbb{E}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\varphi' A^j U_{i,t-1-j}U'_{i,t-1-k}A'^k\varphi\varphi' A^l U_{i,t-1-l}U'_{i,t-1-m}A'^m\varphi\right),$$

and if we define a random variable $w_j = \varphi' A^j U_{i,t-1-j}$ with $\mathbb{E}w_j = 0$ then

$$\begin{aligned}
&\mathbb{E}\left(\varphi' A^j U_{i,t-1-j}U'_{i,t-1-k}A'^k\varphi\varphi' A^l U_{i,t-1-l}U'_{i,t-1-m}A'^m\varphi\right)\\
&= \mathbb{E}\left(w_j w_k w_l w_m\right)\\
&= \mathbb{E}\left(w_j w_k\right)\mathbb{E}\left(w_l w_m\right) + \mathbb{E}\left(w_j w_l\right)\mathbb{E}\left(w_k w_m\right) + \mathbb{E}\left(w_j w_m\right)\mathbb{E}\left(w_k w_l\right) + \mathcal{K}\left(w_j, w_k, w_l, w_m\right),
\end{aligned}$$

where $\mathcal{K}\left(w_j, w_k, w_l, w_m\right)$ is the forth order cumulant of w_j. Note that similarly as $\mathbb{E}z_{i,t}^2$, we have $\mathbb{E}\left(w_j w_k\right) = \sigma^2\varphi' A^j M A'^j\varphi$ if $j = k$ and zero otherwise. Therefore,

$$\begin{aligned}
\left|\mathbb{E}z_{i,t}^4\right| &\leq \left|\mathbb{E}u_{i,t}^4\right|\left\{3\left(\varphi'\left|\sigma^2\sum_{j=0}^{\infty}A^j M A'^j\right|\varphi\right)^2 + \sum_{j,k,l,m=0}^{\infty}\left|\mathcal{K}\left(w_j, w_j, w_j, w_j\right)\right|\right\}\\
&\leq 3\left|\mathbb{E}u_{i,t}^4\right|\left(\varphi'\Gamma_0\varphi\right)^2\\
&+ \left|\mathbb{E}u_{i,t}^4\right|\sum_{r_1,\cdots r_4=1}^{p}\varphi_{r_1}\varphi_{r_2}\varphi_{r_3}\varphi_{r_4}\sum_{j,k,l,m=0}^{\infty}\left|\mathcal{K}_{r_1,\cdots r_4}\left(A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j}\right)\right|\\
&< \infty
\end{aligned}$$

since $\sum_{j,k,l,m=0}^{\infty}\left|\mathcal{K}_{r_1,\cdots r_4}\left(A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j}\right)\right| < \infty$, which is by Lemma 1 in Hahn and Kuersteiner (2002) for $u_{i,t}$ is i.i.d. and it has finite eighth moments (Assumption E). As a consequence, we have $\mathbb{E}((1/\sqrt{T})\sum_{t=1}^{T}z_{i,t})^4 < \infty$, which provides a sufficient condition for the generalized Lindeberg-Feller condition (Phillips and Moon, 1999, Theorem 2). For the variance term, note that

$$\begin{aligned}
\mathbb{E}\left[vec\left(V_{i,t-1}U'_{i,t}\right)\left(vec\left(V_{i,t-1}U'_{i,t}\right)\right)'\right] &= \mathbb{E}\left[\left(I_p\otimes V_{i,t-1}\right)vec\left(U'_{i,t}\right)\right]\left[\left(I_p\otimes V_{i,t-1}\right)vec\left(U'_{i,t}\right)\right]'\\
&= \mathbb{E}\left(I_p\otimes V_{i,t-1}\right)U_{i,t}U'_{i,t}\left(I_p\otimes V'_{i,t-1}\right)\\
&= \mathbb{E}\left[\left(I_p\otimes V_{i,t-1}\right)\left(\mathbb{E}_{t-1}U_{i,t}U'_{i,t}\right)\left(I_p\otimes V'_{i,t-1}\right)\right]\\
&= \sigma^2\mathbb{E}\left[\left(I_p\otimes V_{i,t-1}\right)ee'\left(I_p\otimes V'_{i,t-1}\right)\right]\\
&= \sigma^2\mathbb{E}\left[\left(I_p\otimes V_{i,t-1}\right)\left(e\otimes 1\right)\right]\left[\left(e'\otimes 1\right)\left(I_p\otimes V'_{i,t-1}\right)\right]\\
&= \sigma^2\mathbb{E}\left(e\otimes V_{i,t-1}\right)\left(e'\otimes V'_{i,t-1}\right)\\
&= \sigma^2\left(M\otimes\Gamma_0\right).
\end{aligned}$$

$\mathbb{E}_{t-1}(\cdot)$ is the conditional expectation given $\{u_{i,s} : s \leq t, \textit{for all } i\}$ and $\mathbb{E}_{t-1}\left(U_{i,t}U'_{i,t}\right) = \mathbb{E}\left(U_{i,t}U'_{i,t}\right)$ because $u_{i,t}$ is *i.i.d.* Note that $vec\left(U_{i,t}\right) = vec\left(U'_{i,t}\right) = U_{i,t}$ and we use the identity $vec\left(BC\right) = \left(I\otimes B\right)vec\left(C\right)$. ■

Proof of Lemma A2 We observe that

$$\begin{aligned}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}V_{i,t-1}^{0}U_{i,t}^{0\prime} &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}V_{i,t-1}U_{i,t}^{\prime}-\sqrt{\frac{N}{T}}\cdot\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}V_{i,t-1}U_{i,s}^{\prime} \\ &= C_{31}(N,T)-C_{32}(N,T).\end{aligned}$$

By Lemma B3, $vec(C_{31}(N,T)) \to_d \mathcal{N}\left(0,\sigma^2\left(M\otimes\Gamma_0\right)\right)$ as $N,T\to\infty$. For $C_{32}(N,T)$, it can be shown that $C_{32}(N,T)\to_p \sqrt{\kappa}\sigma^2\left(I_p-A\right)^{-1}M$ as $N,T\to\infty$. This result requires integrability of $(1/T)\sum_{t=1}^{T}\sum_{s=1}^{T}V_{i,t-1}U_{i,s}^{\prime}$ to satisfy the conditions for joint probability limit in Phillips and Moon (1999, Theorem 1), which indeed directly follows from $C_{12}\left(N,T\right)$ in the proof of Lemma A1.[1] Therefore,

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}vec\left(V_{i,t-1}^{0}U_{i,t}^{0\prime}\right)\to_d \mathcal{N}\left(-\sqrt{\kappa}\sigma^2 vec\left(\left(I_p-A\right)^{-1}M\right),\sigma^2\left(M\otimes\Gamma_0\right)\right). \quad \blacksquare$$

References

[1] Hahn, J. and G. Kuersteiner (2002). Asymptotically unbiased inference for a dynamic panel model with fixed effects, *Econometrica*, 70, 1639-1657.

[2] Phillips, P.C.B. and H.R. Moon (1999). Linear Regression Limit Theory for Nonstationary Panel Data, *Econometrica*, 67, 1057-1111.

[1] In the proof of Theorem 2, we state that $\sqrt{N/T}\cdot T(\widehat{\Gamma}_0^{Y|X}-\Gamma_0^{Y|X})A^{\prime}\to_p -\sqrt{\kappa}\Omega A^{\prime}$ as $N,T\to\infty$. In order to prove this result, however, we also need to check the conditions for joint probability limit as in Theorem 1 of Phillips and Moon (1999) since we are considering probability limit of double indexed process. In this case, the integrability of $\sum_{t=1}^{T}(V_{i,t-1}^{0}V_{i,t-1}^{0\prime}-\Gamma_0)$ is to be verified. (From Lemma A1, all other conditions are obviously satisfied.) Note that

$$\begin{aligned}\left\|\mathbb{E}\sum\nolimits_{t=1}^{T}\left(V_{i,t-1}^{0}V_{i,t-1}^{0\prime}-\Gamma_0\right)\right\| &\leq \left\|\sum\nolimits_{t=1}^{T}\mathbb{E}\left(V_{i,t-1}V_{i,t-1}^{\prime}-\Gamma_0\right)\right\|+\left\|\frac{1}{T}\mathbb{E}\left(\sum\nolimits_{t=1}^{T}V_{i,t-1}\right)\left(\sum\nolimits_{t=1}^{T}V_{i,t-1}^{\prime}\right)\right\| \\ &\leq \left\|\left(I_p-A\right)^{-1}\right\|\left\{\left\|I_p-H_TA\right\|+\left\|\left(I_p-H_T\right)A\right\|\right\}\left\|\Gamma_0\right\|<\infty\end{aligned}$$

because Assumptions S and E imply that A, $\left(I_p-A\right)^{-1}$, TH_T and Γ_0 are all bounded for large T since $\left\|\left(I_p-A\right)^{-1}\right\|\leq\sum_{j=0}^{\infty}\left\|A^j\right\|<\infty$ and $\left\|TH_T\right\|\leq\sum_{t=1}^{T}\left\|A^{t-1}\right\|\leq\sum_{j=0}^{\infty}\left\|A^j\right\|<\infty$.