CLASS FIELD THEORY NOTES

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Abstract. This is the note for Class field theory taught by Professor Jeff Lagarias.

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1. Day 1

1.1. Class Field Theory.
It is a theory that describes all abelian Galois extension $K$ of certain kind of fields $k$.

   (1) Local field (local class field theory)
   $k = \text{finite extension of } \mathbb{Q}_p \text{ p-adic fields or } \mathbb{R} \text{ or } \mathbb{C}.$
   (Local algebraic function field of one variable $\mathbb{F}_q[[T]]$, where $q = p^k$, and $p$ is a prime.)

   (2) Global field
   Algebraic number field $K$, finite extension fields of $\mathbb{Q}$
   Algebraic function fields of one variable over finite field $\mathbb{F}_q$.
   [Further extension of CFT: higher local class field theory ($k = \text{infinite extension of } \mathbb{Q}$)]

1.2. ABC conjecture.
Consider the diophantine equation $a + b = c$ in $\mathbb{Z}$, and define the height function

$$H(a, b, c) = \max |a|, |b|, |c|,$$  \text{ assuming } \gcd(a, b, c) = 1,$$
also define the radical function

$$R(a, b, c) = \prod_{p\mid abc} p.$$ 

Then for each $\epsilon > 0$, we have that $\text{Height} \geq \Radical^{1+\epsilon}$ has finitely many solutions.
Note that ABC conjecture implies the finiteness version of Fermat’s theorem for \( n \geq 5 \) easily, since if you consider \( x^y z^n \), then \( \text{height} = z^n \) and \( \text{radical} R \leq xyz \leq z^3 \) and \( z^n \geq z^5 \geq (z^3)^{1+\epsilon} \).

1.3. History.

- Gauss (1777-1855)
  - Study classes of binary quadratic form \( Q(x, y) = ax^2 + bxt + cy^2 \) under equivalence \( SL(2, \mathbb{Z}) \) substitution, studied which integer \( n \) can be represented by \( Q(x, y) \) which primes can be represented.
  - This equivalence relation presents as an invariant \( d = b^2 - 4ac = \text{disc} \), the discriminant of the quadratic form.
  - Gauss showed that finitely many equivalent classes, called this number \( h(d) \) for \( d < 0 \), definite quadratic form
    \[
    \begin{cases}
    d > 0 \neq \square, & \text{indefinite quadratic form}
    \end{cases}
    \]
  - If restrict to primitive quadratic form, i.e. \( \gcd(a, 2b, c) = 1 \), then define a notion of composition of form made the classes of quadratic form into an abelian group.

Example 1.1. \( Q = x^2 + y^2 \), then \( d = -4 \), and there is one such class, \( h(d) = 1 \).

Moreover, composition of forms respect multiplicative.

- Euler
  - \( p \equiv 1 \mod 4 \) represented
  - \( p \equiv 2 \mod 4 \) represented
  - \( p \equiv 3 \mod 4 \) not represented
  - \( p^2 \equiv 3 \mod 4 \) represented
  - Bilinear identity: \( (x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1y_2 + x_2y_1)^2(x_1y_1 + x_2y_2)^2 \).

- Dirichlet (1805-1859)
  - In 1837, he introduce analytic device Dirichlet \( L \)-function: \( L(s, \chi) = \sum \frac{\chi(n)}{n^s} \) where \( \chi(n) \) is the dirichlet character mod \( m \) (\( s \) is real and \( s = \sigma > 1 \)).
  - \( \chi(n) = 0 \) if \( (n, m) \neq 1 \) and \( \chi(1) = 1 \). This implies that
    \[
    L(s, \chi) = \prod (1 - \frac{\chi(n)}{n^s})^{-1}.
    \]
  - Moreover, he takes a limit as \( \sigma \to 1^+ \), then define \( L(1, \chi) = \lim_{\sigma \to 1} L(s, \chi) \).
  - He thus proved a theorem: There exists infinitely many primes in arithmetic progression \( mx + b \) with \( (m, b) = 1 \), and they have the positive Dirichlet density \( 1/\phi(m) \), where \( \phi(n) \) is the Euler totient function.
  - He moreover gives the Dirichlet class number formula

\textbf{Theorem 1.2.} For \( d > 0 \), let \( t > 0, u > 0 \) be the solution to the Pell equation \( t^2 - du^2 = 4 \) for which \( u \) is smallest, and write \( c = \frac{1}{2}(t + u\sqrt{d}) \).

For \( d < 0 \), write \( w \) for the number of automorphisms of quadratic forms of
discriminant $d$, where $w = \begin{cases} 2, & d < -4; \\ 4, & d = -4; \\ 6, & d = -3. \end{cases}$ Then Dirichlet showed that $h(d) = \frac{w\sqrt{|d|}}{2\pi L(1, \chi)}$, $d < 0$; 
\[ \frac{2\sqrt{d}}{\ln \epsilon} L(1, \chi), \quad d > 0. \]

Note: to a positive definite quadratic form we can associate a Dirichlet series

$$Z_Q(s) = \sum_{(x,y) \in \mathbb{Z}^2 \setminus (0,0)} Q(x, y)^{-s}.$$ Moreover,

$$Z_{x^2+y^2}(s) = \sum_{(x,y)} \frac{1}{(x^2 + y^2)^s} = \phi(s) L(s, \chi_{-4})$$

**Theorem 1.3.**

$$L(1, \chi) = \begin{cases} -\frac{\pi}{|d|^{3/2}} \sum_{m=1}^{n-1} m \left( \frac{d}{m} \right), & d < 0; \\ -\frac{\pi}{|d|^{1/2}} \sum_{m=1}^{q-1} \left( \frac{d}{m} \right) \log(\sin \frac{m\pi}{d}), & q > 0. \end{cases}$$

Therefore we obtain a formula for class number $h(d)$.

- **Kummer(1810-1893)**
  - 1843 Fermat’s Last Theorem proof with a bug.
  - Theory of cyclotomic fields $\mathbb{Q}[\phi_m]$
  - Failure of Unique factorization in $\mathbb{Z}[1, \phi_m, \ldots] = \text{Ring of algebraic integers of cyclotomic field}$.
  - Solved: introduce class number for nonunique factor in cyclotomic field. Introduce ideal number to restore unique factorization.
  - (lots of p-adic analysis)
  - reciprocity law for power residues.

- **Dedekind(1831-1910)**
  - 1871 Ideal theory version of ideals. (Supplement to: Dirichlet book of number theory)
  - Discriminant of number
  - Dedekind Zeta function of number field $K/\mathbb{Q}$. $\phi_K(s) = \sum_{\text{ideals in } \mathcal{O}_K} \frac{1}{N(a)^s} = \prod (1 - \frac{1}{p^s})^{-1}$
  - Note for quadratic field $k = \mathbb{Q}(\sqrt{d})$,

$$\phi_K(s) = \sum_{[Q] \text{ primitive class of disc } d = \text{Disc}(K)} Z_Q(s)$$

- **Kronecker(1823-1891)**
  - Theory of divisors
  - Elliptic function, elliptic modular form
  - 1882, ”Foundation of algebraic numbers”.
  - Splitting of prime ideals in number field $\mathbb{Q}(\theta)$ where $\theta$ is a root of polynomial
\( f(x) \in \mathbb{Z}[x] \) is described by factorial of \( f(x) \equiv 0 \pmod{p} \) (away from finite set of bad primes).

- Standard analysis of Epstein zeta function of \( \theta \).

\[
Z_Q(s) = \sum_{(x,y) \neq (0,0)} \frac{1}{Q(m,n)^s} \quad (\text{Q positive definite})
\]

Consider

\[
E(z,s) = \sum_{(x,y) \neq (0,0)} \frac{y^s}{(mz+n)^{2s}}
\]

(Nonholomorphic Eisenstein attached to \( \mathcal{H}\setminus \text{PSL}(2,\mathbb{Z}), z = x + iy \in \mathcal{H} \))

\[
Q(m,n) = a|mz-n|^2
\]

and \( Q = ax^2 + bxy + cy^2 = a(y - \theta x)(y - \overline{\theta} x) \) where \( z \) is the root in the upper half plane.

- Kronecker Limit Formula

\[
E(z,s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 = \log(\sqrt{\eta(z)^2})) + O(s-1)
\]

where \( \eta(z) = g^{-\frac{1}{24}} \pi (1 - q^n) \) is the Dedekind eta function.

2. Day 2

2.1. Basic Objects:
Algebraic number fields (Finite extension \( K/\mathbb{Q} \)). \( K/\mathbb{Q} \) described in part by set of invariants.

\( n_L = \deg[K : \mathbb{Q}] \), \( \mathcal{O}_K = \text{ring of integers of } K \), \( d_K = \text{discriminant of } K \). \( h_K = r_K + 2s_K \), where \( r_K = \text{number of real conjugates} \), \( s_K = \text{number of complex conjugate fields} \).

\( K = \mathbb{Q}(\theta) \), where \( \theta \) is the root of \( f(x) = x^n + \sum_{i=1}^{n-1} a_i x_i = \prod (x - \theta_i) \). Then \( r_K = \text{number of real } \theta_i \), and \( 2s_K = \text{number of the complex conjugate } \theta, \overline{\theta}. \)

We are interested in \( \text{Cl}(K) = \text{ideal class group of } K \) (really of \( \mathcal{O}_K \)) = fractional ideals of \( K/\mathcal{O}_K \). Principal fractional ideals.

**Decomposition law** of prime ideals of \( K/\mathbb{Q} \) ground fields

\[
(p)\mathcal{O}_K = \prod p_i^{e_i}
\]

Usually all the \( e_i = 1 \), which is called the unramified case. If some \( e_i > 1 \), then \( (p) \) ramifies over \( K \), \( \Rightarrow p|d_K \).

**Unit group** = \( \{ \epsilon \in \mathcal{O}_K : \epsilon^{-1} \in \mathcal{O}_K \} = \{ \epsilon \in \mathcal{O}_K : N_{K/\mathbb{Q}} \epsilon = \pm 1 \} \).

2.2. Analogy of Fields with Riemann surfaces.

\( \mathbb{Q} \leftrightarrow \text{Riemann surface sphere} \leftrightarrow \mathbb{C}(x), \text{the field of rational functions on Riemann sphere } \mathbb{P}^1(\mathbb{C}) \).

In analogy:

\( K \text{algebraic number field} \leftrightarrow \text{Ramified curves of Riemann} \leftrightarrow \text{algebraic function } \mathbb{C}(x) \)

Can calculate the ramified covering using the Riemann-Hurwitz formula

Number field \( K \) analogous to Ramified curves of Riemann surface. In this analogy, branch points correspond to the Ramified primes.
Problem: What do the arithmetic primes correspond to? (Some kind of ramification from $\mathbb{R}$ to $\mathbb{C}$)

**Modern Analogy:** suggest that number field be looked at as 3-dim objects. They have one extra "arithmetic" dimension. [Mazur (1970) cohomological dim of global number field in etale cohomology of $\text{Gal}(\overline{K}/K)$]

### 2.1 History

C.F. Gauss (1777-1855). Raised basic questions in number theory.

1. **Diophantine Equations:** Solved Linear and Quadratic Diophantine equations on 1, 2 and some 3 variables.
2. **Studies multiplicative structure:** reduction and composition of binary quadratic forms.
   - [Class number = classes of definite and indefinite binary forms. $\leftrightarrow$ [ideal class structure of the quadratic number field]
   - [Units = automorphism of binary forms] $\leftrightarrow$ [units of real quadratic field]
3. **Reciprocity Law $\leftrightarrow$ decomposition rule for prime ideals in quadratic fields/\mathbb{Q}$$
4. **Periods (Cyclotomy and Gauss sums) $\leftrightarrow$ work of Kummer: that give all subfields of cyclotomid field $\mathbb{Q}(\phi_m)$
5. **Gauss genus theory:** Classify all quadratic binary forms of discriminant $d$ into genera. Given $Q(x, y) = ax^2 + bxy + cy^2$ and if $d = d_1d_2$ where $d_1$, $d_2$ are field discriminants. Then can check if $Q(x, y)$ represents a prime $p$ where $p \nmid d$, then residue symbol $(\frac{d_1}{p}) = (\frac{d_2}{p}) = \xi_{d_1, d_2}(p)$ genus character value depend on $\mathbb{Q}$. [This defines a character of narrow class group]
   - Classify discriminant $D$ quadratic forms into genera by the vanishing of the auxiliary character. [Beginning of the Class field theory]

Note: $D = df^2$.

$D$ is disc quadratic form. $d$ is discriminant of a quadratic form $d \equiv 0, 1 \mod 4$ square free except above (2). where $2 \nmid d$ or $4|d$ or $8|d$ + leftover part is perfect square $f^2$.

Raise questions:

- to which integers are represented by $Q(x, y) = ax^2 + bxy + cy^2$ ($\gcd(a, b, c) = 4$)
- which primes are represented by $Q(x, y)$

   1. for ensemble of all prop primitive form of disc $D = b^2 - 4ac$, answer in terms of quadratic fields. Consider $Q(1, 0) = a$, know discriminant $D = b^2 - 4ac \equiv 0 \mod a$.
   2. $D \equiv b^2 \mod a$, so $(\frac{D}{a}) - 1 \Leftrightarrow (\frac{a}{D}) = (-1)^{(a-1)(D-1)/4} \Leftrightarrow$ congruence conductor $\mod 4D$.

**Example 2.1.** $D = -23$, then

$$
Q_1 = x^2 + xy + 6y^2 \\
Q_2 = 2x^2 + xy + 3y^2 \\
Q_3 = 2x^2 - xy + 3y^2
$$

$\text{Cl}(-23) = \mathbb{Z}/3\mathbb{Z}$.

Convert to ideal theory on quadratic field
Theorem 2.2. Cohen ANT

Let $K = \mathbb{Q}\sqrt{d}$ be the quadratic field with $d < 0$ as the field discriminant, then

1. $a = \mathbb{Z}[\alpha, \beta] \iff N(\alpha x + \beta y) = N_a = ax^2 + bxy + cy^2$

2. $Q = ax^2 + bxy + cy^2 \Rightarrow b_Q = \mathbb{Z}[a, b - \sqrt{D}]$, then $Q_1 = Q_2 \iff b_{Q_1} = b_{Q_2}$

Primitive quadratic form of discriminant $D = df^2$ ↔ (primitive) ideal classes with respect to an order of quadratic field

$\mathcal{O}_d f^2 = \mathbb{Z}[1, f\sqrt{D}]$ This is a ring (closed under multiplication.) It is a subring of $\mathcal{O}_K$ which is the maximal order.

Definition 2.3. order = subring of $\mathcal{O}_K$ containing 1 whose fraction field is $K$.

Exercise: The order $\mathcal{O} \subset \mathcal{O}_K$, $\mathcal{O} \neq \mathcal{O}_K$ are not Dedekind domain.

2.3. ”class fields” of $K$.

General definition:

Definition 2.4. Abelian Galois extension of $K$.

Original definition:

Definition 2.5. Fields whose decomposition of primes $(p)$ is describles in terms of the class group of $K$ (⇒ subfields of the maximal unramified abelian extension of $K$.)

Example 2.6. Quadratic fields.

The primes represented by quadratic forms of field disc $d$ are grouped into genera.

Principle genus: set of all forms with all genus char =1

principal genera of form ↔ $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$ extends $\mathbb{Q}(\sqrt{d})(d = d_1 d_2)$

Decomposition law for $\mathbb{Q}(\sqrt{d}) = K$. $(p)\mathcal{O}_K = p_1 p_2$ iff $(\frac{d}{p}) = 1$.

Kummer: defined ideal classes of subfield of cyclotomic fields. He gave the decomposition rule of factorization of primes in $\mathbb{Q}(\phi_m)$. (involves residue classes mod $m$)

Kronecker: ”Theorem”: the complete set of abelian Galois extension $\mathbb{Q}$ are given by subfields of cyclotomic fields.

Heinneh Weber(1842-1913)

Kronecker-Weber theorem(1886) proof with mistake.

Lehbuuh der Algebra(1890)

He describes class field by the law of decomposition of primes. $k =$ number field, $K/k$=finite extension.

Definition 2.7. $K/k$ is the class field of $k$ if all prime ideals of degree 1 in $K$ correspond to the principal ideal class in $k$.

Definition 2.8. (Congruence ideal class group in $k =$ ray class group)
3. Day 3

3.1. History (Continued).


Weber definition of Class field: \( K/k \) is a class field for \( k \), if all ideals of degree 1 in \( K \) (over \( \mathbb{Q} \)) correspond to principal prime ideals of degree 1 in \( k \) (over \( \mathbb{Q} \)).

Introduce notion of congruence divisor class group of \( k \) (or congruence ideal class group on \( k \)). \( A_m/H_\tilde{m} \mod m \) where \( A_m = \) all fractional ideals of \( k \) relatively prime to ideal \( (m) \) in \( \mathcal{O}_K \), and \( H_\tilde{m} = \) all principal ideals \( (\alpha) \) in \( \mathcal{O}_K \) with \( \alpha \equiv 1 \mod m \).

(\text{It is possible to include conditions at real places of } k \text{ also, } \alpha \text{ has absolute conjugates } \alpha_{\sigma_1}, \alpha_{\sigma_2}, \ldots, \alpha_{\sigma_n} \text{ that at real places may impose extra condition } \alpha_{\sigma_j} \not\equiv 0 \text{ (positivity condition) } \mod \infty.)

Congruence modulo \( \tilde{m} = m \prod_{j \text{ certain}} p_\infty \Rightarrow \) congruence class group \( A_m/H_\tilde{m} \).

**Definition 3.1.** \( K/k \) is a class field with respect to \( A_m/H_\tilde{m} \) if the degree 1 prime ideals of \( K/Q \) correspond to principal prime ideals \( (\pi) \) of \( k \) with a generator \( \pi \in H_\tilde{m} \).

He asserted the following theorems:

- **Isomorphism theorem.**
  A class field \( K/k \) for \( A_m/H_\tilde{m} \) is Galois over \( k \) and \( \text{Gal}(K/k) A_m/H_\tilde{m} \). Hence it is an abelian extension of \( k \).

- **Uniqueness theorem.**
  There is exactly one such class field \( K \).

- **Ordering theorem.**
  If \( H \subseteq H' \) (if \( H_\tilde{m} \subset H'_\tilde{m} \)) \( \iff \) \( K \supseteq K' \).

- **Conjectured: Existence theorem.**
  To every congruence divisor class \( H_\tilde{m} \), there exist a class field \( K/k \).

Reference: Cox, Primes of the form \( x^2 + ny^2 \).

3.1.2. D. Hilbert (1862-1943).

1897 Zahbencht. Reformulate class field theory for fields with class number \( h(k) = 2 \) (of ideal class group) associated class fields, relative quadratic extension \( K/k \) where \( [K:k] = 2 \) abelian extension.

1900 23 problems.

Problem 8: (Analytic NT) RH (including the GRH) and Goldbach.

Problem 9: (General reciprocity law) for number field

Problem 10: Determine of solvability of Diophantine equation over \( \mathbb{Z} \) (\( \mathbb{Q} \)). (Solved over \( \mathbb{Z} \))

Problem 11: Deduce a theory of solvability of quadratic forms in \( n \) variables with algebraic number coefficients.

Problem 12: Extension of Kronecker theorem on Abelian Field to any algebraic realm of rationality.

Asked: Can one find special functions where values generate class field, which play the same role as in exponential functions in the field of rationals and the elliptic modular function in the imaginary quadratic field.

Example: Kronecker weber theorem.
\{\text{Congruence class group mod } mp_{\infty} \text{ for } \mathbb{Q}\} \leftrightarrow \mathbb{Q}(\phi_m)

\{\text{integers prime to } m\}/\{\text{integers } \equiv 1 \mod m\} \leftrightarrow \mathbb{Q}(\phi_m + \bar{\phi_m})

Theory of relatively abelian fields for \( h(d) = 2 \), relative quadratic field.

Show for a general field \( K/\mathbb{Q} \).

- **existence theorem**
  - There exists a relatively abelian extension of degree 2 which is unramified.

- **uniqueness theorem**
  - There exists a unique unramified quadratic extension if \( h(d) = 2 \).

- **decomposition**
  - \( p \) in \( k \) splits completely in \( K/k \) if we have \( N_{K/k}p = (\pi) \) which is a principal prime ideal of degree 1.

- **Principal divisor theorem**
  - In class field \( K/k \) all ideals \( a \) in \( k \) has \( a\mathcal{O}_K = (\alpha) \) a principal ideals in Hilbert class field of \( k \).

- **Discriminant Theorem**
  - The Hilbert class field has relative discriminant 1 (i.e. unramified \( K/\mathbb{Q} \)).

  Hilbert introduced a symbol for \( a, b \in k \),

  \[ (\frac{a,b}{p})_2 = \pm 1 \]

  for principal ideals \( p \) in \( k \) with \( p \nmid ab \), and that

  \[ (\frac{a,b}{p})_2 = 1 \]

  for all primes \( p \) \( \leftrightarrow \) \( a \) is a relative norm from \( k(\sqrt{b})/k \). This has symmetric property

  \[ (\frac{a,b}{p})_2 = (\frac{b,a}{p})_2 \]

  \[ . \]

  Hilbert conjectured that this theory exists to all relative abelian extensions of \( k \).

  \( A_m/H_m \) also has subgroup \( H_m \subseteq H \subseteq A_m \), get

  \[ A_m/H \leftrightarrow \text{subfield of ray class group mod } k \]

  Conjecture also the the hilbert class field has all ideals of \( k \) becomes principal in \( K \).

3.1.3. **Takage(1920,1922).** Established hilbert form of class field theory.

**Definition 3.2.** New definition: \( K/k \) is called a class field to a congruence divisor class group \( A_m/H_m \) if the principal divisor class \( H_m \mod p \) contains all relative norms of divisor in \( K \) prime to \( (m) \), and

\[ [A:H] = |A_m/H_m| = h = [K:k] \]

\[ . \]

\[ [\text{In general all such fields } K \text{ has } h \leq n] \]
3.2. Number fields: Statistical question.
Let $K/\mathbb{Q}$ number fields.

Let $n_K = [K : \mathbb{Q}]$

$d_K = \text{disc}(K/\mathbb{Q})$, where $d_K \equiv 0, 1 \mod 4$.

$h(K) = \text{class number of } K$ (size of wide divisor class group)

$h^*(K) = \text{narrow class number of } K$

Question: How many number fields are there ordered by discriminant?
(There are finitely many number field with a given absolute discriminant $d$).

Conjecture: $\# \{ K : |\text{disc}(K)| \leq x \} < c_x x^{1+\epsilon}$ for any $\epsilon > 0$

Example: quadratic field

$\# \{ K : |\text{disc}(K)| \leq x \} \leq x$ for any $\epsilon > 0$ as $x \to \infty$

Question: Are there infinitely many number field of class number 1? Also, are there infinitely many Galois extensions of $\mathbb{Q}$ with class number 1?

Conjecture: yes, even in degree $[k : \mathbb{Q}] = 2$. Cohen-Lenstra Heuristics (predict).

By Brauer-Siegel theorem, series of fields $\{k_n : n \geq 1\}$ such that

$$\frac{[k_n : \mathbb{Q}]}{\log d_{k_n}} \to 0$$

, and thus

$$\frac{\log h(k_n) R(k_n)}{\log \sqrt{|d_{k_n}|}} \to 1$$

as $n \to \infty$, i.e. $h(k_n) R(k_n) \sim (d_{k_n})^{0.5+\epsilon}$

**Example 3.3.** $k_n = \mathbb{Q}(p_n)$, where $p_n \equiv 1 \mod 4$. Then $d_n = p_n [k_n : \mathbb{Q}] = 2$.

So $\frac{2}{\log p_n} \to 0$ as $n \to \infty$. So can conclude from Brauer-Siegel, $h(\mathbb{Q}(\sqrt{p_n})) \log |\epsilon_{p_n}| \sim (p_n)^{1/2+\epsilon}$.

To make the class number small, the unit $\epsilon_{p_n}$ must be exponentially large in $p_n$, can compute the unit in the continued fraction expression of $\sqrt{p_n} >> \frac{p_n^{1/2}}{\log p_n}$ terms.

**Example 3.4.** $K = \mathbb{Q}(\sqrt{377}) = \mathbb{Q}(\sqrt{13 \cdot 29})$, in this example, $Cl(K) = \mathbb{Z}/2\mathbb{Z}$, and the narrow class group $Cl^+(K) = \mathbb{Z}/4\mathbb{Z}$

$$K = \mathbb{Q}(\sqrt{13}, \sqrt{29}, \sqrt{\eta})$$ (Narrow Hilbert Class Field)

$$H = \mathbb{Q}(\sqrt{13}, \sqrt{29})$$ (Genus field, also HCF in this example)

$$k = \mathbb{Q}(\sqrt{377})$$

Notice that in this example, we have that $K$ is the narrow Hilbert class field which is the maximal abelian extension which is unramified at all finite prime, $H$ is the Hilbert class field which is maximal unramified extension which is also unramified at the archimedian place, i.e.
it is totally real. Moreover, $H$ is the genus field, i.e., it is the maximal unramified extension of $k$ which is Galois over $\mathbb{Q}$, and abelian over $\mathbb{Q}$. 