Math 116 — Practice for Exam 3

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2013	2	5	pneumonia	10	
Winter 2013	3	4	tree	10	
Winter 2014	2	5	telemarketing	7	
Fall 2012	2	8	internet cafe	14	
Fall 2013	2	9	coffee	7	
Winter 2012	2	9	wire	14	
Winter 2014	2	6		6	
Total				68	

Recommended time (based on points): 64 minutes

5. [10 points] Consider a group of people who have received a new treatment for pneumonia. Let t be the number of days it takes for a person with pneumonia to fully recover. The probability density function giving the distribution of t is

$$f(t) = \frac{10}{(1+at)^2}, \qquad \text{for } t > 0,$$

for some positive constant a.

a. [2 points] Give a practical interpretation of the quantity $\int_3^{10} f(t)dt$. You do not need to compute the integral.

Solution: The fraction of the people who recovered in between three and ten days after the treatment.

b. [5 points] Find a formula for the cumulative distribution function $F(t)$ of $f(t)$ for $t > 0$. Show all your work. Your answer may include a, but it should not contain any integrals.

Solution: $F(t) = \int_0^t \frac{10}{(1+ax)^2}dx = -\frac{10}{a(1+ax)}\Big|_0^t = \frac{10}{a} - \frac{10}{a(1+at)}$

c. [3 points] Determine the value of a. Show all your work.

Solution: Since $f(t)$ is a probability density function, then $1 = \int_0^{\infty} \frac{10}{(1+ax)^2}dx$. Hence

$$\int_0^{\infty} \frac{10}{(1+ax)^2}dx = \lim_{b\to\infty} \int_0^{b} \frac{10}{(1+ax)^2}dx = \lim_{b\to\infty} \frac{10}{a} - \frac{10}{a(1+ab)} = \frac{10}{a}.$$

Hence $a = 10$.

4. [10 points] The lifetime t (in years) of a tree has probability density function

$$f(t) = \begin{cases} \dfrac{a}{(t+1)^p} & \text{for} \quad t \geq 0. \\ \\ 0 & \text{for} \quad t < 0. \end{cases}$$

where $a > 0$ and $p > 1$.

a. [4 points] Use the comparison method to find the values of p for which the average lifetime M is finite ($M < \infty$). Properly justify your answer.

Solution: The average lifetime M is given by the formula $M = \displaystyle\int_0^\infty t\frac{a}{(t+1)^p}dt$.
Since

$$t\frac{a}{(t+1)^p} \leq t\frac{a}{t^p} = \frac{a}{t^{p-1}} \qquad \text{for } t > 0,$$

then

$$\int_1^\infty t\frac{a}{(t+1)^p}dt \leq \int_1^\infty \frac{a}{t^{p-1}}dt$$

We know that $a\displaystyle\int_1^\infty \frac{1}{t^{p-1}}$ converges precisely when $p-1 > 1$ ($p > 2$) by the p-test, so the first integral converges precisely when $p > 2$. This implies that the average lifetime M is finite for $p > 2$.
Note:

- We use the inequality $\displaystyle\int_1^\infty t\frac{a}{(t+1)^p}dt \leq \int_1^\infty \frac{a}{t^{p-1}}dt$ since the inequality $\displaystyle\int_0^\infty t\frac{a}{(t+1)^p}$ $\displaystyle\int_0^\infty \frac{a}{t^{p-1}}dt$ is not useful $\left(\displaystyle\int_0^\infty \frac{a}{t^{p-1}}dt = \infty \text{ for all values of } p\right)$.
- You do not need to discuss the convergence of the integral $\displaystyle\int_0^1 \frac{at}{(t+1)^p}dt$ since this integral is not an improper integral.

b. [4 points] Find a formula for a in terms of p. Show all your work.

Solution: We know that

$$1 = \int_0^\infty \frac{a}{(t+1)^p} dt.$$

We use u-substition with $u = t + 1$ to calculate the integral:

$$\begin{aligned}
\int_0^\infty \frac{a}{(t+1)^p} dt &= \lim_{b\to\infty} \int_0^b \frac{a}{(t+1)^p} dt \\
&= \lim_{b\to\infty} \int_1^{b+1} \frac{a}{u^p} du = a \lim_{b\to\infty} \int_1^{b+1} u^{-p} du \\
&= a \lim_{b\to\infty} \frac{u^{-p+1}}{(-p+1)}\Big|_1^{b+1} = a \lim_{b\to\infty} \frac{1}{(-p+1)u^{p-1}}\Big|_1^{b+1} \\
\text{(since } p > 1\text{)} \quad &= \frac{a}{p-1}.
\end{aligned}$$

Therefore $1 = \dfrac{a}{p-1}$, so $a = p - 1$.

c. [2 points] Let $C(t)$ be the cumulative distribution function of $f(t)$. For a given tree, what is the practical interpretation of the expression $1 - C(30)$?

Solution: $1 - C(30)$ is the probability that a given tree lives at least 30 years.

5. [7 points] The Terrible Telemarketing corporation has realized that people often hang up on their telemarketing calls. After collecting data they found that the probability that someone will hang up the phone at time t seconds after the call begins is given by the probability density function $p(t)$. The formula for $p(t)$ is given below.

$$p(t) = \begin{cases} 0 & t < 0 \\ te^{-ct^2} & t \geq 0 \end{cases}$$

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a. [5 points] Find the value of c so that $p(t)$ is a probability density function.

Solution:
We must have $\int_0^\infty te^{-ct^2}\,dt = 1$. Let $u = ct^2$ then $du = 2ctdt$. Thus we get the integral $\frac{1}{2c}\int_0^\infty e^{-u}du = \frac{1}{2c}\lim_{N\to\infty} -e^{-u}|_0^N = \frac{1}{2c}$. Thus we have $1 = \frac{1}{2c}$ so $c = \frac{1}{2}$.

b. [2 points] What is the probability that someone will stay on the phone with a telemarketer for more than 4 seconds?

Solution:
The probability is $\int_4^\infty te^{-\frac{1}{2}t^2}\,dt = e^{-8}$.

6. [6 points] Consider the probability density function $q(t)$ shown below.

$$q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2} & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

a. [4 points] What is the cumulative distribution function $Q(t)$ of the density given by $q(t)$? Write your final answer in the answer blanks provided.

Solution:
$Q(t) = \int_{-\infty}^t q(s)\,ds = \int_0^t q(s)\,ds$. If $t < 0$ then $Q(t) = 0$ and if $0 \leq t < 2$ then $Q(t) = \int_0^t q(s)\,ds = \int_0^t \frac{s}{2}\,ds = t^2/4$. Then it follows that $Q(t) = 1$ for $t \geq 2$.

If $t < 0$ then $Q(t) = 0$

If $0 \leq t < 2$ then $Q(t) = t^2/4$

If $t \geq 2$ then $Q(t) = 1$

b. [2 points] What is the median of the distribution?

Solution: The median is the number T such that $Q(T) = \frac{1}{2}$. Thus we want $T^2/4 = \frac{1}{2}$. Therefore $T = \sqrt{2}$.

8. [14 points] A coffee shop offers only one hour of free internet access to all its customers. The time t in hours a customer uses the internet at the coffee shop has a probability density function

$$p(t) = \begin{cases} at\sqrt{1-t^2} & 0 \leq t \leq 1. \\ 0 & \text{otherwise.} \end{cases}$$

where a is a constant.

a. [4 points] For what value of a is $p(t)$ a probability density function? Find its value without using your calculator.

Solution:

$$1 = \int_0^1 at\sqrt{1-t^2}dt = -\frac{a}{2}\int_1^0 \sqrt{u}du = -\frac{a}{2}\frac{2}{3}u^{3/2}\Big|_1^0 = -\frac{a}{2}\left(-\frac{2}{3}\right) = \frac{a}{3}.$$

So, $a = 3$.

b. [4 points] Find the cumulative distribution function $P(t)$ of $p(t)$. Make sure to indicate the value of $P(t)$ for all values of $-\infty < t < \infty$. Your final answer should not contain any integrals.

Solution: $P(t) = \int_{-\infty}^t p(x)dx$, so if $t \leq 0$ then $P(t) = 0$, if $t \geq 1$ then $P(t) = 1$. If $0 < t < 1$,

$$P(t) = \int_0^t 3x\sqrt{1-x^2}dx = -\frac{3}{2}\int_1^{1-t^2} \sqrt{u}du = -\, u^{3/2}\Big|_1^{1-t^2} = 1-(1-t^2)^{3/2}.$$

c. [3 points] Find the the probability that a customer is still using the internet after 40 minutes (without using your calculator).

Solution: The probability that a customer users the internet for 40 minutes or less is $P(40/60) = P(2/3)$. So the probability of using the internet for more than 40 minutes is

$$1 - P(2/3) = 1 - \left(1 - (1 - (2/3)^2)^{3/2}\right) = \left(1 - \frac{4}{9}\right)^{3/2} = \frac{\sqrt{125}}{27}.$$

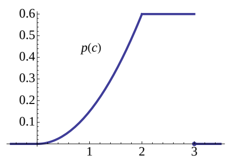
d. [3 points] Find an expression for the mean of this distribution. Use your calculator to compute its value.

Solution:

$$\int_0^1 3t^2\sqrt{1-t^2}dt \approx 0.589 \text{ hours.}$$

9. [7 points]
Thanks to the Math Department's acquisition of a coffee tank in October, there are now 300 cups of coffee available to the graduate students each day.
The department wants to assess how much of the coffee is drunk and how much is wasted. Let c be the amount of coffee drunk in one day, measured in hundreds of cups of coffee. The probability density function for c is given by

$$p(c) = \begin{cases} \frac{3}{20}c^2 & \text{for } 0 \le c \le 2 \\ \frac{3}{5} & \text{for } 2 \le c \le 3 \\ 0 & \text{otherwise.} \end{cases}$$



a. [4 points] Find the mean of the amount of coffee drunk in one day. Include units. Show all your work.

Solution: The mean is
$$\int_{-\infty}^{\infty} c\ p(c)dc = \int_0^3 c\ p(c)dc.$$
Since $p(c)$ is defined piecewise, we break the integral into the two pieces:
$$\int_0^3 cp(c)dc = \int_0^2 c \cdot \tfrac{3}{20}c^2 dc + \int_2^3 c \cdot \tfrac{3}{5} dc$$
$$= \tfrac{3}{80}c^4\Big|_0^2 + \tfrac{3}{10}c^2\Big|_2^3 = \tfrac{3}{80} \cdot (16-0) + \tfrac{3}{10}(9-4)$$
$$= 2.1 \text{ or } \tfrac{168}{80}.$$
So, the mean amount of coffee drunk in one day is **210 cups of coffee**.

b. [3 points] Find the median of the amount of coffee drunk in one day. Include units. Show all your work.

Solution: We want to find M such that
$$\int_0^M p(c)dc = \int_M^3 p(c)dc = \frac{1}{2}.$$
From the graph, we see that the area between $c = 2$ and $c = 3$ is $\frac{3}{5}$ (since it is rectangular), which is already more than $\frac{1}{2}$. So, the median will be in the interval $2 \le c \le 3$, and we can use the second part of the piecewise formula:
$$\tfrac{1}{2} = \int_M^3 \tfrac{3}{5} dc = \tfrac{3}{5}(3 - M),$$
so $M = 3 - \frac{5}{6} \approx 2.17$. So, the median amount of coffee drunk in one day is **217 cups**.

(**Note**: the area in the first interval is $\int_0^2 \frac{3}{20}c^2 dc = \frac{1}{20} \cdot 2^3 = \frac{2}{5}$, which is 0.1 less than $\frac{1}{2}$. So, we could instead solve $\int_2^M \frac{3}{5} dc = 0.1$.)

9. [14 points] A machine produces copper wire, and occasionally there is a flaw at some point along the wire. The length x of wire produced between two consecutive flaws is a continuous variable with probability density function

$$f(x) = \begin{cases} c(1+x)^{-3} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work in order to receive full credit.

a. [5 points] Find the value of c.

Solution: Since $f(x)$ is a density function $\int_{-\infty}^{\infty} f(x)dx = 1$. Then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} c(1+x)^{-3}dx = \lim_{b\to\infty} \int_{0}^{b} c(1+x)^{-3}dx$$
$$= \lim_{b\to\infty} \frac{-c}{2(1+x)^2}\Big|_0^b = \frac{c}{2} = 1$$

Hence $c = 2$.

b. [3 points] Find the cumulative distribution function $P(x)$ of the density function $f(x)$. Be sure to indicate the value of $P(x)$ for **all** values of x.

Solution:

$$P(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} c(1+t)^{-3}dt = \frac{-c}{2(1+t)^2}\Big|_0^x = \frac{c}{2} - \frac{c}{2(1+x)^2} = 1 - \frac{1}{(1+x)^2}.$$

$$P(x) = \begin{cases} 1 - \frac{1}{(1+x)^2} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

c. [5 points] Find the mean length of wire between two consecutive flaws.

Solution:

$$\text{mean} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} c\frac{x}{(1+x)^3}dx = c\lim_{b\to\infty}\int_0^b \frac{x}{(1+x)^3}dx$$

$$u = x \qquad v' = (1+x)^{-3}$$

$$u' = 1 \qquad v = \frac{-1}{2(1+x)^2}$$

$$= c\lim_{b\to\infty} \frac{-x}{2(1+x)^2}\Big|_0^b + \int_0^b \frac{1}{2(1+x)^2}dx$$

$$= c\lim_{b\to\infty} \frac{-x}{2(1+x)^2} - \frac{1}{2(1+x)}\Big|_0^b = \frac{c}{2} = 1.$$

or

$$\text{mean} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} c\frac{x}{(1+x)^3}dx = c\lim_{b\to\infty}\int_0^b \frac{x}{(1+x)^3}dx$$

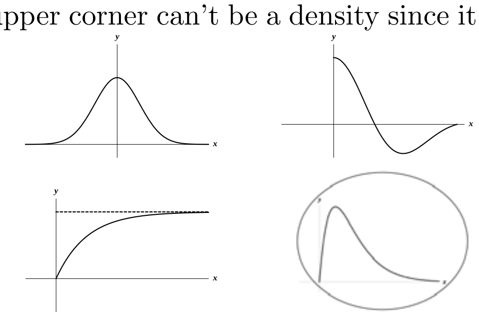
$$u = 1+x$$

$$= c\lim_{b\to\infty}\int_1^{b+1} \frac{u-1}{u^3}dx = c\lim_{b\to\infty}\int_1^{b+1} u^{-2} - u^{-3}dy$$

$$= c\lim_{b\to\infty} -u^{-1} + \frac{u^{-2}}{2}\Big|_1^{b+1} = \frac{c}{2} = 1.$$

d. [1 point] A second machine produces the same type of wire, but with a different probability density function (pdf). Which of the following graphs could be the graph of the pdf for the second machine? Circle all your answers.

Solution: The graph on the left upper corner can't be the density since$x$ is the distance between flaws, hence the probability density function can't be positive for $x < 0$.
The graph on the left lower corner can't be the density since the area under the curve for $x \geq 0$ is infinite (it has a positive horizontal asymptote).
The graph on the right upper corner can't be a density since it is negative on an interval.



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