Secret Calculus Note

Abstract

This is the secret calculus I note taken by Yiwang Chen. Please notify me at yiwchen@umich.edufor any typo (I am sure there will be some somewhere).

1 General Functions

Definition 1.1. A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number.
The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.
The input is called the **independent variable** and the output is called the **dependent variable**.

Definition 1.2. The **Rule of Four**: Tables, Graphs, Formulas, and Words

Definition 1.3. We denote the **graph of a function** to be a set of points (x, y) with an extra property $y = f(x)$.

Definition 1.4. A function f is **increasing** if the values of $f(x)$ increase as x increases.
A function f is **decreasing** if the values of $f(x)$ decrease as x increases.
Moreover, A function f(x) is *monotonic* if it increases for all x or decreases for all x.

Remark. The graph of an increasing function climbs as we move from left to right. The graph of a decreasing function falls as we move from left to right.

Definition 1.5. The graph of function is **Concave up** if it bends upwards as we move left to right ("Smile Face").
The graph of function is **Concave down** if it bends downwards as we move left to right ("Sad Face").

Definition 1.6. Composition of function is basically combining of functions so that the output from one function becomes the input for the next function. We usually denote $f(g(x))$ as "f composed with g of x".

Definition 1.7. Let $f : A \to B$ be a function. If for any $b \in f(A)$, there is a unique $a \in A$ such that $b = f(a)$. Then we see that given any b we can track back to see what a it corresponds to. Thus we have another function $f^{-1} : B \to A$ such that $f^{-1}(b) = a$. Therefore,

$$f^{-1}(b) = a \iff f(a) = b$$

and we have $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
A function f is **invertible** if whenever $f(a_1) = f(a_2)$, it necessarily follows that $a_1 = a_2$. (This is so-called the horizontal test.

Definition 1.8. A function $f(x)$ is said to be **periodic** with period p if $f(x) = f(x+np)$ for any n which is an integer. (We denote the least p to be the period.)

Interpretation: Period is the smallest time needed for the function to run a complete cycle.

Definition 1.9. A periodic function come with three property, Amplitute, Midline, Period, where Amplitute A is defined as

$$A = (\max - \min)/2$$

, Midline $y = m$ is defined as

$$y = m = (\max + \min)/2$$

Example 1.1. $f(t) = \sin t$ is of period 2π, amplitute $A = 1$, and midline is $y = 0$.
$f(t) = \cos t$ is of period 2π, amplitute $A = 1$, and midline is $y = 0$.
$f(t) = \tan t$ is of period π.

Definition 1.10. A function f is defined on an interval around c (not necessarily defined on c). The **limit** of the function $f(x)$ as x approaches c, written $\lim_{x \to c} f(x)$, to be a number L such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write $\lim_{x \to c} f(x) = L$

Remark. Informally, we write $\lim_{x \to c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c.

Remark. Assuming all the limits on the right hand side exist:

1. If b is a constant, then $\lim_{x \to c}(bf(x)) = b \lim_{x \to c} f(x)$.
2. $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
3. $\lim_{x \to c}(f(x)g(x)) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$
4. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$
5. $\lim_{x \to c} k = k$, for k a constant
6. $\lim_{x \to c} x = c$

Definition 1.11. Formal definition for **Asymptotes**.
If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \to \infty$ or $x \to -\infty$, then the line $y = L$ is called a horizontal asymptote, i.e. The **horizontal asymptotes** are

$$y = \lim_{x\to\infty} f(x)$$

, and

$$y = \lim_{x\to-\infty} f(x)$$

if they exist.
If the graph of $y = f(x)$ approaches a vertical line $x = K$ as $x \to K^+$ or $x \to K^-$, then the line $x = K$ is called a vertical asymptote, i.e. The **vertical asymptotes** are all the $x = K$ such that either of the following holds.
$\lim_{x\to K^+} f(x) = \infty$ or $\lim_{x\to K^+} f(x) = -\infty$ or $\lim_{x\to K^-} f(x) = \infty$ or $\lim_{x\to K^-} f(x) = -\infty$

Example 1.2.

$$\frac{b_t x^t + \ldots + b_1 x + b_0}{c_r x^r + \ldots + c_1 x + c_0} \text{ has HA: } \begin{cases} y = \frac{b_t}{c_r} & r = t \\ y = 0 & r > t \\ \text{None} & \text{else} \end{cases}$$

and has VA at $x = K$, if K is a root of $a_n x^n + \ldots + a_1 x + a_0$.

2 Continuity

The function f is **continuous** at $x = c$ if f is defined at $x = c$ and if $\lim_{x\to c} f(x) = f(c)$.
The function is continuous on an interval $[a, b]$ if it is continuous at every point in the interval.
Suppose that f and g are continuous on an interval and that b is a constant. Then, on that same interval,

1. $bf(x)$ is continuous.
2. $f(x) + g(x)$ is continuous.
3. $f(x)g(x)$ is continuous.
4. $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

3 Shifting and Stretches

Given function $y = f(x)$. There are six transformation onto the function.
Vertical Transformation.

1. Vertical Shift.

   To shift the function $y = f(x)$ vertically by k, we have the new function $y = f(x) + k$. If $k > 0$, we shift the graph upwards and if $k < 0$, we shift the graphs downwards.

2. Vertical compression or stretch.

   To vertically compress/stretch the function $y = f(x)$ by a factor of a, we have the new function as $y = af(x)$. Here, a is always positive. If $0 < a < 1$, we are compressing the graph vertically by a. If $a > 1$ we are stretching the graph vertically by a.

3. Reflection along x-axis.

   We reflect a function along x-axis by multiplying a -1 outside our function.

Horizontal Transformation.

1. Horizontal shift.

   To shift the function $y = f(x)$ horizontally by t, we have the new function as $y = f(x - k)$. Note we are subtracting t here. If $t > 0$, we are shifting the graph towards right and if $k < 0$, we are shifting the graphs towards left.

2. Horizontal compression/stretch.

   To horizontally compress/stretch the function $y = f(x)$ by a factor of r, we have the new function as $y = f(\frac{x}{r})$. Note that we are dividing r here.

   if $r > 1$ we are horizontally compressing the graph horizontally, and if $0 < r < 1$ we are horizontally stretching the graph horizontally.

3. Reflection along y-axis.

   We reflect a function along y-axis by multiplying a -1 inside our function.

4 Linear Functions

Definition 4.1. A **linear function** has the form $y = f(x) = b + mx$. Its graph is a line such that

- m is the **slope, or rate of change** of y with respect to x. b is the **vertical intercept**, or value of y when x is zero.

Remark. Two different points on the graph determine a unique linear function that has the function graph through them.

Definition 4.2. y is proportional to x if there is a nonzero constant k such that $y = kx$. This k is called the constant of proportionality

5 Exponential Functions and Logarithm

Definition 5.1. A **Exponential Functions** has the form $P(t) = P_0 \cdot a^t = e^{kt} = (1+r)^t$.

Definition 5.2. The logarithm of x to base b, denoted $\log_b(x)$, is the unique real number y such that $b^y = x$. Therefore, logarithmic function is the inverse function for exponential function.
In the case where $b = e$, we denote it as $\ln(x)$, i.e. $\log_e(x) = \ln(x)$.
In the case where $b = 10$, we denote it as $\log(x)$, i.e. $\log_{10}(x) = \log(x)$

Remark. Properties of Log Function:
(1) $\log_a(a^x) = x$; (2) $a^{\log_a(y)} = y$;
(3) $\log_a(xy) = \log_a(x) + \log_a(y)$; (4) $\log_a(1) = 0$;
(5) $\log_a(1/x) = \log_a(x)$; (6) $\log_a(x) = ln(x)/ln(a)$.

Definition 5.3. The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
The **doubling time** of an exponentially increasing quantity is the time required for the quan tity to double.

6 Trigonometric Function

Definition 6.1. Unit circle Consider the set of points $\{(x, y) : x^2 + y^2 = 1\}$. It is called the unit circle centered at the origin. (Warning: this is not a graph of a function since it fails the vertical line test.)

Definition 6.2. Angle θ is defined as the arc length on the unit circle measured counterclockwise from the x-axis along the arc of the unit circle.

Definition 6.3. Thus angle is a real number and it is of the unit (**radian**). The **angle in radian and the angle in degree is connected in the following way**,

$$360^\circ = 2\pi(\text{radian})$$

Here I bracket the radian since radian is considered as unit-free.

Remark. arc formula connecting the arc length with the radius and angle, i.e.

$$\text{Arc length} = l = r\theta$$

Definition 6.4. cos and sin is the horizontal coordinate of the arc endpoint and the endpoint coordinate of the arc endpoint. Moreover, we define tangent of the angle θ as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, and secant of the angle θ as $\sec(\theta) = 1/\cos(\theta)$.

Remark. We have those identity coming from the definition.

1. $\cos^2(\theta) + \sin^2(\theta) = 1$,
2. $1 + \tan^2(\theta) = \sec^2(\theta)$,
3. $\cos(x) = \cos(x + 2\pi)$,
4. $\sin(x) = \sin(x + 2\pi)$,
5. $\tan(x) = \tan(x + \pi)$,
6. $\cos(t + \pi) = -\cos(t)$,
7. $\sin(t + \pi) = -\sin(t)$,
8. $\sin(x + \pi/2) = \cos(x)$,
9. $\cos(x - \pi/2) = \sin(x)$,
10. $\cos(x) = \cos(-x)$,
11. $\sin(x) = -\sin(-x)$,
12. $\tan(x) = -\tan(-x)$

Remark. Since the function graph does not pass the vertical line test, we cannot define an inverese trigonometry function in general. However, restricted the function to some certain domain, we can define the inverse of the trigonometry function.

7 Power Function, Polynomials, and Rational Function

Definition 7.1. A function f is called a **Power functions** if we can write $f(x) = kx^p$ where k, p are fixed constants.

Definition 7.2. A **Polynomials** is a function that is in the form

$$f(x) = a_n x^n + a_{n1} x^{n1} + \ldots + a_1 x + a_0$$

where $n \geq 0$ is a fixed integer and $a_n, a_{n1}, \ldots, a_1, a_0$ are fixed constants. Moreover, we call $a_n x^n$ as its leading coefficient.

Definition 7.3. A **Rational functions.** is a function $f(x) = \frac{p(x)}{q(x)}$ for all x in its domain, with $p(x)$ and $q(x)$ are both fixed polynomials.

Theorem 7.1. Fundamental Theorem of Algebra
Any polynomial with degree n can be factored as $f(x) = a_n(x - r_1)\ldots(x - r_N)(x^2 + b_1 x + c_1)\ldots(x^2 + b_M x + c_M)$ with $n = N + 2M$ and each $x^2 + b_j x + c_j$ has no real roots.

8 Derivative

Definition 8.1.

Average rate of change of f over the interval from a to (a + h) $= \dfrac{f(a+h)-f(a)}{h}$

Definition 8.2. The derivative of f at a is defined as

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If the limit exists, then f is said to be differentiable at a.

Remark. Differentiable function is continuous function, while continuous function fails to be differentiable function easily!

Definition 8.3. The derivative function
For any function f, we dene the derivative function, f', by

$$f'(x) = \text{Rate of change of f at x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

For every x-value for which this limit exists, we say f is differentiable at that x-value.
If the limit exists for all x in the domain of f, we say f is differentiable everywhere.

Remark. If $f > 0$ on an interval, then f is increasing over that interval.
If $f < 0$ on an interval, then f is decreasing over that interval.

Theorem 8.1. Power Function. If $f(x) = x^n$, then $f(x) = nx^{n-1}$.

Interpretation of derivatives

- NEVER use mathematical terminologies to write down a sentence interpretations.
- The statement $y = f'(x)$ should be interpreted as if x is increased by *(1,0.1,0.01,..., whatever is comparably small in the context), the resulting increase of f is approximately $* \times y$. (With units)
- The statement $y = (f^{-1})'(x)$ should be interpreted as if x is increased by *(1,0.1,0.01,..., whatever is comparably small in the context), the resulting increase of f^{-1} is approximately $* \times y$. (With units) You need to find out the meaning of f^{-1} as well in this case.

Definition 8.4. The second derivative: For a function f, the derivative of its derivative is called the second derivative, and written f'' (read f double-prime). If $y = f(x)$, the second derivative can also be written as $\frac{d^2y}{dx^2}$, which means $\frac{d}{dx}(\frac{dy}{dx})$, the derivative of $\frac{dx}{dy}$.

Proposition 8.1. *Property of second derivative:*
If $f'' > 0$ on an interval, then f' is increasing, so the graph of f is concave up there.
If $f'' < 0$ on an interval, then f' is decreasing, so the graph of f is concave down there.
If the graph of f is concave up on an interval, then $f'' \geq 0$ there.
If the graph of f is concave down on an interval, then $f'' \leq 0$ there.

Remark. **Important:** a straight line is neither concave up nor concave down, and has second derivative zero.

Remark. **Derivative Rules:**

1. General rules:

   (a) **Derivative of a constant times a function**

   $$\frac{d}{dx}[cf(x)] = cf'(x)$$

   (b) **Derivative of Sum and Difference**

   $$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$
   $$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

   (c) **Product Rule and Quotient Rule**
   Product Rule:
   $$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$
   or equivalently,
   $$\frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right) = f\left(x\right)\frac{d}{dx}g\left(x\right) + \frac{d}{dx}f\left(x\right)g\left(x\right)$$
   Quotient Rule:
   $$(\frac{f}{g})'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$$
   or equivalently,
   $$\frac{d}{dx}\left(\frac{f\left(x\right)}{g\left(x\right)}\right) = \frac{\frac{d}{dx}f\left(x\right)g\left(x\right) - f\left(x\right)\frac{d}{dx}g\left(x\right)}{g^2\left(x\right)}$$

   (d) **Chain rule:**
   $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.$$
   or equivalently,
   $$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

(e) **Derivative of general inverse function**

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}.$$

2. Rules for specific type functions:

(a) **Power Rule**

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(b) **Exponential Rule**

$$\frac{d}{dx}(a^x) = \ln a a^x$$

(c) **Trignometric Rules**

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$

(d) **Logarithmetic Rules**

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

(e) **Anti-Trig Rules**

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) - = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

3. **Implicit function derivative.**