Math 116 — Practice for Exam 3

Generated December 1, 2020

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2019	3	1	race	14	
Fall 2015	3	11		7	
Winter 2015	2	1	spinning	16	
Winter 2019	3	2	castar	8	
Fall 2018	3	4	butterfly	9	
Fall 2017	3	2		8	
Fall 2016	2	7		9	
Total				71	

Recommended time (based on points): 78 minutes

1. [14 points] Hannah Haire and Ryan Rabbit meet for one last race. Once again, they both start at the west side of a large square field that is 10 km wide; it will end when one reaches the east side. The racers' (x, y) positions are given by the parametric equations below, where $(0, 0)$ represents the southwest corner of the field, x represents kilometers east of this corner, y represents kilometers north of this corner, and $t \geq 0$ is measured in hours after the race begins.

Hannah Haire: $\begin{cases} x = t^2 \\ y = \dfrac{t^2}{2} + 2 \end{cases}$ $x'(t) = 2t$, $y'(t) = t$

Ryan Rabbitt: $\begin{cases} x = 4t - t^2 \\ y = t^2 - t + 1 \end{cases}$ $x'(t) = 4 - 2t$, $y'(t) = 2t - 1$

Be sure to justify your answers to the following questions algebraically.

a. [2 points] Who is going faster two hours into the race?

Hannah:
Speed $= \sqrt{x'(2)^2 + y'(2)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$

Ryan:
Speed $= \sqrt{x'(2)^2 + y'(2)^2} = \sqrt{0^2 + 3^2} = 3$

Answer: Hannah

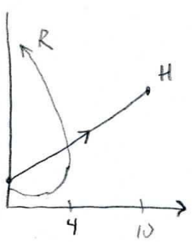
b. [3 points] The race ends when the first racer reaches the east side of the field. When does the race end? Who wins?

Hannah finishes when:
$t^2 = 10 \Rightarrow t = \sqrt{10}$

Ryan finishes when:
$4t - t^2 = 10 \Rightarrow t^2 - 4t + 10 = 0 \Rightarrow t = \dfrac{4 \pm \sqrt{16 - 40}}{2}$.
No real solutions, so Ryan never finishes.

Answer: Race ends at $t = \sqrt{10}$ **Winner:** $\boxed{\text{Hannah}}$ Ryan Tie

c. [3 points] Write an integral representing the distance, in km, that Ryan runs during the race.



$\int_0^{\sqrt{10}} (\text{Ryan's speed})\, dt$

Answer: $\int_0^{\sqrt{10}} \sqrt{(4-2t)^2 + (2t-1)^2}\, dt$

d. [3 points] Find all times at which Ryan and Hannah are in the same spot on the field. If there are none, write "none".

X-values match when $t^2 = 4t - t^2 \Rightarrow 0 = 4t - 2t^2 = 2t(2-t)$
$\Rightarrow t = 0$ or $t = 2$.

But y-values match at neither $t = 0$ nor $t = 2$.

Answer: $t =$ NONE

e. [3 points] Find all times at which Ryan is facing directly northeast (that is, halfway between directly north and directly east). If there are none, write "none".

NE $\Rightarrow \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dt} = \dfrac{dx}{dt} > 0$.

So $4 - 2t = 2t - 1 \Rightarrow 5 = 4t \Rightarrow t = 1.25$

At that time,

$\dfrac{dx}{dt} = 4 - 2.5 = 1.5$, $\dfrac{dy}{dt} = 2.5 - 1 = 1.5$, so both are positive.

Answer: $t =$ 1.25

11. [7 points] Two squirrels, Zini and Aladar, are quickly scavenging for their last acorns before returning to their dens for winter. At a time t seconds after they begin running, Zini's position on the diag is given by

$$x(t) = t, \qquad y(t) = t - 3$$

and Aladar's position is given by

$$x(t) = 4t, \qquad y(t) = t^2$$

for $0 \le t \le 5$. Assume $x(t)$ and $y(t)$ are measured in meters.

a. [3 points] Find Aladar's **speed** 1 second after the squirrels begin running. Remember to include units.

Solution: Aladar's speed at time $t = 1$ is

$$\begin{aligned} \sqrt{\left(\left.\frac{dx}{dt}\right|_{t=1}\right)^2 + \left(\left.\frac{dy}{dt}\right|_{t=1}\right)^2} &= \sqrt{(4)^2 + (2)^2} \\ &= \sqrt{20}\text{ m/s} \end{aligned}$$

b. [4 points] Find the x- and y-coordinates of the point(s) where their **paths** intersect, if any.

Solution: The paths intersect if the two x-coordinates and two y-coordinates are equal at (possibly different) time values. That is, solutions to the system of equations

$$\begin{aligned} t &= 4s \\ t - 3 &= s^2 \end{aligned}$$

Subsituting $t = 4s$ into the second equation gives $s^2 - 4s + 3 = 0$, so $s = 1, 3$. Plugging in, we get the two possible intersection points (4,1) and (12,9). However, Zini would be at the point (12,9) when $t = 12$, which is outside of the domain, thus the only intersection of their paths is at the point (4,1).

1. [16 points] Carla and Bobby run a race after spinning in circles for a good amount of time to make themselves dizzy. They start at the origin in the xy-plane and they race to the line $y = 5$. Assume the units of x and y are meters.
Bobby's position in the xy-plane t seconds after the races starts is

$$(-\sqrt{3}t\cos t, \frac{1}{\sqrt{3}}t\sin t)$$

and Carla's position in the xy-plane t seconds after the race starts is

$$(t\sin t, -t\cos t).$$

a. [4 points] Write an integral that gives the distance that Carla travels during the first two seconds of the race. Do not evaluate your integral.

Solution: We have

$$\frac{dx}{dt} = \sin(t) + t\cos(t),$$
$$\frac{dy}{dt} = -\cos(t) + t\sin(t).$$

The distance traveled by Carla in the first two seconds of the race is then given by $\int_0^2 \sqrt{(\sin(t) + t\cos(t))^2 + (t\sin(t) - \cos(t))^2}dt$.

b. [3 points] Find Carla's speed at $t = \pi$.

Solution: We have that Carla's speed is given by the function $\sqrt{(\sin(t) + t\cos(t))^2 + (t\sin(t) - \cos(t))^2}$, and so we need only plug in $t = \pi$ which gives us the value below.

Carla's speed at $t = \pi$ is $\sqrt{\pi^2 + 1}$ **m/sec**

c. [4 points] Carla and Bobby are so dizzy that they run into each other at least once during the race. Find the first time $t > 0$ that they run into each other, and give the point (x, y) where the collision occurs.

Solution: Setting the x and y coordinate functions equal gives us that collisions will occur when $\tan(t) = -\sqrt{3}$. The first time for $t > 0$ when this occurs is $2\pi/3$. Plugging this t value into either the equations for Bobby's or Carla's position will give the (x, y) coordinates given below for where the collision occurs.

They first run into each other at $t =$ $\frac{2\pi}{3}$

The collision occurs at $(x, y) =$ $(\frac{\sqrt{3}}{3}\pi, \frac{\pi}{3})$

d. [5 points] Bobby's phone flies out of his pocket at $t = \pi/2$. It travels in a straight line in the same direction as he was moving at $t = \pi/2$. Find the equation of this line in Cartesian coordinates.

Solution: Plug in $t = \frac{\pi}{2}$ to the parametric equations for Bobby's position to get that Bobby is at the point $P = \left(0, \frac{\pi}{2\sqrt{3}}\right)$ at $t = \frac{\pi}{2}$. We can find the slope of the curve at that point

$$\frac{dy}{dx}\bigg|_P = \frac{\frac{dy}{dt}|_{t=\pi}}{\frac{dx}{dt}|_{t=\pi}} = \frac{\frac{1}{\sqrt{3}}\sin(\frac{\pi}{2}) + \frac{\pi}{2\sqrt{3}}\cos(\frac{\pi}{2})}{-\sqrt{3}\cos(\frac{\pi}{2}) + \frac{\sqrt{3}\pi}{2}\sin(\frac{\pi}{2})} = \frac{2}{3\pi}.$$

Since the line that gives the path of the phone is the same as the tangent line to the curve giving Bobby's motion at the point P, the equation of the line we want is that given below.

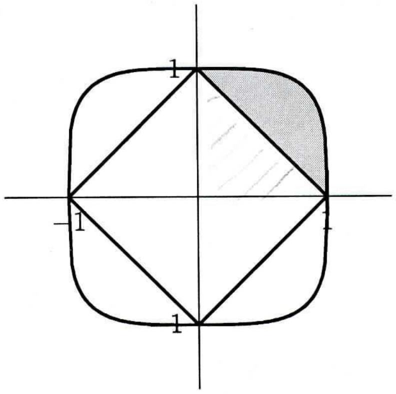
The equation for the line is $y = \frac{2}{3\pi}x + \frac{\pi}{2\sqrt{3}}$

2. [8 points]

The *castar*, a coin widely used in Middle-Earth, allegedly has the shape graphed to the right. The outer perimeter can be modeled by the implicit equation

$$x^4 + y^4 = 1$$

and the perimeter of the hole in the middle is a square. To help his fellow Hobbits detect counterfeit coins, Samwise Gamgee, the Mayor of the Shire, is working on obtaining the specifications of a genuine castar. Sam needs your help.



a. [2 points] Find a function $f(\theta)$ so that the outer edge of the castar is given by the function $r = f(\theta)$.

$x = r\cos\theta$, $y = r\sin\theta$, so

$(r\cos\theta)^4 + (r\sin\theta)^4 = 1$

$\Rightarrow r^4\cos^4\theta + r^4\sin^4\theta = 1$

Answer: $f(\theta) = \left(\dfrac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/4}$

b. [3 points] Write an expression involving one or more integrals that gives the total area of the quarter of a castar in the first quadrant (shaded above).

Area of shaded triangle $= \frac{1}{2}$, so

area of quarter coin $= \frac{1}{2}\int_0^{\pi/2} f(\theta)^2\, d\theta - \frac{1}{2}$

Answer: $\dfrac{1}{2}\displaystyle\int_0^{\pi/2} \left(\frac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/2} d\theta - \frac{1}{2}$

c. [3 points] Approximate the area of a castar by estimating your integral(s) from part (b) using TRAP(2). Write out all the terms in your sum(s).

For $\int_0^B g(x)\,dx$, TRAP(2) $= \Delta x\left[\frac{1}{2}g(0) + g\left(\frac{B}{2}\right) + \frac{1}{2}g(B)\right]$, where $\Delta x = \frac{B}{2}$.

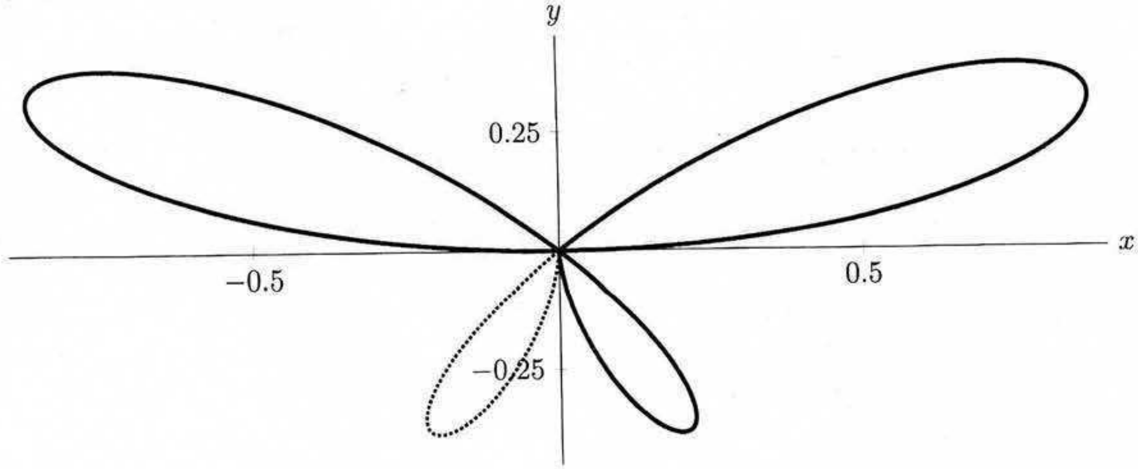
In our case $B = \frac{\pi}{2}$ so $\frac{B}{2} = \Delta x = \frac{\pi}{4}$.

four quarters

Answer: $4\left[\frac{1}{2}\cdot\frac{\pi}{4}\left[\frac{1}{2}\left(\frac{1}{\cos^4(0)+\sin^4(0)}\right)^{1/2} + \left(\frac{1}{\cos^4(\frac{\pi}{4})+\sin^4(\frac{\pi}{4})}\right)^{1/2} + \frac{1}{2}\left(\frac{1}{\cos^4(\frac{\pi}{2})+\sin^4(\frac{\pi}{2})}\right)^{1/2}\right] - 1/2\right]$

$= 4\left[\frac{\pi}{8}(1+\sqrt{2}) - \frac{1}{2}\right] = \frac{\pi}{2}(1+\sqrt{2}) - 2 \approx 1.79$

4. [9 points] The polar curve $r = \sin(4\theta)\cos(\theta)$ for $0 \leq \theta \leq \pi$ is shown below.



Note that there are two "large loops" and two "small loops".

For reference, note that for this curve, $\frac{dr}{d\theta} = 4\cos(\theta)\cos(4\theta) - \sin(\theta)\sin(4\theta)$

a. [3 points] For what values of θ does the polar curve $r = \sin(4\theta)\cos(\theta)$ trace once around the "small loop" in the third quadrant? (This portion of the curve is indicated by the dotted line.) Give your answer as an interval of θ values between 0 and π.

Look at signs of x and y to determine quadrant of points:

θ	$\sin\theta$	$\cos\theta$	$\sin 4\theta$	$r = \sin 4\theta\cos\theta$	$x = r\cos\theta$	$y = r\sin\theta$	
0 to $\pi/4$	+	+	+	+	+	+	Q1
$\pi/4$ to $\pi/2$	+	+	−	−	−	−	$\boxed{Q3}$
$\pi/2$ to $3\pi/4$	+	−	+	−	+	−	Q4
$3\pi/4$ to π	+	−	−	+	−	+	Q2

Answer: $\boxed{\frac{\pi}{4} < \theta < \frac{\pi}{2}}$

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the total arc length of the two small loops.

$$\text{Arclen} = 2\int_{\pi/4}^{\pi/2} \sqrt{r^2 + (r')^2}\, d\theta$$

Answer: Arc Length = $2\int_{\pi/4}^{\pi/2} \sqrt{(\sin 4\theta\cos\theta)^2 + (4\cos 4\theta\cos\theta - \sin 4\theta\sin\theta)^2}\, d\theta$

c. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the area of the region that is enclosed by the polar curve $r = 2$ but is outside the curve $r = \sin(4\theta)\cos(\theta)$.

$|r| = |\sin 4\theta| \cdot |\cos\theta| \leq 1 \cdot 1 < 2$, So the butterfly is contained in the circle of radius 2.

Area inside butterfly $= \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_0^{\pi} \sin^2 4\theta\cos^2\theta\, d\theta$

Answer: Area = $\boxed{4\pi - \frac{1}{2}\int_0^{\pi} \sin^2 4\theta\cos^2\theta\, d\theta}$

2. [8 points] For this problem, consider the family of polar curves described for each positive integer $n \geq 1$ by

$$r = \frac{\cos(2n\theta)}{n}$$

for $0 \leq \theta \leq 2\pi$.

a. [2 points] Consider the polar curve described by $r = \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$. (Note that this is the case of $n = 1$.) Find all values of θ between 0 and 2π for which the curve $r = \cos(2\theta)$ passes through the origin.

At origin $\iff r=0 \iff \cos(2\theta)=0$

$\iff 2\theta = m\pi + \frac{\pi}{2}$ for some integer m

$\iff \theta = m\frac{\pi}{2} + \frac{\pi}{4}$ for some integer m

Answer: $\theta = \boxed{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

b. [3 points] For $n \geq 1$, find all x-intercepts of the polar curve $r = \dfrac{\cos(2n\theta)}{n}$. Your answer(s) may involve n.

x-intercept $\iff 0 = y = r\sin\theta \iff r=0$ or $\sin\theta = 0$.
r can be 0, as when $\theta = \pi/4n$. And $\sin\theta = 0$ when $\theta = m\pi$ for some integer m. In that case, $r = \frac{\cos(2nm\pi)}{n} = \frac{1}{n}$ and $\cos\theta = \begin{cases} 1 & \text{if } m \text{ even} \\ -1 & \text{if } m \text{ odd.} \end{cases}$

So $x = \pm\frac{1}{n}$.

Answer: $x = \boxed{-\frac{1}{n}, 0, \frac{1}{n}}$

c. [3 points] For $n \geq 1$, let A_n be the arclength of the polar curve $r = \dfrac{\cos(2n\theta)}{n}$ for $0 \leq \theta \leq 2\pi$. Write, but do not evaluate, an expression involving one or more integrals that gives the value of A_n.

$$A_n = \int_0^{2\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

where $f(\theta) = \frac{1}{n}\cos(2n\theta)$

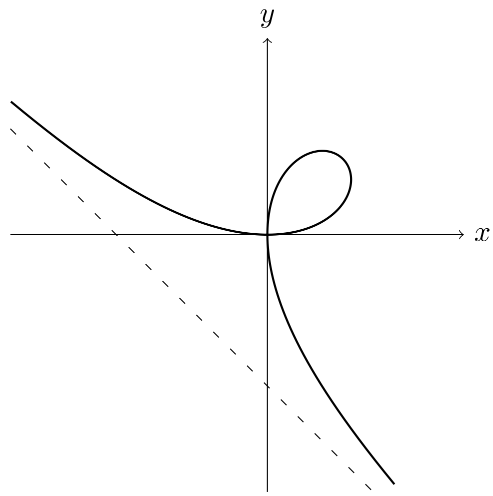
$f'(\theta) = -2\sin(2n\theta)$

Answer: $A_n = \boxed{\int_0^{2\pi} \sqrt{\frac{1}{n^2}\cos^2(2n\theta) + 4\sin^2(2n\theta)}\, d\theta}$

7. [9 points] For $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, consider the polar curve

$$r = \frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}.$$

The curve has an asymptote, the dashed line in the picture, as θ approaches $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$.



a. [4 points] Write down, but do **not** evaluate, an integral that gives the area inside the loop in the first quadrant.

Solution: The area is given by

$$\frac{1}{2}\int_0^{\pi/2} \left(\frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}\right)^2 d\theta.$$

b. [2 points] Find a formula for the quantity $x + y$ in terms of the variable θ. Write your answer in the space provided.

Solution:

$$x + y = \frac{\sin(2\theta)}{\cos(\theta) + \sin(\theta)}(\cos(\theta) + \sin(\theta)) = \sin(2\theta)$$

c. [2 points] Find the limit of $x + y$ as $\theta \to \left(\frac{3\pi}{4}\right)^-$. No justification is needed.

Solution: The specified limit is

$$\lim_{\theta \to (3\pi/4)^-} \sin(2\theta) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

d. [1 point] Write down the Cartesian equation of the asymptote. No justification is needed.

Solution: The asymptote is given by $x + y = -1$.