Math 116 — Practice for Exam 3

Generated April 19, 2021

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2015	2	1	spinning	16	
Winter 2013	2	5	heart	12	
Fall 2014	1	6	soup	11	
Winter 2016	2	3	O ghan	13	
Fall 2016	2	10	tracking chip	14	
Total				66	

Recommended time (based on points): 59 minutes

1. [16 points] Carla and Bobby run a race after spinning in circles for a good amount of time to make themselves dizzy. They start at the origin in the xy-plane and they race to the line $y = 5$. Assume the units of x and y are meters.
Bobby's position in the xy-plane t seconds after the races starts is

$$(-\sqrt{3}t\cos t, \frac{1}{\sqrt{3}}t\sin t)$$

and Carla's position in the xy-plane t seconds after the race starts is

$$(t\sin t, -t\cos t).$$

a. [4 points] Write an integral that gives the distance that Carla travels during the first two seconds of the race. Do not evaluate your integral.

Solution: We have

$$\frac{dx}{dt} = \sin(t) + t\cos(t),$$
$$\frac{dy}{dt} = -\cos(t) + t\sin(t).$$

The distance traveled by Carla in the first two seconds of the race is then given by $\int_0^2 \sqrt{(\sin(t) + t\cos(t))^2 + (t\sin(t) - \cos(t))^2}dt$.

b. [3 points] Find Carla's speed at $t = \pi$.

Solution: We have that Carla's speed is given by the function $\sqrt{(\sin(t) + t\cos(t))^2 + (t\sin(t) - \cos(t))^2}$, and so we need only plug in $t = \pi$ which gives us the value below.

Carla's speed at $t = \pi$ is $\sqrt{\pi^2 + 1}$ **m/sec**

c. [4 points] Carla and Bobby are so dizzy that they run into each other at least once during the race. Find the first time $t > 0$ that they run into each other, and give the point (x, y) where the collision occurs.

Solution: Setting the x and y coordinate functions equal gives us that collisions will occur when $\tan(t) = -\sqrt{3}$. The first time for $t > 0$ when this occurs is $2\pi/3$. Plugging this t value into either the equations for Bobby's or Carla's position will give the (x, y) coordinates given below for where the collision occurs.

They first run into each other at $t =$ $\frac{2\pi}{3}$

The collision occurs at $(x, y) =$ $(\frac{\sqrt{3}}{3}\pi, \frac{\pi}{3})$

d. [5 points] Bobby's phone flies out of his pocket at $t = \pi/2$. It travels in a straight line in the same direction as he was moving at $t = \pi/2$. Find the equation of this line in Cartesian coordinates.

Solution: Plug in $t = \dfrac{\pi}{2}$ to the parametric equations for Bobby's position to get that Bobby is at the point $P = \left(0, \dfrac{\pi}{2\sqrt{3}}\right)$ at $t = \dfrac{\pi}{2}$. We can find the slope of the curve at that point

$$\left.\frac{dy}{dx}\right|_P = \frac{\frac{dy}{dt}|_{t=\pi}}{\frac{dx}{dt}|_{t=\pi}} = \frac{\frac{1}{\sqrt{3}}\sin(\frac{\pi}{2}) + \frac{\pi}{2\sqrt{3}}\cos(\frac{\pi}{2})}{-\sqrt{3}\cos(\frac{\pi}{2}) + \frac{\sqrt{3}\pi}{2}\sin(\frac{\pi}{2})} = \frac{2}{3\pi}.$$

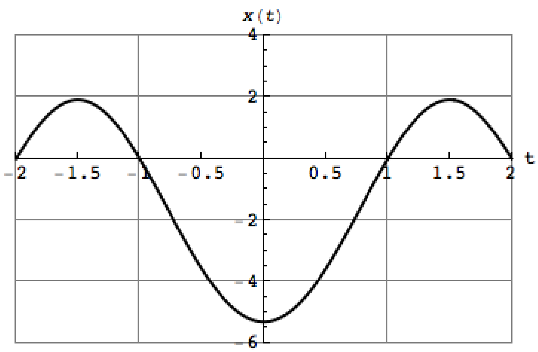
Since the line that gives the path of the phone is the same as the tangent line to the curve giving Bobby's motion at the point P, the equation of the line we want is that given below.

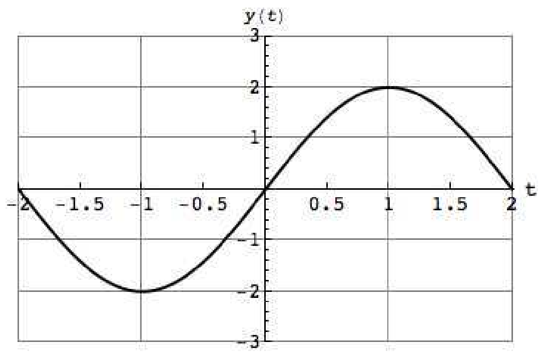
The equation for the line is $y = \dfrac{2}{3\pi}x + \dfrac{\pi}{2\sqrt{3}}$

5. [12 points] A particle moves according to the following parametric equations

$$x = x(t) \quad \text{and} \quad y = y(t) \quad \text{for} \quad -2 \le t \le 2,$$

where the graphs of $x(t)$ and $y(t)$ are shown below.





a. [2 points] Is there a value of t at which the particle is at the point $(0, 2)$? If so, find the value of t where this happens.

Solution: $t = 1$.

b. [3 points] At which value(s) of t is the particle on the x-axis?

Solution: $t = -2, 0, 2$.

c. [4 points] At what points (x, y) does the curve traveled by the particle have a horizontal tangent line? Include the times for each point.

Solution: $y'(t) = 0$ when $t = 1$, $(x, y) = (0, 2)$ and $t = -1$, $(x, y) = (0, -2)$.

d. [3 points] For which of values of t is the slope of the tangent line to the curve positive?

Solution: Slope$=\frac{y'(t)}{x'(t)} > 0$ if x' and y' have the same sign. This occurs at $(0, 1)$, $(-1.5, -1)$ and $(1.5, 2)$.

6. [11 points] Franklin, your robot, is zipping around the kitchen making his famous "Definitely Not Poison!" soup. His coordinates in the xy-plane are given by the parametric equations

$$x = t^2 - t \qquad y = -\sin(\pi t)$$

t seconds after he starts making soup. Assume that both x and y are measured in meters.

a. [2 points] Calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = \underline{\quad 2t - 1 \quad} \qquad \frac{dy}{dt} = \underline{\quad -\pi\cos(\pi t) \quad}$$

b. [2 points] Find all times t when Franklin's velocity is zero.

Solution: Franklin comes to a stop at all times t when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

- $\frac{dx}{dt} = 2t - 1 = 0$ when $t = 1/2$.
- $\frac{dy}{dt} = -\pi\cos(\pi t) = 0$ when $t = 1/2, 3/2, 5/2$, etc.

So Franklin comes to a stop when $t = 1/2$.

$t =$ $\underline{\quad 1/2 \quad}$

c. [3 points] Find Franklin's **speed** when $t = 2$ seconds. Include units.

Solution:

$$\text{Franklin's speed} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

When $t = 2$:

- $\frac{dx}{dt} = 2(2) - 1 = 3$
- $\frac{dy}{dt} = -\pi\cos(2\pi) = -\pi$

Franklin's speed when $t = 2$ is $\sqrt{3^2 + \pi^2} \approx 4.34$ meters per second.

Franklin's speed= $\underline{\quad \sqrt{3^2+\pi^2} \approx 4.34 \ \mathbf{m/s} \quad}$

d. [4 points] Write an integral which gives the distance traveled by Franklin during his first five seconds of zipping around. Do not evaluate this integral.

Solution:

$$\int_0^5 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \int_0^5 \sqrt{(2t-1)^2 + (-\pi\cos(\pi t))^2}dt$$

3. [13 points] O-guk's playful son, O-ghan, is running on the xy-plane. His position t seconds after he begins running is

$$x = \sqrt{t} - 1 \qquad y = \sin(t) + 1.$$

Assume x and y are in meters.

a. [3 points] Does O-ghan pass though the origin? Briefly justify.

Solution: $x = 0$ when $\sqrt{t} - 1 = 0$ so when $t = 1$. For this value of t, $y = \sin(1) + 1 \neq 0$. So he didn't pass through the origin.

b. [4 points] How fast is O-ghan running at $t = 5$? Give your answer in **exact** form (i.e. no decimal approximations). Include **units**.

Solution:

$$\sqrt{\left(\frac{1}{2\sqrt{5}}\right)^2 + (\cos(5))^2} \qquad \frac{m}{s}$$

c. [6 points] Find an equation, in xy-coordinates, of the tangent line to his path at $t = 1$.

Solution: The slope of the tangent line is given by

$$m = \frac{dy/dt}{dx/dt} = \frac{\cos(1)}{\frac{1}{2\sqrt{1}}} = 2\cos(1)$$

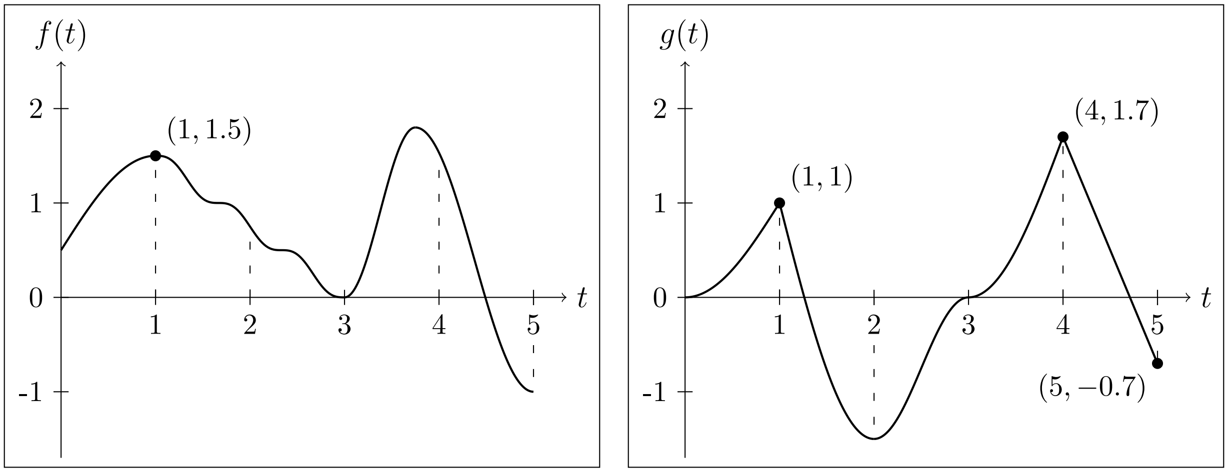
The equation of the tangent line is $y - (\sin(1) + 1) = 2\cos(1)(x - 0)$ or equivalently

$$y = 2\cos(1)x + \sin(1) + 1$$

10. [14 points] Fearing that she is losing authority over her robot ward, Dr. Durant has installed a tracking chip in Steph's mainframe. The chip gives Steph's location separately in x- and y-coordinates, where the units of the axes are miles, Dr. Durant's office corresponds to the origin $(x, y) = (0, 0)$, the positive y-axis points north, and the positive x-axis points east. On night 1, Dr. Durant noticed unusual levels of activity; t hours after midnight, Steph began moving according to the parametric equations

$$x = f(t) \qquad\qquad y = g(t),$$

where $f(t)$ and $g(t)$ are plotted below for $0 \le t \le 5$.



a. [2 points] When was Steph farthest north and south on night 1? Write your answers in the blanks provided. You do **not** need to show your work.

Solution: **North:** 4 **a.m.** **South:** 2 **a.m.**

b. [3 points] What was Steph's speed at $t = 4.9$ on night 1? You may use the fact that $f'(4.9) = -1$. Include units.

Solution: Her speed is $\sqrt{(-1)^2 + (-2.4)^2} = \sqrt{6.76}$ mi/hr.

c. [2 points] What direction was Steph moving at $t = 2$ on night 1? Circle only one answer.

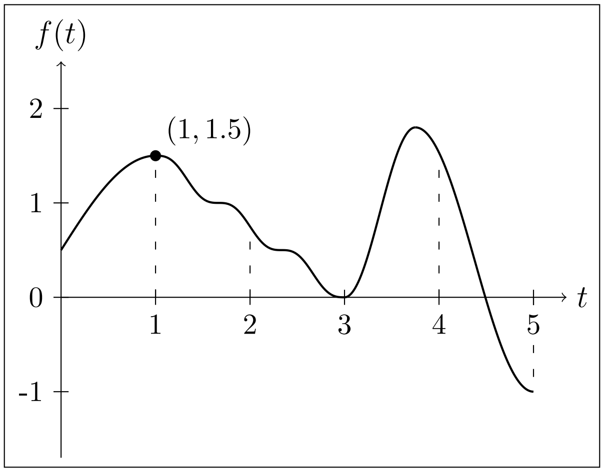
NORTH AND EAST **EAST ONLY** **SOUTH AND EAST**

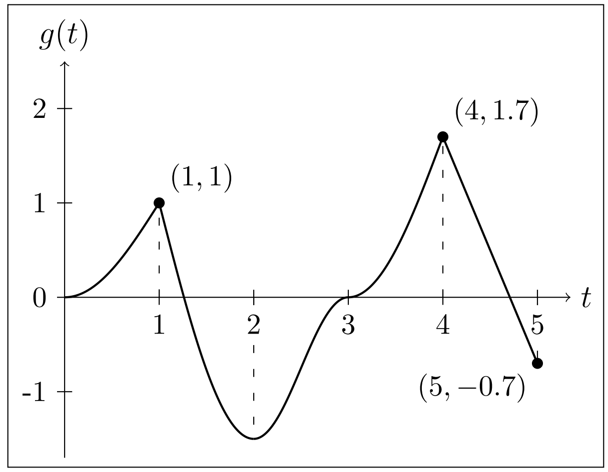
NORTH AND WEST $\boxed{\textbf{WEST ONLY}}$ **SOUTH AND WEST**

10 (continued). Recall that on night 1, Steph's position was given by the parametric equations

$$x = f(t) \qquad\qquad y = g(t),$$

where $f(t)$ and $g(t)$ are plotted below for $0 \le t \le 5$. As before, Dr. Durant's office is at the origin $(x, y) = (0, 0)$, the positive y-axis points north, and the positive x-axis points east.





d. [3 points] How far away was Steph from Dr. Durant's office at $t = 1$ on night 1?

Solution: Steph was $\sqrt{1^2 + 1.5^2} = \sqrt{3.25}$ mi away.

On night 2, Steph's movements were even stranger, following the parametric equations

$$x = \int_0^t f(s)\,ds \qquad\qquad y = \int_0^t g(s)\,ds.$$

e. [2 points] What direction was Steph moving at $t = 2$ on night 2? Circle only one answer.

NORTH AND EAST	**EAST ONLY**	$\boxed{\textbf{SOUTH AND EAST}}$
NORTH AND WEST	**WEST ONLY**	**SOUTH AND WEST**

f. [2 points] Did Steph come to a stop between midnight and 5 a.m. on night 2? If so, at what time(s) did she come to a stop?

Solution: Yes; she came to a stop at 3 a.m.