# MATH 116 — PRACTICE FOR EXAM 3

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NAME: SOLUTIONS

INSTRUCTOR: ______________________ SECTION NUMBER: __________

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

| Semester | Exam | Problem | Name | Points | Score |
|---|---|---|---|---|---|
| Winter 2010 | 3 | 3 | | 12 | |
| Winter 2012 | 3 | 5 | | 8 | |
| Winter 2013 | 3 | 8 | | 8 | |
| Fall 2014 | 3 | 8 | | 7 | |
| Winter 2015 | 3 | 5 | | 10 | |
| Total | | | | 45 | |

**Recommended time (based on points): 54 minutes**

**3**. [12 points] For each of the following series, determine the interval of convergence and write it on the space provided to the right of the series. Be sure to show all appropriate work to justify your answer.

**a**. [6 points] $\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{n}$ $\underline{\qquad 1 < x \leq 3 \quad}$

*Solution:* By using the ratio test we see that

$$\lim_{n\to\infty} \frac{|x-2|^{n+1}(n)}{|x-2|^n(n+1)} = |x-2|.$$

So the center is at $x = 2$ and the radius is 1. We need to check the endpoints $x = 1$ and $x = 3$. We see it converges when $x = 3$ by alternating series, and diverges at $x = 1$ by the p-test.

**b**. [6 points] $\sum_{n=1}^{\infty} \frac{n!x^n}{n^{10}}$ $\underline{\qquad x = 0 \quad}$

*Solution:* By using the ratio test we get

$$\lim_{n\to\infty} \frac{(n+1)!x^{n+1}n^{10}}{n!x^n(n+1)^{10}} = \infty.$$

So it has radius of convergence 0 and converges only at the center which is $x = 0$.

**5**. [8 points] Consider

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n}.$$

**a**. [2 points] Does the series converge for $x = 2$? Justify your answer.

*Solution:* At $x = 2$

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n} = \sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, 4^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

The series diverge since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

**b**. [2 points] Based only on your answer from part **a**, what can you say about $R$, the radius of convergence of the series? Circle your answer.

*Solution:* $R = 2$ $\quad R > 2$ $\quad R < 2$ $\quad \boxed{R \leq 2}$ $\quad R \geq 2$

since it is possible for $x = 2$ to be one of the endpoints in the interval of convergence.

**c**. [4 points] Find the interval of convergence of the series.

*Solution:*

$$\lim_{n\to\infty} \frac{\left|\frac{n+1}{4^{n+1}(n+2)}\, x^{2n+2}\right|}{\left|\frac{n}{4^n(n+1)}\, x^{2n}\right|} = x^2 \lim_{n\to\infty} \frac{(n+1)^2}{4n(n+2)} = \frac{x^2}{4}.$$

If $\frac{x^2}{4} < 1$, then the series converges. Hence Ratio test states that the series converges if $-2 < x < 2$. We need to check the endpoints $x = \pm 2$. We already checked $x = 2$. For $x = -2$,

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n} = \sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, 4^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges by part **a**. The interval of convergence of the series is $-2 < x < 2$.

**8**. [8 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(x-5)^n.$$

In the following questions, you need to support your answers by stating and properly justifying the use of the test(s) or facts you used to prove the convergence or divergence of the series. Show all your work.

**a**. [2 points] Does the series converge or diverge at $x = 3$?

*Solution:* At $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the alternating series test, since $1/\sqrt{n}$ is decreasing and converges to 0.

**b**. [2 points] What does your answer from part (**a**) imply about the radius of convergence of the series?

*Solution:* Because it converges at $x = 3$, we know that the radius of convergence $R \geq 2$.

**c**. [4 points] Find the interval of convergence of the power series.

*Solution:* Using the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n+1}\sqrt{n+1}}|x-5|^{n+1}}{\frac{1}{2^n\sqrt{n}}|x-5|^n} = \frac{1}{2}|x-5| = L,$$

so the radius of convergence is 2. Now we have to check the endpoints. We know from part (a) that it converges at $x = 3$. For $x = 7$, we get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges. Thus, the interval of convergence is $3 \leq x < 7$.

**8**. [7 points] Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

**a**. [2 points] At which $x$-value is the interval of convergence of this power series centered?

*Solution:* This power series is centered on $x = -2$.

**b**. [5 points] The radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$ is 3. Find the interval of convergence for this power series. Thoroughly justify your answer.

*Solution:* Since the radius of convergence for this power series is 3 and it is centered on $x = 2$, the interval of convergence contains the open interval $(-2-3, -2+3) = (-5, 1)$. Now we only need to check the endpoints $x = -5$ and $x = 1$.

- For $x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(1+2)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the $p$-test with $p = 1$ (this is the harmonic series).
- For $x = -5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-5+2)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test.

Therefore, the interval of convergence for this power series is $[-5, 1)$.

**9**. [5 points] Find the radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^{2n}$

*Solution:* Let the $n$-th term be denoted by $a_n$

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \left|\frac{(2(n+1))!x^{2(n+1)}}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!x^{2n}}\right| \\ &= \left|\frac{(2n+2)(2n+1)x^2}{((n+1)^2}\right|\end{aligned}$$

Therefore, we can use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)x^2}{((n+1)^2}\right| = 4x^2.$$

So this series converges for $x$ with $4x^2 < 1$, or rather with $x^2 < \frac{1}{4}$ which implies that the radius of convergence is $1/2$.

**5**. [10 points]

**a**. [5 points] Determine the **radius** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{4n}}{n^5 (16)^n}.$$

*Solution:* The above series will converge for $x$ values such that

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} < 1$$

by the ratio test. We have

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} = \frac{1}{16}|x-5|^4.$$

and so the desired inequality holds if $\frac{1}{16}|x-5|^4 < 1$. This is equivalent to $|x-5| < 2$. Thus, the radius of convergence is 2.

The radius of convergence is $\underline{2}$.

**b**. [5 points] The power series $\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ has radius of convergence 1. Determine the **interval** of convergence for this power series.

*Solution:*

For $x = 1$, we get the series $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$. We have that $\lim_{n\to\infty} \frac{\frac{n+2}{n^4+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4+2n^3}{n^4+1} = 1$.

The limit comparison test then tells us that the series $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges if and only if the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p > 1$ it converges, and so $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges.

For $x = -1$, we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{n^4+1}$. By the work shown above, this series converges absolutely.

The interval of convergence for $\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ is then $-1 \le x \le 1$

The interval of convergence is $\underline{[-1, 1]}$.