Math 116 — Practice for Exam 3

Generated December 2, 2020

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2019	3	7		11	
Fall 2018	3	10		6	
Winter 2016	3	11	Hanoi Tower	5	
Fall 2015	3	5		3	
Winter 2015	3	6		9	
Total				34	

Recommended time (based on points): 41 minutes

7. [11 points] Some values of a function $m(x)$ and its derivatives are given below.

x	0	2
$m(x)$	4	1
$m'(x)$	-1	0
$m''(x)$	0	0
$m'''(x)$	3	-2
$m''''(x)$	5	8

a. [4 points] Find a formula for $P_4(x)$, the Taylor polynomial of degree 4 for $m(x)$ about $x = 2$.

Answer: $P_4(x) =$ $1 - \frac{2}{3!}(x-2)^3 + \frac{8}{4!}(x-2)^4$

b. [3 points] Use your answer to approximate the value of $\int_1^3 m(x)\,dx$. Show your work.

Solution:

$$\int_1^3 1 - \frac{2}{3!}(x-2)^3 + \frac{8}{4!}(x-2)^4 dx = x - \frac{2}{4!}(x-2)^4 + \frac{8}{5!}(x-2)^5 \Big|_1^3$$
$$= (1 - \frac{2}{4!} + \frac{8}{5!}) - (-1 - \frac{2}{4!} - \frac{8}{5!})$$
$$= 2 + \frac{16}{5!} = \frac{32}{15}$$

Answer: $\int_1^3 m(x)\,dx \approx$ $\frac{32}{15}$

c. [4 points] Let $G(x)$ be the antiderivative of the function $g(x) = m(3x^2)$ with $G(0) = 5$. Find the first three nonzero terms of the Taylor series for $G(x)$ about $x = 0$.

Solution:
Using the table, we see that the first 3 nonzero terms of the Taylor series for $m(x)$ about $x = 0$ are $4 - x + \frac{1}{3!}x^3$. Then the first 3 nonzero terms for the Taylor series for $m(3x^2)$ about $x = 0$ are $4 - 3x^2 + \frac{1}{3!}(3x^2)^3$. To get the Taylor series for an antiderivative of $m(3x^2)$ about $x = 0$, we take an antiderivative of the Taylor series we found above: $C + 4x - \frac{3}{3}x^3$. Since $G(0) = 5$, we must have $C = 5$.

Answer: $5 + 4x - x^3$

10. [6 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for $-e^{-1} < x < e^{-1}$ and is given below.

$$F(x) = \sum_{n=0}^{\infty} \frac{(n+1)^n}{n!} x^n \quad \text{for } -\frac{1}{e} < x < \frac{1}{e}.$$

a. [2 points] What is $F^{(2018)}(0)$? Make sure your answer is exact. You do not need to simplify.

$$\frac{F^{(2018)}(0)}{2018!} = \text{coefficient of } x^{2018} = \frac{(2018+1)^{2018}}{(2018)!}$$

Answer: $F^{(2018)}(0) =$ 2019^{2018}

b. [4 points] Use appropriate Taylor series for $F(x)$ and $\cos(x)$ to compute the following limit:

$$\lim_{x\to 0} \frac{(F(x)-1)(\cos(x)-1)}{x^3}$$

Show your work carefully.

$$F(x) = \frac{(0+1)^0}{0!}x^0 + \frac{(1+1)^1}{1!}x^1 + \frac{(2+1)^2}{2!}x^2 + \dots$$

$$= 1 + 2x + \frac{9}{2}x^2 + \dots$$

So $F(x) - 1 = 2x + \frac{9}{2}x^2 + \dots$

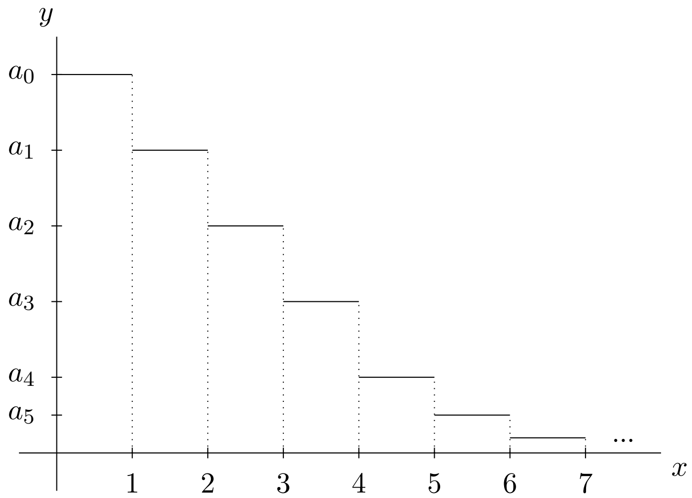
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

So $\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \dots$

so we have $\lim_{x\to 0} \frac{(2x + \frac{9}{2}x^2 + \dots)(-\frac{x^2}{2} + \frac{x^4}{24} - \dots)}{x^3} = \lim_{x\to 0} \frac{-x^3 + (\text{powers of } x \text{ bigger than } 3)}{x^3}$

Answer: $\lim_{x\to 0} \frac{(F(x)-1)(\cos(x)-1)}{x^3} =$ -1

11. [5 points] The Hanoi tower is made by rotating the region depicted below around the y-axis. The region is made up of infinitely many adjacent rectangles. The nth rectangle has width 1 and height $a_n = \dfrac{1}{n!(2n+1)}$ where $n = 0, 1, 2, 3, \ldots$. The rectangle touching the y-axis corresponds to $n = 0$. Note that the y-axis is not to scale.



Compute the volume of the Hanoi Tower. Give an **exact** answer.

Solution: To compute the volume of the Hanoi Tower, we focus on each rectangle separately. The volume of the object made by the revolution of the nth rectangle is given by

$$[\pi(n+1)^2 - \pi n^2] \cdot a_n = \pi(2n+1)\frac{1}{n!(2n+1)} = \frac{\pi}{n!}$$

The total volume is given by adding the volume of all those objects for $n = 0, 1, 2, 3, \ldots$.

$$\sum_{n=0}^{\infty} \frac{\pi}{n!} = \pi \cdot \sum_{n=0}^{\infty} \frac{1}{n!} = \pi e$$

4. [8 points] Let $f(x) = \sqrt[3]{1+2x^2}$.

a. [5 points] Find the first 3 nonzero terms of the Taylor series for f centered at $x = 0$.

Solution: Using the Taylor series for $(1+y)^{1/3}$ centered at $y = 0$,

$$\begin{aligned}\sqrt[3]{1+y} &= 1 + \frac{1}{3}y + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}y^2 + \dots \\ &= 1 + \frac{y}{3} - \frac{y^2}{9} + \dots\end{aligned}$$

Substituting $y = 2x^2$,

$$\sqrt[3]{1+2x^2} = 1 + \frac{2x^2}{3} - \frac{4x^4}{9} + \dots$$

b. [3 points] For what values of x does the Taylor series converge?

Solution: The binomial series for $\sqrt[3]{1+y}$ converges when $-1 < y < 1$. Substituting $y = 2x^2$, this converges when $1 < 2x^2 < 1$, or $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

5. [3 points] Determine the **exact** value of the infinite series

$$-1 + \frac{1}{3!} - \frac{1}{5!} + \dots + \frac{(-1)^{n+1}}{(2n+1)!} + \dots$$

No decimal approximations are allowed. You **do not** need to show your work.

Solution:

$$-1 + \frac{1}{3!} - \frac{1}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{(2n+1)!} = \sin(-1).$$

6. [9 points]

a. [3 points] Find the first three nonzero terms in the Taylor series for $\dfrac{1}{\sqrt{1-x^2}}$ centered at $x = 0$.

Solution: We have $(1+y)^{-1/2} = 1 - \dfrac{1}{2}y + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}y^2 + \cdots$. Substituting $y = -x^2$ gives us $\dfrac{1}{\sqrt{1-x^2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{3}{8}x^4 + \cdots$

b. [4 points] Use your answer from part (a) to find the first three nonzero terms in the Taylor series for $\arcsin(2x)$ centered at $x = 0$. Recall that $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Solution: Integrating the series from part (a) termwise gives us

$$\begin{aligned}\arcsin(x) &= \int 1dx + \int \frac{1}{2}x^2dx + \int \frac{3}{8}x^4dx + \cdots \\ &= C + x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots\end{aligned}$$

Since $\arcsin(0) = 0$, we must have $C = 0$. Then $\arcsin(x) = x + \dfrac{1}{6}x^3 + \dfrac{3}{40}x^5 + \cdots$, and from there a substitution gives us that $\arcsin(2x) = 2x + \dfrac{4}{3}x^3 + \dfrac{12}{5}x^5 + \cdots$.

c. [2 points] Find the values of x for which the Taylor series from part (b) converges.

Solution: The steps we took to get a Taylor series expansion for $\arcsin(x)$ do not change the radius of convergence. So, the Taylor series we found for $\arcsin(x)$ converges for $-1 < x < 1$. Then substituting $2x$ for x gives us that the series which gives our answer to (b) converges for $-1 < 2x < 1$, and so $-\frac{1}{2} < x < \frac{1}{2}$.