Math 116 — Practice for Exam 2

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2012	3	3		12	
Fall 2013	3	2		11	
Winter 2014	3	4		10	
Fall 2014	3	10		10	
Winter 2015	3	1		10	
Winter 2016	3	2		12	
Winter 2017	3	3		9	
Winter 2018	2	6		12	
Winter 2013	3	9		14	
Total				100	

Recommended time (based on points): 117 minutes

3. [12 points]

a. [6 points] State whether each of the following series converges or diverges. Indicate which test you use to decide. Show all of your work to receive full credit.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: The function $\frac{1}{n\sqrt{\ln n}}$ is decreasing and positive for $n \geq 2$, then the Integral test says that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ behaves as $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}} dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}} du = \lim_{b\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{\ln b} = \infty.$$

Hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$

Solution: Since $0 \leq \dfrac{\cos^2(n)}{\sqrt{n^3}} \leq \dfrac{1}{n^{\frac{3}{2}}}$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by p-series test ($p = \frac{3}{2} > 1$), then comparison test yields the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$.

b. [6 points] Decide whether each of the following series converges absolutely, converges conditionally or diverges. Circle your answer. No justification required.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$

Converges absolutely **Converges conditionally** Diverges

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{3n}}{5^n}$

Converges absolutely Converges conditionally **Diverges**

Solution: $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right| = \sum_{n=0}^{\infty}\frac{\sqrt{n^2+1}}{n^2+n+8}$ behaves as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ since

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n^2+1}}{n^2+n+8}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n\sqrt{n^2+1}}{n^2+n+8} = 1 > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series test $p=1$), then by limit comparison test $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right|$ diverges.

The convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$ follows from alternating series test since for $a_n = \frac{\sqrt{n^2+1}}{n^2+n+8}$:

- $\lim\limits_{n\to\infty} a_n = 0.$
- a_n is decreasing

$$\frac{d}{dn}\left(\frac{\sqrt{n^2+1}}{n^2+n+8}\right) = \frac{-1+6n-n^3}{\sqrt{1+n^2}(n^2+n+8)^2} < 0$$

for n large.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^{3n}}{5^n} = \sum_{n=0}^{\infty}\left(-\frac{8}{5}\right)^n$ is a geometric series with ratio $r = -\frac{8}{5} < -1$, hence it diverges.

2. [11 points] Determine the convergence or divergence of the following series. In parts (a) and (b), support your answers by stating and properly justifying any test(s), facts or computations you use to prove convergence or divergence. Circle your final answer. Show all your work.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ CONVERGES **DIVERGES**

Solution:

$$\lim_{n\to\infty} \frac{9n}{e^{-n}+n} = \lim_{n\to\infty} \frac{9n}{n} = 9 \neq 0.$$

Since $\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ diverges.

b. [4 points] $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$. **CONVERGES** DIVERGES

Solution: The function $f(n) = \frac{4}{n(\ln n)^2}$ is positive and decreasing for $n > 2$, then by Integral Test the convergence or divergence of $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$ can be determined with the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{4}{x(\ln x)^2} dx$

$$\int \frac{4}{x(\ln x)^2} dx = \int \frac{4}{u^2} du \qquad \text{where } u = \ln x.$$

$$= -\frac{4}{u} + C = -\frac{4}{\ln x} + C$$

Hence

$$\int_2^{\infty} \frac{4}{x(\ln x)^2} dx = \lim_{b\to\infty} -\frac{4}{\ln x}\Big|_2^b = -\frac{4}{\ln 2} \qquad \text{converges.}$$

or

$$\int_2^{\infty} \frac{4}{x(\ln x)^2} dx = 4\int_{\ln 2}^{\infty} \frac{1}{u^2} du \qquad \text{converges by } p\text{-test with } p = 2 > 1.$$

c. [4 points] Let r be a **real** number. For which values of r is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^r+4}$ absolutely convergent? Conditionally convergent? No justification is required.

Solution:

Absolutely convergent if : $r > 3$

Conditionally convergent if : $2 < r \leq 3$

Justification (not required):

- Absolute convergence:
  The series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^2}{n^r+4}\right| = \sum_{n=1}^{\infty}\frac{n^2}{n^r+4}$ behaves like $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^r} = \sum_{n=1}^{\infty}\frac{1}{n^{r-2}}$. The last series is a p-series with $p = r-2$ which converges if $r-2>1$. Hence the series converges absolutely if $r>3$.

- Conditionally convergence:
  The function $\dfrac{n^2}{n^r+4}$ is positive and decreasing (for large values of n) when $r>2$. Hence by the Alternating series test $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^r+4}$ converges in this case.

4. [10 points] Determine whether the following series converge or diverge. Show all of your work and justify your answer.

a. [5 points] $\displaystyle\sum_{n=1}^{\infty} \frac{8^n + 10^n}{9^n}$

Solution: $\lim_{n\to\infty} \frac{8^n+10^n}{9^n} = \infty$ therefore by the n^{th} term test the series diverges.

b. [5 points] $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^3 + n^2\cos(n)}$

Solution: $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^3 + n^2\cos(n)} \leq \sum_{n=4}^{\infty} \frac{1}{n^2(n-1)} \leq \sum_{n=4}^{\infty} \frac{1}{n^2}$. The final series is a convergent p series since $p = 2 > 1$. Therefore the original series converges by comparison.

10. [10 points] Determine whether the following series converge or diverge. Justify your answers.

a. [5 points] $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$

Solution: Note that this series is alternating and that the absolute values of the terms $\dfrac{|\ln(n)|}{|n|}$ form a decreasing sequence that converges to 0 as n approaches ∞. Therefore, this series converges by the alternating series test.

b. [5 points] $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3}+2}$

Solution: This can be done with a comparison or limit comparison test. For comparison:

$$\frac{n}{\sqrt{n^3}+2} > \frac{n}{3\sqrt{n^3}} = \left(\frac{1}{3}\right)\frac{1}{\sqrt{n}}$$

By the p-test with $p = 1/2 < 1$, we have that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. By the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3}+2}$ also diverges.

1. [10 points] Show that the following series converges. Also, determine whether the series converges conditionally or converges absolutely. Circle the appropriate answer below. **You must show all your work and indicate any theorems you use to show convergence and to determine the type of convergence**.

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

$\boxed{\text{CONVERGES CONDITIONALLY}}$ CONVERGES ABSOLUTELY

Solution:
The series we obtain when we take the absolute value of the terms in the series above is $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. Now consider the integral $\int_2^{\infty} \frac{\ln(x)}{x} dx$. By making a change of variables we see that

$$\begin{aligned}\int_2^{\infty} \frac{\ln(x)}{x} dx &= \lim_{b\to\infty} \int_2^{b} \frac{\ln(x)}{x} dx \\ &= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} u du \\ &= \lim_{b\to\infty} \frac{(\ln(b))^2}{2} - \frac{(\ln(2))^2}{2} \\ &= +\infty\end{aligned}$$

and so the integral above diverges. Thus, the integral test implies that $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ diverges.

Since $\frac{\ln(n+1)}{n+1} \le \frac{\ln(n)}{n}$ and $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, we have that $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the alternating series test.

Altogether we have shown that the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is conditionally convergent.

2. [12 points] In this problem **you must give full evidence supporting your answer, showing all your work and indicating any theorems about series you use.**

a. [7 points] Show that the following series **converges**. Does it converge conditionally or absolutely? Justify.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! + 2^n}$$

Solution: We notice that

$$\left| \frac{(-1)^n}{n! + 2^n} \right| = \frac{1}{n! + 2^n} \leq \frac{1}{2^n} \quad \text{for all } n \geq 1 \quad (*)$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges because it is a geometric series with ratio $1/2$ and $|1/2| < 1$. Thus, using the inequality $(*)$, the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n! + 2^n} \right|$$

converges by the comparison test. Since this series converges, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! + 2^n}$$

converges absolutely. (This also shows that it converges).

b. [5 points] Determine whether the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution: The function $f(x) = \frac{1}{x \ln x}$ is decreasing and has $\lim\limits_{x \to \infty} f(x) = 0$. Moreover,

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\, du = \lim_{b \to \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$$

where in the second equality we used the substitution $u = \ln x$. So the integral $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges. Therefore, the integral test applies and tells us that the series in question diverges as well.

3. [9 points] In this problem you must give full evidence supporting your answer, showing all your work and indicating any theorems or tests about series you use. (Remark: You **cannot** use any results about convergence from the team homework without justification.)

a. [4 points] Determine whether the series below converges or diverges, and circle your answer clearly. Justify your answer as described above.

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$$

Converges $\boxed{\textbf{Diverges}}$

Limit compare with $\sum \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\sin(1/\sqrt{n})}{1/\sqrt{n}} \overset{LH}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{\sqrt{n}}\right)\cdot \cancel{-\frac{1}{2}(n)^{-3/2}}}{\cancel{-\frac{1}{2}(n)^{-3/2}}}$$

(Top and bottom approach 0 as $n\to\infty$, so can use L'Hôpital's Rule) $= \lim_{n\to\infty} \cos\left(\frac{1}{\sqrt{n}}\right) = \cos(0) = 1.$

$\sum_{1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-test ($p=\frac{1}{2}$), so $\sum \sin\left(\frac{1}{\sqrt{n}}\right)$ diverges by limit comparison.

b. [5 points] Determine if the following infinite series converges absolutely, converges conditionally, or diverges, and circle your answer clearly. Justify your answer as described above.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$$

Converges Absolutely $\boxed{\textbf{Converges Conditionally}}$ **Diverges**

Alternates ✓
|terms| decreasing ✓
terms $\to 0$ ✓
So series converges by the alternating series test.

But: $\sum \left|\frac{(-1)^n}{n\ln(n)}\right| = \sum \frac{1}{n\ln(n)}$ can be resolved using the integral test:

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{b\to\infty} \ln\ln x \Big|_2^b = \lim_{b\to\infty} \ln\ln b - \ln\ln 2 = \infty.$$

So $\sum$ |terms| diverges by the integral test.

6. [12 points] Determine whether the following series converge or diverge.
Fully justify your answer. Show all work and indicate any convergence tests used.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + n\cos(n)}{\sqrt{n^8 - n + 1}}$ $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: Since

$$\frac{n^2 + n\cos(n)}{\sqrt{n^8 - n + 1}} \approx \frac{1}{n^2},$$

we use the limit comparison test comparing with $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{n^2+n\cos(n)}{\sqrt{n^8-n+1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4 + n^3\cos(n)}{\sqrt{n^8 - n + 1}} = 1.$$

We can see this is true by using domination arguments: the numerator is dominated by n^4, while $\sqrt{n^8 - n + 1}$ is dominated by $\sqrt{n^8}$ Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test ($p = 2$), our original series converges by the LCT.

b. [6 points] $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{e^n}$ $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: The terms in this series are not positive, but it is also not an alternating series. We will consider the series

$$\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right| = \sum_{n=0}^{\infty} \frac{|\sin(n)|}{e^n}. \tag{1}$$

Since $|\sin(n)| \leq 1$ we have

$$\sum_{n=0}^{\infty} \frac{|\sin(n)|}{e^n} \leq \sum_{n=0}^{\infty} \frac{1}{e^n}.$$

The larger series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by the geometric series test since the common ratio $1/e$ is less than 1. (Note that there are many other ways to show that this series converges.) Therefore $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right|$ converges by comparison. Since $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right|$ converges, our original series $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{e^n}$ converges absolutely, and, specifically, must itself converge (this is sometimes called the absolute convergence test).

9. [14 points] Determine the convergence or divergence of the following series. In questions (**a**) and (**b**) you need to support your answers by stating and properly justifying the use of the test(s) or facts you used to prove the convergence or divergence of the series. Circle your answer. Show all your work.

a. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+1}}$ Converges Diverges

Solution: You can use either the limit comparison test or the comparison test. We simply use the comparison test. We know that

$$0 < \frac{2n}{\sqrt{n^5+1}} \le \frac{2n}{n^{5/2}} \le 2\frac{1}{n^{3/2}}.$$

Because $\displaystyle\sum \frac{1}{n^{3/2}}$ converges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+1}}$ converges by the comparison test.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ Converges Diverges

Solution: Since the function $f(x) = x^2 e^{-x^3}$ is positive and decreasing for $x > 1$, we can use the integral test to determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$. To do this, we use u-substitution. Let $u = -x^3$, $du = -3x^2 dx$. Therefore

$$\int_0^{\infty} x^2 e^{-x^3} dx = \lim_{b\to\infty} \int_0^b x^2 e^{-x^3} dx = \lim_{b\to\infty} \frac{1}{3} \int_{-b^3}^0 e^u du$$
$$= \lim_{b\to\infty} \frac{1}{3} e^u \Big|_{-b^3}^0 = \frac{1}{3}.$$

Hence $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by the integral test.

c. [6 points] Determine if the following series converge absolutely, conditionally or diverge. Circle your answers. No justification is required.

a). $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(3n)}{n^6+1}$

Converges absolutely Converges conditionally Diverges

b). $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+1}$

Converges absolutely Converges conditionally **Diverges**