Math 116 — Practice for Exam 3

Generated April 19, 2021

Name: SOLUTIONS

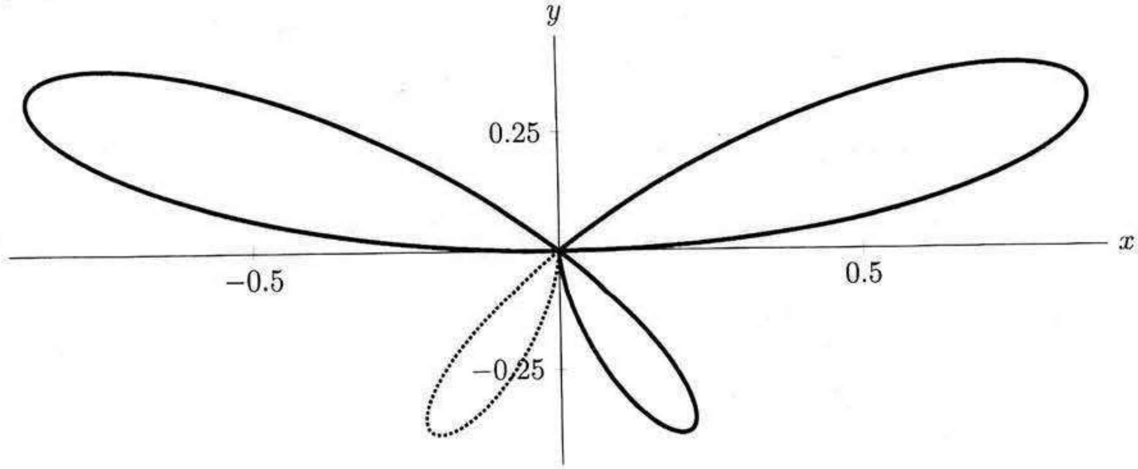
Instructor: ______________________ Section Number: ________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2018	3	4	butterfly	9	
Winter 2019	3	2	castar	8	
Fall 2010	2	2	cardioid	14	
Fall 2015	2	4	moon	10	
Fall 2014	3	5	wild chickens	11	
Total				52	

Recommended time (based on points): 55 minutes

4. [9 points] The polar curve $r = \sin(4\theta)\cos(\theta)$ for $0 \le \theta \le \pi$ is shown below.



Note that there are two "large loops" and two "small loops".

For reference, note that for this curve, $\dfrac{dr}{d\theta} = 4\cos(\theta)\cos(4\theta) - \sin(\theta)\sin(4\theta)$

a. [3 points] For what values of θ does the polar curve $r = \sin(4\theta)\cos(\theta)$ trace once around the "small loop" in the third quadrant? (This portion of the curve is indicated by the dotted line.) Give your answer as an interval of θ values between 0 and π.

Look at signs of x and y to determine quadrant of points:

θ	$\sin\theta$	$\cos\theta$	$\sin 4\theta$	$r=\sin 4\theta\cos\theta$	$x=r\cos\theta$	$y=r\sin\theta$	
0 to $\pi/4$	+	+	+	+	+	+	Q1
$\pi/4$ to $\pi/2$	+	+	−	−	−	−	$\boxed{Q3}$
$\pi/2$ to $3\pi/4$	+	−	+	−	+	−	Q4
$3\pi/4$ to π	+	−	−	+	−	+	Q2

Answer: $\boxed{\frac{\pi}{4} < \theta < \frac{\pi}{2}}$

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total arc length of the two small loops.

$$\text{Arclen} = 2\int_{\pi/4}^{\pi/2} \sqrt{r^2 + (r')^2}\, d\theta$$

Answer: Arc Length = $2\int_{\pi/4}^{\pi/2} \sqrt{(\sin 4\theta\cos\theta)^2 + (4\cos 4\theta\cos\theta - \sin 4\theta\sin\theta)^2}\, d\theta$

c. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the area of the region that is enclosed by the polar curve $r = 2$ but is outside the curve $r = \sin(4\theta)\cos(\theta)$.

$|r| = |\sin 4\theta|\cdot|\cos\theta| \le 1\cdot 1 < 2$, So the butterfly is contained in the circle of radius 2.

Area inside butterfly $= \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_0^{\pi} \sin^2 4\theta\cos^2\theta\, d\theta$

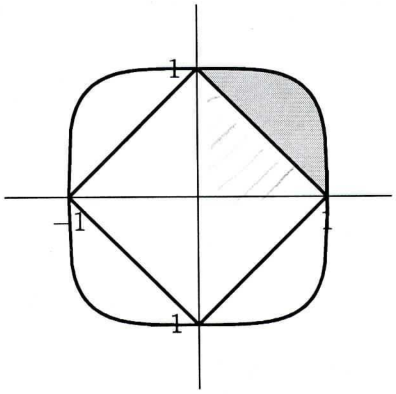
Answer: Area = $\boxed{4\pi - \frac{1}{2}\int_0^{\pi} \sin^2 4\theta\cos^2\theta\, d\theta}$

2. [8 points]

The *castar*, a coin widely used in Middle-Earth, allegedly has the shape graphed to the right. The outer perimeter can be modeled by the implicit equation

$$x^4 + y^4 = 1$$

and the perimeter of the hole in the middle is a square. To help his fellow Hobbits detect counterfeit coins, Samwise Gamgee, the Mayor of the Shire, is working on obtaining the specifications of a genuine castar. Sam needs your help.



a. [2 points] Find a function $f(\theta)$ so that the outer edge of the castar is given by the function $r = f(\theta)$.

$x = r\cos\theta$, $y = r\sin\theta$, so

$(r\cos\theta)^4 + (r\sin\theta)^4 = 1$

$\Rightarrow r^4\cos^4\theta + r^4\sin^4\theta = 1$

Answer: $f(\theta) = \left(\dfrac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/4}$

b. [3 points] Write an expression involving one or more integrals that gives the total area of the quarter of a castar in the first quadrant (shaded above).

Area of shaded triangle $= \frac{1}{2}$, so

area of quarter coin $= \frac{1}{2}\int_0^{\pi/2} f(\theta)^2\, d\theta - \frac{1}{2}$

Answer: $\dfrac{1}{2}\displaystyle\int_0^{\pi/2} \left(\frac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/2} d\theta - \frac{1}{2}$

c. [3 points] Approximate the area of a castar by estimating your integral(s) from part (b) using TRAP(2). Write out all the terms in your sum(s).

For $\int_0^B g(x)\,dx$, $\text{TRAP}(2) = \Delta x\left[\frac{1}{2}g(0) + g\left(\frac{B}{2}\right) + \frac{1}{2}g(B)\right]$, where $\Delta x = \frac{B}{2}$.

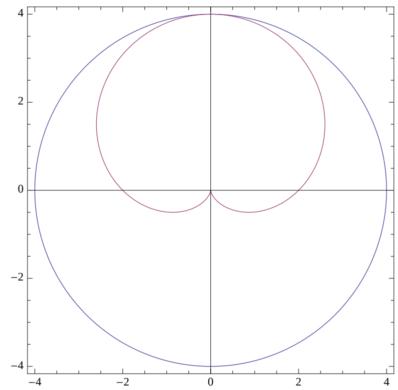
In our case $B = \frac{\pi}{2}$ so $\frac{B}{2} = \Delta x = \frac{\pi}{4}$.

four quarters

Answer: $4\left[\frac{1}{2}\cdot\frac{\pi}{4}\left[\frac{1}{2}\left(\frac{1}{\cos^4(0)+\sin^4(0)}\right)^{1/2} + \left(\frac{1}{\cos^4(\frac{\pi}{4})+\sin^4(\frac{\pi}{4})}\right)^{1/2} + \frac{1}{2}\left(\frac{1}{\cos^4(\frac{\pi}{2})+\sin^4(\frac{\pi}{2})}\right)^{1/2}\right] - \frac{1}{2}\right]$

$= 4\left[\frac{\pi}{8}(1+\sqrt{2}) - \frac{1}{2}\right] = \frac{\pi}{2}(1+\sqrt{2}) - 2 \approx 1.79$

2. [14 points] The graph of the circle $r = 4$ and and the cardioid $r = 2\sin\theta - 2$ are shown below.



a. [3 points] Write a formula for the area inside the circle and outside the cardioid in the first quadrant.

Solution: Area of the quarter of a circle$= 4\pi$

Area of cardioid$=\int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$

Area$=4\pi - \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$

b. [7 points] At what angles $0 \le \theta < 2\pi$ is the minimum value of the y coordinate on the cardioid attained? No credit will be given for answers without proper mathematical justification.

Solution:

$y(\theta) = (2\sin\theta - 2)\sin\theta$

$y'(\theta) = 2\cos\theta\sin\theta + (2\sin\theta - 2)\cos\theta = 4\cos\theta\sin\theta - 2\cos\theta$

Critical points $\quad (4\sin\theta - 2)\cos\theta = 0$

$\cos\theta = 0 \quad$ or $\quad \sin\theta = \frac{1}{2} \quad$ then $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

Minimum y coordinate at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

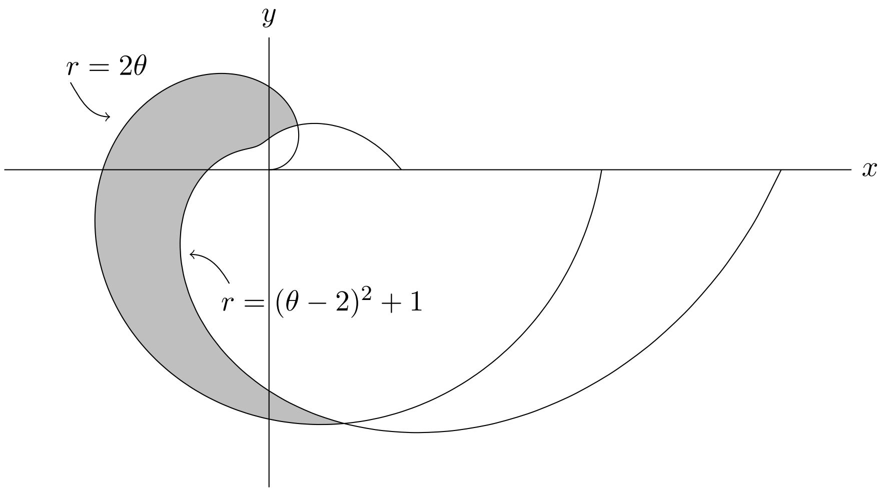
c. [4 points] Write an integral that computes the value of the length of the piece of the cardioid lying below the x-axis.

Solution:

$x(\theta) = (2\sin\theta - 2)\cos\theta \qquad x'(\theta) = 2\cos^2\theta - (2\sin\theta - 2)\sin\theta$

$L = \int_0^{\pi} \sqrt{(2\cos^2\theta - (2\sin\theta - 2)\sin\theta)^2 + (4\cos\theta\sin\theta - 2\cos\theta)^2} d\theta$

4. [10 points] The visible portion of the strangely-shaped moon of the planet Thethis during its waxing crescent phase is in the shape of the region bounded between the polar curves $r = 2\theta$ and $r = (\theta-2)^2+1$. The region is pictured below. Assume x and y are measured in thousands of miles.



a. [6 points] Write an expression involving integrals which gives the area of the visible portion of this moon. Include the units of the integral in your answer. Do not evaluate any integrals.

Solution: The two curves intersect when $\theta = 1, 5$. Therefore the area of the moon is

$$\left(\int_1^5 \frac{1}{2}(2\theta)^2 d\theta - \int_1^5 \frac{1}{2}((\theta-2)^2+1)^2 d\theta\right) \text{ (thousand miles)}^2$$
$$= \left(\int_1^5 \frac{1}{2}(2\theta)^2 d\theta - \int_1^5 \frac{1}{2}((\theta-2)^2+1)^2 d\theta\right) \text{million miles}^2.$$

b. [4 points] Find the slope of the tangent line to the polar curve $r = (\theta-2)^2+1$ at $\theta = \pi$.

Solution: Converting to parametric equations, we have

$$x = r\cos\theta = ((\theta-2)^2+1)\cos\theta$$
$$y = r\sin\theta = ((\theta-2)^2+1)\sin\theta$$

Thus

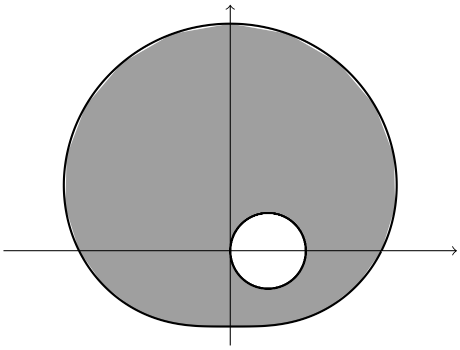
$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{\left.\frac{dy}{d\theta}\right|_{\theta=\pi}}{\left.\frac{dx}{d\theta}\right|_{\theta=\pi}} = \frac{2(\pi-2)\sin\pi + ((\pi-2)^2+1)\cos\pi}{2(\pi-2)\cos\pi - ((\pi-2)^2+1)\sin\pi} = \frac{(\pi-2)^2+1}{2(\pi-2)}.$$

5. [11 points] Franklin's robot army is surrounding you!

a. [6 points] Consider the polar curves

$$r = \cos(\theta) \qquad\qquad r = \sin(\theta) + 2$$

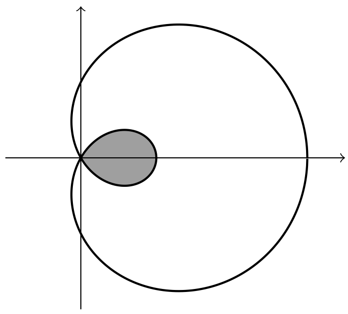
Franklin's robot army occupies the shaded region between these two curves. Write an expression involving integrals that gives the **area** occupied by Franklin's robot army. Do not evaluate any integrals.



Solution:

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\sin(\theta)+2)^2 d\theta - \frac{1}{2}\int_0^{\pi} (\cos(\theta))^2 d\theta$$

b. [5 points] Your friend, Kazilla, pours her magic potion on the ground. Suddenly, a flock of wild chickens surrounds you. The chickens occupy the shaded region enclosed within the polar curve $r = 1 + 2\cos(\theta)$ as shown below. Write an expression involving integrals that gives the **perimeter** of the region occupied by the flock of wild chickens. Do not evaluate any integrals.



Solution: We use the arc length formula:

$$\text{Arc Length} = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2} d\theta$$

Note that $r'(\theta) = -2\sin(\theta)$. Also, the shaded region of lies between $\theta = 2\pi/3$ and $\theta = 4\pi/3$ (you can see this by setting $r(\theta) = 0$, and testing that $r(\pi) = -1$, so it lies on the boundary of the shaded region.)

$$\text{Arc Length} = \int_{2\pi/3}^{4\pi/3} \sqrt{(1+2\cos(\theta))^2 + (-2\sin(\theta))^2} d\theta$$