MATH 116 — PRACTICE FOR EXAM 2

NAME: SOLUTIONS

INSTRUCTOR: ______________________ SECTION NUMBER: __________

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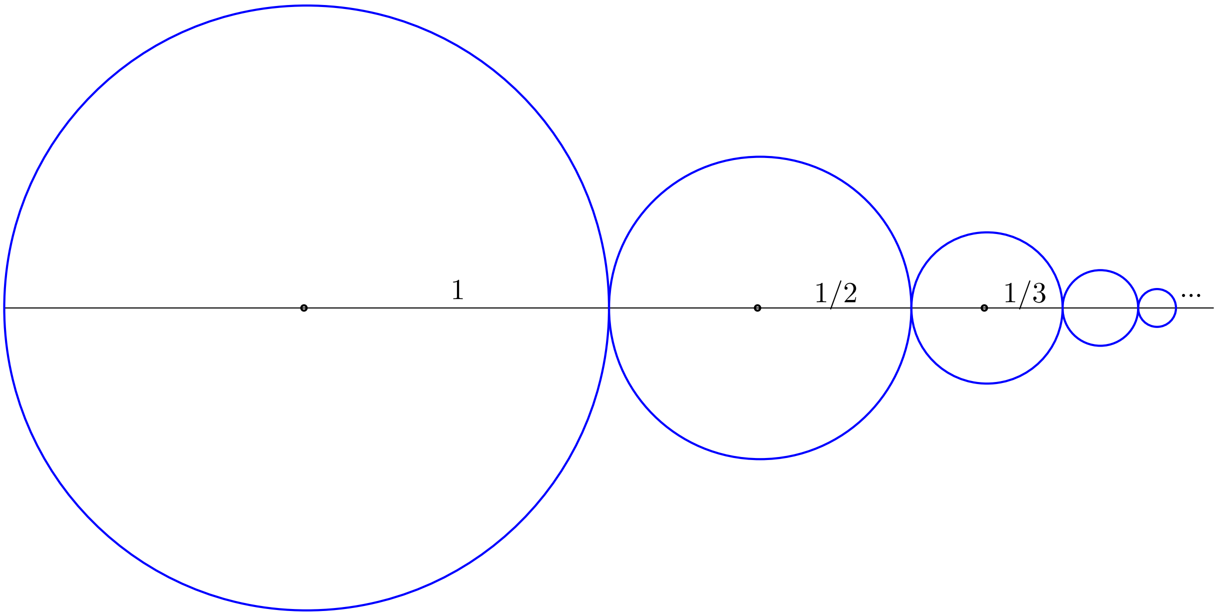
1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	**Exam**	**Problem**	**Name**	**Points**	**Score**
Winter 2016	3	5	pizzas	6	
Winter 2016	3	10		14	
Winter 2018	2	11		10	
Fall 2017	2	10	gamma	9	
Total				39	

Recommended time (based on points): 42 minutes

5. [6 points] O-guk is eating pizzas! All is well now, so he got hungry. He has put them next to each other, as depicted below, so that he can devour them one after another. There are infinitely many pizzas, and they have radii $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
The following figure shows the first five pizzas.



a. [4 points] Write infinite series for the total area and the total perimeter of the pizzas. You must write your series in sigma notation.

Total area: $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{n^2}$

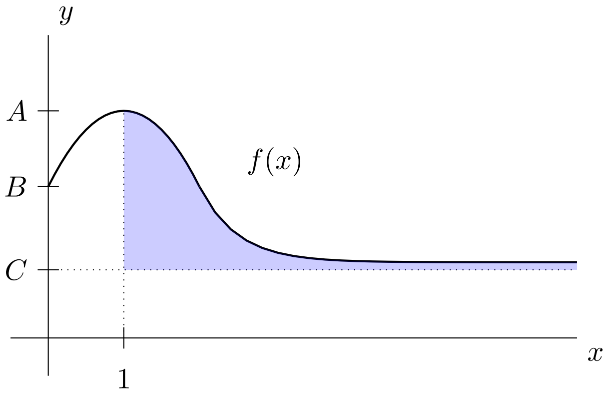
Total perimeter: $\displaystyle\sum_{n=1}^{\infty} \frac{2\pi}{n}$

b. [2 points] In the next two questions **circle** the correct answer.

Is the total area a finite number?
[✓ **YES**] **NO**

Is the total perimeter a finite number?
YES [✓ **NO**]

10. [14 points] A function f has domain $[0, \infty)$, and its graph is given below. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



a. [5 points] Determine the convergence of the improper integral below. **You must give full evidence supporting your answer, showing all your work and indicating any theorems about integrals you use**.

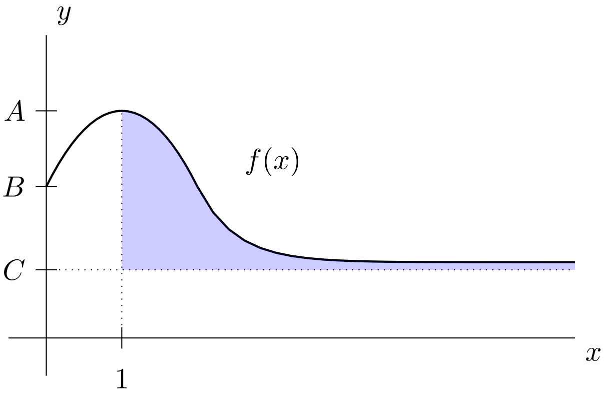
$$\int_0^1 \frac{f(x)}{x}\,dx$$

Solution: We have that for $0 < x \leq 1$

$$\frac{f(x)}{x} \geq \frac{B}{x}$$

The improper integral $\int_0^1 \frac{B}{x}\,dx = B\int_0^1 \frac{1}{x}\,dx$ diverges by the p-test with $p = 1$. Thus, the integral $\int_0^1 \frac{f(x)}{x}\,dx$ diverges by the comparison test.

10. (continued) For your convenience, the graph of f is given again. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



b. [3 points] **Circle** the correct answer. The value of the integral $\int_1^\infty f(x)f'(x)\,dx$

is $C - A$ [✓ is $\frac{C^2-A^2}{2}$] is $B - A$ cannot be determined diverges

c. [3 points] **Circle** the correct answer. The value of the integral $\int_1^\infty f'(x)\,dx$

[✓ is $C - A$] is $\frac{C^2-A^2}{2}$ is C cannot be determined diverges

d. [3 points] Determine, with justification, whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}(f(n) - C)$$

Solution: We notice that the function $f(x) - C$ is decreasing, positive with $\lim_{x\to\infty}(f(x) - C) = 0$. By the integral test, the series

$$\sum_{n=1}^{\infty}(f(n) - C)$$

converges if and only if the improper integral

$$\int_1^\infty (f(x) - C)$$

converges. But this integral gives exactly the shaded area, which we know that it is finite. So this integral converges and therefore the series converges as well.

11. [10 points] You work for a temp agency. Today you fill in for Russ Weterson, doing important work for the city. On Mr. Weterson's desk you find the following problems with a note: "Russ, the Mayor needs these problems done yesterday. -Brontel"

Suppose $f(x)$ and $g(x)$ are positive, continuous, decreasing functions such that

1. $\int_1^\infty f(x)\,dx$ converges, and
2. $0 \leq g(x) \leq 9$ for all real numbers x.

Determine whether the following expressions must converge, must diverge, or whether convergence cannot be determined. **No justification required.**

a. [2 points] $\displaystyle\int_1^\infty \frac{1}{f(x)}\,dx$

Converges [✓ Diverges] Cannot be determined

b. [2 points] $\displaystyle\sum_{n=1}^{\infty} f(n)$

[✓ Converges] Diverges Cannot be determined

c. [2 points] $\displaystyle\int_1^\infty f(x)g(x)\,dx$

[✓ Converges] Diverges Cannot be determined

d. [2 points] $\displaystyle\sum_{n=1}^{\infty} f(n)^{g(n)}$

Converges Diverges [✓ Cannot be determined]

e. [2 points] $\displaystyle\int_1^\infty g(x)\,dx$

Converges Diverges [✓ Cannot be determined]

10. [9 points] The sequence $\{\gamma_n\}$ is defined according to the formula

$$\gamma_n = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k}.$$

(You may recall this sequence from team homework 5.) This sequence converges to a positive number γ (which happens to be $\gamma \approx 0.5772156649$).

a. [2 points] Does the sequence $\{\gamma_n^2\}$ converge or diverge? If this sequence converges, compute the value to which this sequence converges, either in terms of the constant γ or with five decimal places of accuracy.

Solution: Yes, this sequence converges.

$$\lim_{n\to\infty} \gamma_n^2 = \left(\lim_{n\to\infty} \gamma_n\right)^2 = \gamma^2 \approx 0.33318.$$

b. [3 points] Does the series $\sum_{n=1}^{\infty} \gamma_n$ converge or diverge? Briefly explain your answer, and if this series converges, compute the value to which the series converges either in terms of the constant γ or with five decimal places of accuracy.

Solution: This series diverges by the n^{th} term test for divergence. The terms of this series do not approach 0; instead the terms approach γ, i.e.

$$\lim_{n\to\infty} \gamma_n = \gamma \approx 0.5772156649 \neq 0.$$

c. [4 points] Let $h_n = \sum_{k=1}^{n} \frac{1}{k}$. Find the value of $\lim_{n\to\infty} \frac{e^{h_n}}{n}$.

You may give your answer either in terms of the constant γ or with five decimal places of accuracy.

Hint: First consider $\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right)$.

Solution: Following the hint, we first compute

$$\ln\left(\frac{e^{h_n}}{n}\right) = \ln\left(\frac{e^{\sum_{k=1}^{n} 1/k}}{n}\right) = \ln(e^{\sum_{k=1}^{n} 1/k}) - \ln(n) = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k} = \gamma_n.$$

Next, we take the limit of this expression.

$$\lim_{n\to\infty} \ln\left(\frac{e^{h_n}}{n}\right) = \lim_{n\to\infty} \gamma_n = \gamma \approx 0.5772156649.$$

Finally, the limit we are looking for is found by exponentiating this result (in order to "undo" the natural log from before).

$$\lim_{n\to\infty} \frac{e^{h_n}}{n} = e^{\gamma} \approx e^{0.5772156649} \approx 1.78107.$$