Math 116 — Practice for Exam 3

Name: SOLUTIONS

Instructor: ______________ Section Number: ______

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2011	3	4	savings	9	
Fall 2012	3	7	banks	14	
Winter 2012	3	10	drug	9	
Fall 2013	3	10	movie tickets	13	
Winter 2011	3	4	signal fire	8	
Winter 2015	3	11	checkers	12	
Fall 2014	3	4	robot army	8	
Total				73	

Recommended time (based on points): 88 minutes

4. [9 points] Ramon starts depositing \$10,000 each year at his 25th birthday into a retirement account and continues until his 45th birthday. After this point, he does not touch the account until he is 65. The retirement account accrues interest at a rate of 3% compounded annually.

a. [3 points] Let R_n be the amount of money *in thousands* of dollars in Ramon's retirement account after n years from his initial deposit. Find an expression for R_0, R_1 and R_2.

Solution:
$R_0 = 10$, since Ramon deposits \$10,000 initially.
$R_1 = 10 + 10(1.03)$, since Ramon deposits another \$10,000 and the previous year's deposit accrues interest.
$R_2 = 10 + [10 + 10(1.03)]\,(1.03) = 10 + 10(1.03) + 10(1.03)^2$, since all of R_1 accrues interest.

b. [3 points] Find a closed form expression (an expression that does not involve a long summation) for how much money Ramon has in his retirement account at his 45th birthday.

Solution: From the calculations in part (a), we can see that

$$R_n = 10 + 10(1.03) + 10(1.03)^2 + \ldots + 10(1.03)^n.$$

Ramon's 45th birthday corresponds to $n = 20$, so

$$R_{20} = 10 + 10(1.03) + 10(1.03)^2 + \ldots + 10(1.03)^{20} = \sum_{k=0}^{20} 10(1.03)^k.$$

As this is a finite geometric series with 21 terms, a closed form expression is

$$R_{20} = \frac{10(1 - 1.03^{21})}{1 - 1.03},$$

where the amount is given in thousands of dollars.

c. [3 points] Find a closed form expression for how much money Ramon has in his retirement account when he is 65 years old. Compute its value.

Solution: Since Ramon stops depositing money after his 45th birthday, his account is just accumulating interest (at an annual rate of 3%) for the next 20 years. Thus, when he is 65 years old, his account balance is

$$R_{20}\,(1.03)^{20} = \frac{10(1 - 1.03^{21})}{1 - 1.03}\,(1.03)^{20} \approx 517.92923$$

in thousands of dollars (\$517,929.23).

7. [14 points] You want to open a savings account to deposit 1000 dollars. Three banks offer the following options:

a. [3 points] Bank A offers its clients a savings account that earns 1.5% per year compounded annually. Define the sequence A_n to be the amount of money in the savings account n years after you deposit your 1000 dollars. Find a formula for A_n.

Solution: $A_n = 1000(1.015)^n$

b. [7 points] Bank B offers its clients a savings account that earns 2% per year compounded annually. At the end of each year, after the bank deposits the interest you earned, it withdraws a 1 dollar service fee from the account. Define the sequence B_n to be the amount of money, right after the service fee deduction, in the savings account n years after you deposit your 1000 dollars. Find B_1, B_2, B_3 and a **closed form** formula for B_n.

Solution:

$$
\begin{aligned}
B_1 &= 1000(1.02) - 1 = 1019. \\
B_2 &= (1000(1.02) - 1)(1.02) - 1 = 1000(1.02)^2 - (1 + 1.02) = 1038.38. \\
B_3 &= \left(1000(1.02)^2 - (1 + 1.02)\right)(1.02) - 1 = 1000(1.02)^3 - (1 + 1.02 + 1.02^2) \\
&= 1058.15. \\
&\vdots \\
B_n &= 1000(1.02)^n - (1 + 1.02 + 1.02^2 + \cdots + 1.02^{n-1}) = 1000(1.02)^n - \frac{1 - 1.02^n}{1 - 1.02}
\end{aligned}
$$

c. [4 points] Bank C offers its clients a savings account that earns interest continuously at a rate of 1.5% of the current balance per year. At the same time, the bank withdraws a service fee from the account at a rate of 1 dollar per year continuously. Let $M(t)$ be the amount of money in the savings account t years after you deposit your 1000 dollars. Write the differential equation satisfied by $M(t)$. Include initial conditions.

Solution: $\dfrac{dM}{dt} = 0.015M - 1, \qquad M(0) = 1000.$

10. [9 points] A patient takes a drug in doses of 100 mg once every 24 hours. The half-life of the drug in the patient's body is 12 hours. Let D_n be the amount of the drug in the patient immediately after taking the nth dose of the drug. Be sure to include units.

a. [3 points] Find D_1, D_2 and D_3.

Solution: Since the half-life is 12 hours, after 24 hours, $\frac{1}{4}$ of the drug remains in the body.

$$D_1 = 100\text{mg}$$
$$D_2 = 100 + 100\left(\tfrac{1}{4}\right) = 100\left(1 + \tfrac{1}{4}\right) = 125\text{mg}$$
$$D_3 = 100\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = 100\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = 131.25 \text{ mg}$$

b. [4 points] Find a closed form expression (an expression that does not involve a long summation or a recursive formula) for D_n.

Solution: From part **a**

$$D_n = 100 + 100\left(\tfrac{1}{4}\right) + 100\left(\tfrac{1}{4}\right)^2 + \cdots + 100\left(\tfrac{1}{4}\right)^{n-1}.$$

This is a finite geometric series with first term 100 and the common ratio between terms is $\frac{1}{4}$. So we have

$$D_n = \sum_{k=1}^{n} 100\left(\tfrac{1}{4}\right)^{k-1} = \frac{100\left(1 - \left(\frac{1}{4}\right)^n\right)}{\frac{3}{4}} = \frac{400}{3}\left(1 - \left(\tfrac{1}{4}\right)^n\right) \text{ mg}$$

c. [2 points] What is $\lim_{n\to\infty} D_n$?

Solution: Taking the limit, we obtain

$$\lim_{n\to\infty} \frac{400}{3}\left(1 - \left(\tfrac{1}{4}\right)^n\right) = \frac{400}{3} \approx 133.3 \text{ mg}$$

10. [13 points] The blockbuster action movie *Mildred's Adventures with Calculus!* was just released. During the first week after the premiere, 2.5 million people went to see it. The studio has conducted a study to gauge the impact of the film on audiences, and found that: *the number of tickets sold in a given week is 60% of the number of tickets sold the previous week.* Assume that this process repeats every week.

a. [5 points] Let p_k be the number of movie tickets, in millions, sold during the kth week after the premiere of the movie. Determine p_2, p_3 and a formula for p_k.

Solution:

$$\begin{aligned} p_1 &= 2.5 \\ p_2 &= 2.5(0.6) \\ p_3 &= 2.5(0.6)^2 \\ p_k &= 2.5(0.6)^{k-1}. \end{aligned}$$

b. [6 points] A movie ticket costs \$8. Let T_n be the total amount of money earned in ticket sales, in millions of dollars, during the first n weeks the movie has been exhibited. Determine T_3 and a closed formula for T_n. Show all your work.

Solution:

$$\begin{aligned} T_1 &= 8(2.5) \\ T_2 &= 8(2.5 + 2.5(0.6)) \\ T_3 &= 8(2.5 + 2.5(0.6) + 2.5(0.6)^2) \\ T_n &= 8(2.5 + 2.5(0.6) + 2.5(0.6)^2 + \cdots + 2.5(0.6)^{n-1}) \\ T_n &= 8(2.5)\frac{1-(0.6)^n}{1-.6} = 50(1-(0.6)^n) \end{aligned}$$

c. [2 points] Determine the value of $\lim_{n\to\infty} T_n$.

Solution: $\lim_{n\to\infty} T_n = \lim_{n\to\infty} 50(1-(0.6)^n) = 50$

4. [8 points] You are trapped on an island, and decide to build a signal fire to alert passing ships. You start the fire with 200 pounds of wood. During the course of a day, 40% of the wood pile burns away (so 60% remains). At the end of each day, you add another 200 pounds of wood to the pile. Let W_i denote the weight of the wood pile immediately after adding the i^{th} load of wood (the initial 200-pound pile counts as the first load).

a. [3 points] Find expressions for W_1, W_2 and W_3.

Solution:

$$W_1 = 200$$
$$W_2 = 200 + 200(0.6)$$
$$W_3 = 200 + 200(0.6) + 200(0.6)^2$$

b. [3 points] Find a closed form expression for W_n (*a closed form* expression means that your answer should not contain a large summation).

Solution:

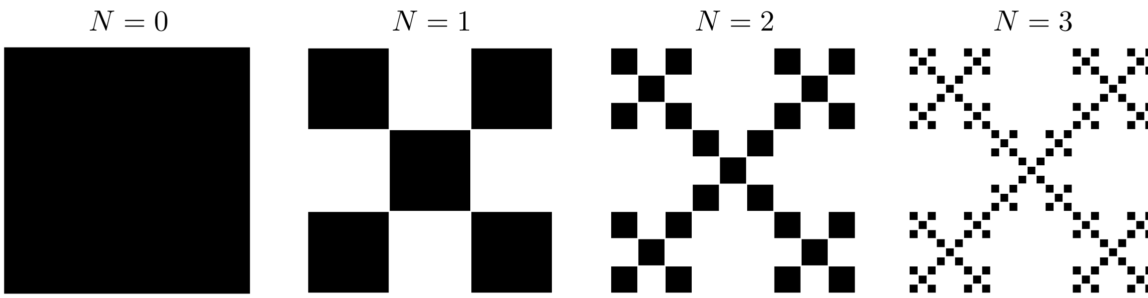
$$W_n = \frac{200(1 - 0.6^n)}{1 - 0.6}$$

c. [2 points] Instead of starting with 200 pounds of wood and adding 200 pounds every day, you decide to start with P pounds of wood and add P pounds every day. If you plan to continue the fire indefinitely, determine the largest value of P for which the weight of the wood pile will never exceed 1000 pounds.

Solution:

$$\frac{P}{1 - 0.6} = 1000$$
$$P = 400$$

11. [12 points] You construct a snowflake by starting with a square piece of paper of side length 3 inches. You divide the square into a three by three grid of squares of side length one and remove the four squares in the grid that share a side with the center square in the grid. For each remaining square in the grid, subdivide each of them into 9 equally sized squares and remove the four squares in each of these new grids that share a side with the center square in the grid. You continue in this manner for a long time.



a. [3 points] Write a formula that gives the perimeter, P_N, of the black squares that make up the snowflake after N steps.

Solution: $P_N = 12\left(\frac{5}{3}\right)^N$

b. [2 points] Find $\lim_{N\to\infty} P_N$.

Solution: P_N tends to infinity as $N \to \infty$.

c. [3 points] Suppose $N \geq 1$. Write a sum that gives the area, A_N of all the squares you have **removed** after N steps.

Solution: $\displaystyle\sum_{j=0}^{N-1} 4\left(\frac{5}{9}\right)^j$

d. [2 points] Write a closed form expression for A_N.

Solution: $A_N = 4\dfrac{1-\left(\frac{5}{9}\right)^N}{1-\frac{5}{9}}$

e. [2 points] Find the limit as $N \to \infty$ of your expression in (d).

Solution: $\lim_{N\to\infty} A_N = 9$

4. [8 points] Franklin's robots start building more robots to replace their deactivated comrades. The initial number of robots in Franklin's army is 800. Each minute, the number of robots increases by 15%. At the end of each minute, you fire an EMP which immediately deactivates 50 robots.

a. [3 points] Let R_n denote the number of active robots in Franklin's army immediately after the EMP is fired for the n-th time. Find R_1 and R_2.

Solution:

$$R_1 = (1.15)800 - 50$$
$$R_2 = (1.15)((1.15)800 - 50) - 50$$

b. [4 points] Find a closed form expression for R_n (i.e. evaluate any sums and solve any recursion).

Solution:

$$R_n = 800(1.15)^n - \sum_{i=0}^{n-1} 50(1.15)^i$$
$$= 800(1.15)^n - \frac{50(1 - 1.15^n)}{1 - 1.15}$$

c. [1 point] Find $\lim_{n\to\infty} R_n$. No justification is necessary.

Solution:

$$\lim_{n\to\infty} R_n = \infty$$