MATH 116 — PRACTICE FOR EXAM 3

Generated December 8, 2020

NAME: SOLUTIONS

INSTRUCTOR: ____________________ SECTION NUMBER: ________

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1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	**Exam**	**Problem**	**Name**	**Points**	**Score**
Winter 2019	3	7		13	
Winter 2019	3	3	Ultron	10	
Fall 2019	3	8	party horn	9	
Winter 2016	3	12		8	
Winter 2020	1	9		10	
Winter 2018	3	10	juice	8	
Fall 2018	3	11		9	
Total				67	

Recommended time (based on points): 78 minutes

7. [13 points] The parts of this problem are unrelated.

a. [3 points] Consider the function

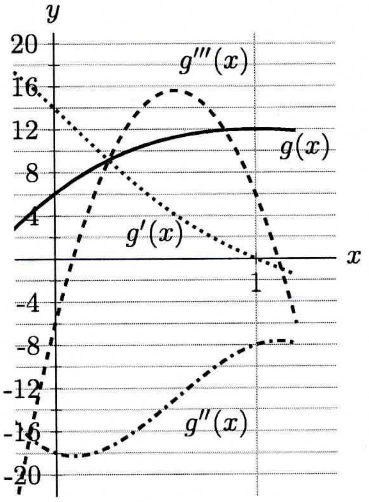
$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{\sin x}{x} - \cos x & \text{for } x \neq 0 \end{cases}$$

Find the Taylor series for $f(x)$ centered at $x = 0$. Write your answer as a single sum using sigma notation.

$$f(x) = \frac{1}{x}\sin(x) - \cos(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(2n+1)!} - \frac{1}{(2n)!}\right]x^{2n}$$

Answer: $f(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\left[\frac{-2n}{(2n+1)!}\right]x^{2n}$

b. [4 points] Part of the graphs of $g(x), g'(x), g''(x)$, and $g'''(x)$ are given to the right.
Find the third-degree Taylor polynomial for $g(x)$ near $x = 1$.



$g(1) = 12$
$g'(1) = 0$
$g''(1) = -8$
$g'''(1) = 6$

$$P_3(x) = g(1) + g'(1)(x-1) + \tfrac{1}{2}g''(1)(x-1)^2 + \tfrac{1}{6}g'''(1)(x-1)^3$$

$$= 12 + (0)(x-1) + \tfrac{1}{2}(-8)(x-1)^2 + \tfrac{1}{6}(6)(x-1)^3$$

Answer: $12 - 4(x-1)^2 + (x-1)^3$

c. [6 points] Find the exact value (in closed form) of the following series. You do not need to justify your answers.

i. $0.1 + \dfrac{0.01}{2} + \dfrac{0.001}{3} + \dfrac{0.0001}{4} + \cdots =$ $-\ln(.9) = \ln 10 - \ln 9$

ii. $\dfrac{\pi}{2} - \dfrac{3}{\pi} + \dfrac{18}{\pi^3} - \dfrac{108}{\pi^5} + \cdots =$ $\dfrac{\pi}{2}\left[1 - \dfrac{6}{\pi^2} + \dfrac{6^2}{\pi^4} - \dfrac{6^3}{\pi^6} + \cdots\right]$ $\pi^3/(2\pi^2+12)$

iii. $\dfrac{1}{2} - 2e^2 + \dfrac{2^3e^4}{3!} - \dfrac{2^5e^6}{5!} + \cdots =$ $\dfrac{1}{2} - e\left[2e - \dfrac{2^3e^3}{3!} + \dfrac{2^5e^5}{5!} - \cdots\right]$ $\dfrac{1}{2} - e\sin(2e)$

i) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

So $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

So $-\ln(1-x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$

So $-\ln(1-.1) = .1 + \dfrac{.01}{2} + \dfrac{.001}{3} + \dfrac{.0001}{4} + \cdots$

ii) is geometric with $a = \frac{\pi}{2}$, $x = \frac{-6}{\pi^2}$

so $\dfrac{\pi/2}{1 - \frac{-6}{\pi^2}} \cdot \dfrac{2\pi^2}{2\pi^2} = \dfrac{\pi^3}{2\pi^2+12}$

iii) Looks like sin:

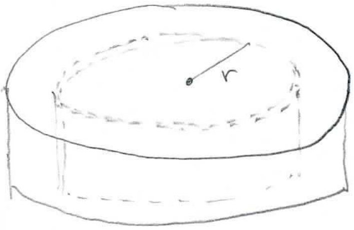
$\sin(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$

3. [10 points] A group of scientists of S.H.I.E.L.D. are investigating the Battle of Sokovia, trying to understand how Ultron lifted the capital city of Sokovia up into the sky. They use data available to them to model the situation.

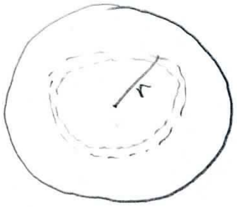
Pay careful attention to the units involved in the data they use.

a. [5 points] The scientists find that they can model the part of the city that was lifted by the shape of a cylinder of radius 2 kilometers and height 100 meters. The density $\delta(r)$, in kilograms per cubic meter, is a function of distance r meters away from the central axis of the cylinder. Let M be the total mass, in kilograms, of the part of the city that was lifted. Write an expression involving one or more integrals that gives the value of M.

Density in kg/m³, so convert km to m



Top view



Shape of slice = cylindrical shell

radius " " = r $(0 \le r \le 2000 \text{ m})$

height " " = 100 m

area " " = $(2\pi r)(\text{height}) = 200\pi r$ m^2

thickness " " = Δr m

Volume " " = $200\pi r\,\Delta r$ m^3

mass " " = (volume)(density) = $200\pi r\,\delta(r)\Delta r$ kg

Answer: $M =$ $\int_0^{2000} 200\pi r\,\delta(r)\,dr$ kg

b. [5 points] You may use M and $\delta(r)$ from part **a.** for this part.

Ultron lifted the city at a constant rate of 2 meters per second to a height of 1000 meters above the ground. While he lifted it, a small portion of the city kept detaching from the rising part at a constant rate of p kilograms per second. Write an expression involving one or more integrals that gives the total work, in Joules, it takes to complete the lifting process. Your answer may be in terms of $m, g, \delta(r)$, and M, where g is the gravitational constant, $g \approx 9.8$ m/s^2.

Total Time to lift = $1000 \text{ m} / 2 \text{ m/s} = 500$ s

mass at time t = $M - pt$ kg

weight " " " = $(M-pt)g$ N

Distance lifted from time t to $t+\Delta t$ = $2\Delta t$ m

work done from time t to $t+\Delta t$ = (weight)(distance) = $2(M-pt)g\,\Delta t$ J

Answer: $\int_0^{500} 2(M-pt)g\,dt$ joules

8. [9 points] Derivative Girl and Gradi-Ant are excited for the end of the semester. To celebrate, they decide to make an Infinite Party Horn. In this problem, x and y are measured in meters. (In Derivative Girl's world, infinite objects are possible.)

a. [4 points] They decide to make the horn by rotating the region bounded by the positive x-axis, the positive y-axis, and the function $y = \dfrac{1}{2(x+1)^2}$ about the line $y = -1$. Write, but do not evaluate, an expression involving <u>one</u> integral that gives the volume, in cubic meters, of the Infinite Party Horn.

Answer: $\pi \int_0^\infty \left(\frac{1}{2(x+1)^2} + 1\right)^2 - 1\,dx$

b. [5 points] Derivative Girl will use her favorite continuous and differentiable functions f and g to make a banner for the Infinite Party Horn. She loves the functions f and g because they have the properties:

- $\dfrac{d}{dx}\left(\dfrac{1+x}{g(x)}\right) = f(x)$,
- $g(1) = 15$,
- $\lim_{x\to\infty} g(x) = \infty$,
- $\lim_{x\to\infty} g'(x) = 5$,

and the area of the banner, in square meters, is given by

$$\int_1^\infty 20f(x)\,dx.$$

Does the banner have finite area? If so, what is the banner's area? Show all work and indicate any theorems you use.

Solution: Using the information we've been given, we find

$$\begin{aligned}\int_1^\infty 20f(x)dx &= \lim_{b\to\infty}\int_1^b 20f(x)dx\\ &= \lim_{b\to\infty}\frac{20+20b}{g(b)} - \frac{40}{g(1)}\\ &= \lim_{b\to\infty}\frac{20}{g'(b)} - \frac{40}{15} \text{ by L'Hopital's Rule}\\ &= \frac{20}{5} - \frac{40}{15}\end{aligned}$$

Answer *(Circle one):* **Infinite area** [✓ **Finite area:**] $\frac{4}{3}$ m^2

12. [8 points] Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges when $x = 0$ and diverges when $x = 9$. In this problem, you do not need to show your work.

a. [4 points] Which of the following could be the interval of convergence? Circle **all** that apply.

$\boxed{[0, 8]}$ $\qquad [0, 7] \qquad$ $\boxed{(-1, 9)}$ $\qquad (-2, 10) \qquad (0, 8]$

b. [2 points] The limit of the sequence a_n is 0.

[✓ **ALWAYS**] **SOMETIMES** **NEVER**

c. [2 points] The series $\sum_{n=0}^{\infty}(-5)^n a_n$ converges.

ALWAYS [✓ **SOMETIMES**] **NEVER**

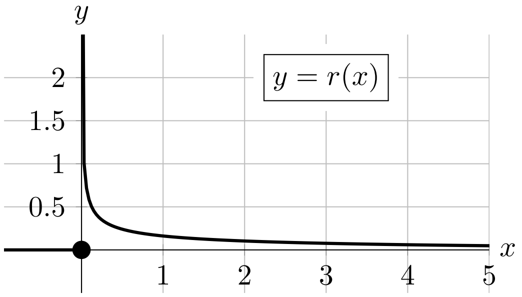
9. [10 points]

It has been suggested that the probability density function given by

$$r(x) = \begin{cases} 0 & \text{if } \; x \le 0 \\ \dfrac{e^{-0.1x}}{\sqrt{10\pi x}} & \text{if } \; x > 0 \end{cases}$$

models the size of rainfalls. That is, on a given rainy day, this pdf models the amount x (measured in millimeters) of rain that falls.

A graph of $y = r(x)$ is shown below.



Note that even though $r(x)$ has a vertical asymptote as $x \to 0^+$, it is still a valid pdf.

a. [1 point] Use the formula above and the fact that $r(x)$ is a pdf to find the value of $\int_0^\infty r(x)\, dx$. (You do not need to show any work.)

Answer: $\int_0^\infty r(x)\, dx =$ 1

b. [4 points] Write out all the terms of a MID(4) approximation to the integral $\int_3^5 r(x)\, dx$. Do not evaluate the sum, but the letters r and x should not appear in your answer.

Solution: With 4 subdivisions, we have $\Delta x = \frac{5-3}{4} = 0.5$. Our four midpoints are at $x = 3.25, x = 3.75, x = 4.25$, and $x = 4.75$. Hence our sum is

$$\frac{e^{(-0.1)(3.25)}}{\sqrt{(10\pi)(3.25)}}0.5 + \frac{e^{(-0.1)(3.75)}}{\sqrt{(10\pi)(3.75)}}0.5 + \frac{e^{(-0.1)(4.25)}}{\sqrt{(10\pi)(4.25)}}0.5 + \frac{e^{(-0.1)(4.75)}}{\sqrt{(10\pi)(4.75)}}0.5$$

or

$$0.5\left(\frac{e^{-0.325}}{\sqrt{32.5\pi}} + \frac{e^{-0.375}}{\sqrt{37.5\pi}} + \frac{e^{-0.425}}{\sqrt{42.5\pi}} + \frac{e^{-0.475}}{\sqrt{47.5\pi}}\right)$$

c. [2 points] Is the answer to part **b.** an overestimate or underestimate of $\int_3^5 r(x)\, dx$? Circle your choice below. You do not need to explain.

Circle one: OVERESTIMATE [✓ UNDERESTIMATE] NOT ENOUGH INFORMATION

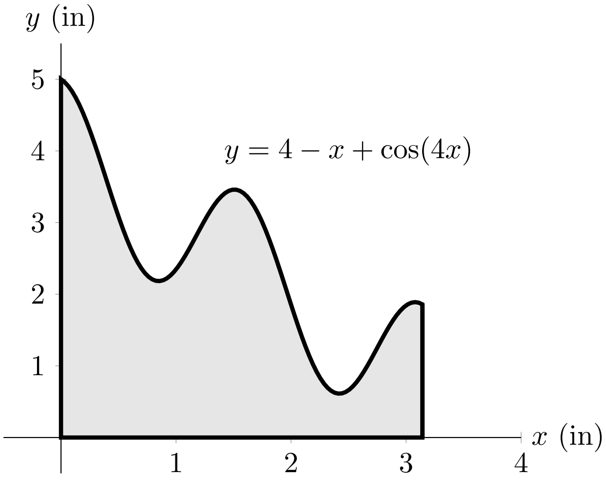
d. [3 points] Let $q(x)$ be the cumulative distribution function for $r(x)$. Which of the following expressions give the fraction of rainfalls that result in between 2 and 4 millimeters of rain? *Circle* ALL *correct answers.*

i. $r(4) - r(2)$
ii. $r'(4) - r'(2)$
iii. [✓ $q(4) - q(2)$]
iv. $q'(4) - q'(2)$
v. [✓ $\int_2^4 r(x)\, dx$]
vi. $\int_2^4 r'(x)\, dx$
vii. $\int_2^4 q(x)\, dx$
viii. [✓ $\int_2^4 q'(x)\, dx$]
ix. NONE OF THESE

10. [8 points] Recently Debra McQueath was thinking about all the great things she used to make at `Print.juice` by revolving regions around the y-axis. Those were the good days, weren't they?

a. [4 points]

There was that one time she designed the Juice Titan$^{\text{TM}}$ formed by rotating the region in the first quadrant bounded by $x = \pi$ and $y = 4 - x + \cos(4x)$ around the y-axis. The density $\delta(x)$ of the plastic was a function of the distance from the center of the juicer, although Debra cannot quite remember what it was. Help Debra write an integral that represents the total mass of the Juice Titan$^{\text{TM}}$. Your integral may include the density function $\delta(x)$.

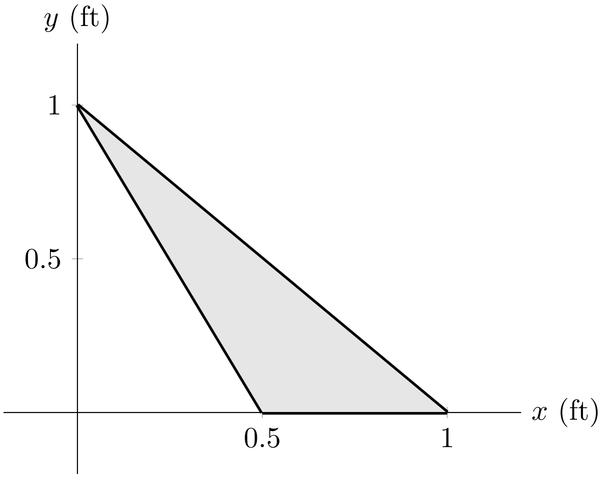


Solution: Use the shell method.

Answer: $\int_0^{\pi} 2\pi x \delta(x)(4 - x + \cos(4x))\, dx$

b. [4 points]

On Debra's last day at `Print.juice` her team made her a commemorative hat containing a hollow chamber filled with juice by rotating the region bounded by $y = 0$, $y = 1 - x$ and $y = 1 - 2x$ around the y-axis. The juice-filled hat still sits on her kitchen table; she sometimes wonders how much juice is in the hat. Write an integral that represents the total volume of juice in the hat. **Note:** juice fills the solid formed by rotating the shaded region.



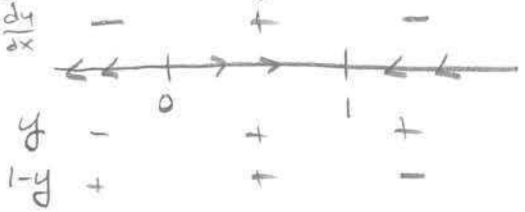
Solution: Use the washer method.

Answer: $\int_0^1 \pi\left((1-y)^2 - ((1-y)/2)^2\right) dy$

11. [9 points] Provide an example for each of the following. Your example must clearly satisfy the given properties. If no example exists then write DOES NOT EXIST and briefly explain why no such example exists.

a. [3 points] A differential equation that has at least one equilibrium solution that is stable and at least one equilibrium solution that is not stable. (A complete answer consists of the differential equation and both of these equilibrium solutions.)

$$\frac{dy}{dx} = y(1-y)$$

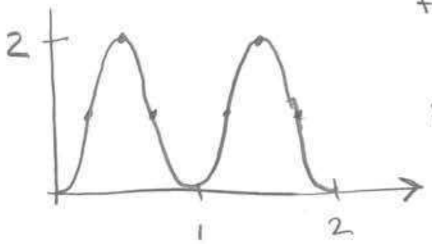


Answer: Differential Equation: dy/dx = $y(1-y)$

Equilibrium Solutions: Stable: 1 Not Stable: 0

b. [3 points] A continuous function $f(x)$ such that $\text{LEFT}(2) \le \text{RIGHT}(2) \le \int_0^2 f(x)\,dx$ where LEFT(2) and RIGHT(2) are, respectively, the left- and right- hand Riemann sum estimates for $\int_0^2 f(x)\,dx$ with two equal subintervals.

You may describe your function f by giving a formula or by drawing a clear and well-labeled graph. Then briefly explain why your function is indeed such an example.



$f(x) = 1 - \cos(2\pi x)$

$\text{LEFT}(2) = (1)\,[f(0) + f(1)] = 0$

$\text{RIGHT}(2) = (1)\,[f(1) + f(2)] = 0$

But $\int_0^2 f(x)\,dx = x - \frac{1}{2\pi}\sin(2\pi x)\Big|_0^2 = [2-0] - [0-0] = 2$

Answer: $f(x) =$ $1 - \cos(2\pi x)$

Brief explanation:

If f is neither increasing nor decreasing, it's possible for both LEFT and RIGHT to be over or underestimates. In this case, the sample points happen to be the lowest values of the function.

c. [3 points] Give an example of a power series whose radius of convergence is 0. (You must use sigma notation for full credit.)

How about: $\sum_{n=0}^{\infty} n!\,x^n$. By the ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$

$= \lim_{n\to\infty} (n+1)|x|$ which diverges unless $x = 0$.

Answer: Power Series: $\sum_{n=0}^{\infty} n!\,x^n$