Math 116 — Practice for Exam 1

Generated February 15, 2021

Name: SOLUTIONS

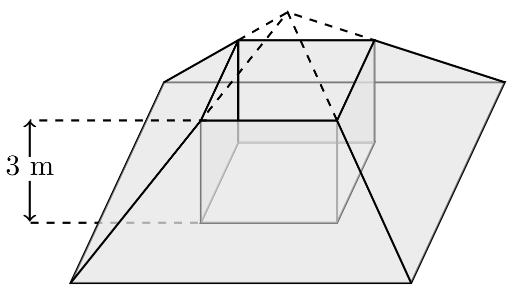
Instructor: ______________________ Section Number: __________

1. This exam has 8 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

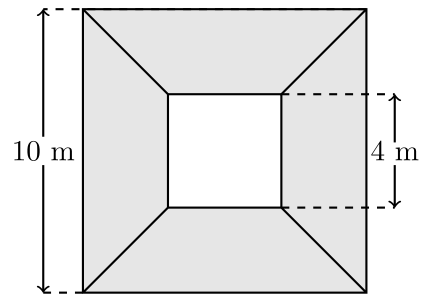
Semester	Exam	Problem	Name	Points	Score
Fall 2016	1	3	fishtank	11	
Winter 2018	1	3	Print.juice	10	
Fall 2015	3	12	pyramid	9	
Winter 2015	1	5	money pool	11	
Winter 2013	1	6	swimming pool	11	
Fall 2013	1	6	triangular tank	11	
Winter 2016	1	9	chocolate prism	9	
Winter 2018	1	9	muck tank	9	
Total				81	

Recommended time (based on points): 76 minutes

3. [11 points] During a trip to the local aquarium, Steph becomes curious and decides to taste the fish food. The fish food tank is completely filled with food, and it is in the shape of a pyramid with a vertical hole through its center, illustrated below (the dashed lines are not part of the tank). The tank itself is 3 m tall, and the pyramid base is a square of side length 10 m. The top and bottom of the hole are squares of side length 4 m. The food is contained in the shaded region only, **not** in the hole.



SIDE VIEW



TOP VIEW

a. [5 points] Write an expression that gives the approximate volume of a slice of fish food of thickness Δh meters, h meters from the bottom of the tank.

Solution: The approximate volume is

$$((10-2h)^2 - 4^2)\Delta h \qquad \text{m}^3.$$

b. [3 points] Suppose that the mass density of fish food is a constant δ kg/m^3. Write, but do **not** evaluate, an expression involving integrals that gives the mass of fish food in the tank.

Solution: The mass of fish food in the tank is given by

$$\delta \int_0^3 ((10-2h)^2 - 4^2)\, dh \qquad \text{kg}.$$

c. [3 points] Write an expression involving integrals that gives $\overline{h}$, the h-coordinate of the center of mass of the fish food, where h is defined as above. Do **not** evaluate your expression.

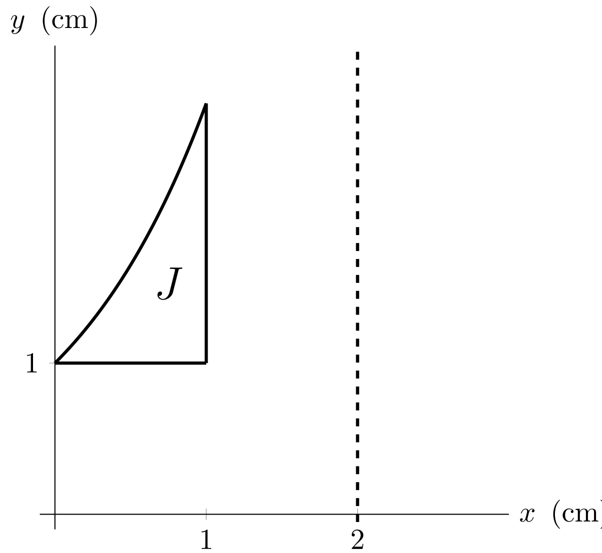
Solution: We have

$$\overline{h} = \frac{\int_0^3 h((10-2h)^2 - 4^2)\, dh}{\int_0^3 ((10-2h)^2 - 4^2)\, dh} \qquad \text{m}.$$

3. [10 points]

Debra McQueath hooked you up with an interview at `Print.juice`. Being a legitimate tech start-up, the `Print.juice` interview consists of answering technical questions on the spot. Debra gave you the following questions for practice.

The region J is a common `Print.juice` shape. It is bounded by $x = 1$, $y = 1$, and $y = e^x$.



a. [3 points] First, consider the solid with base J and square cross sections perpendicular to the x-axis. If the density of the solid is a function of the x-coordinate $a(x)$ g/cm^3, write an integral that represents the total mass of the solid in grams.

Solution: The height of a cross-section is $e^x - 1$, thus the total mass is

$$\int_0^1 a(x)(e^x - 1)^2\, dx.$$

For b. and c., consider the solid made by rotating J around the line $x = 2$.

b. [3 points] If the density of the solid is a function of the y-coordinate $b(y)$ g/cm^3, write an integral that represents the total mass of the solid in grams.

Solution: Using the washer method we compute the total mass to be

$$\int_1^e b(y)\pi((2 - \ln(y))^2 - 1^2)\, dy.$$

c. [4 points] If the density of the solid is a function of the distance r cm from the axis of rotation $c(r)$ g/cm^3, write an integral that represents the total mass of the solid in grams.

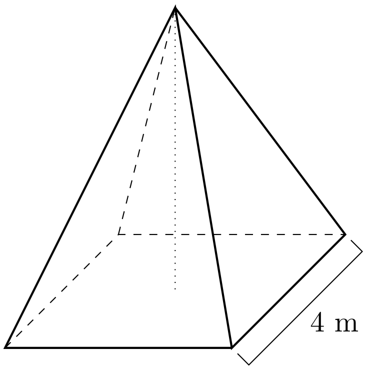
Solution: Using the shell method we can either compute the mass in terms of x or r. In terms of r we get

$$\int_1^2 c(r)2\pi r(e^{2-r} - 1)\, dr,$$

and in terms of x we get

$$\int_0^1 c(2 - x)2\pi(2 - x)(e^x - 1)\, dx.$$

12. [9 points] An oil tank has the shape of a pyramid with a square base of side length 4 meters and height 10 meters. The top of the pyramid lies directly above the center of the base. **Be sure to include units in your answers**. Recall that the gravitational constant is $g = 9.8$ m/s^2.



The tank is filled with oil up to a height of 6 meters.

a. [4 points] Write an expression approximating the mass of a thin horizontal slice of thickness Δy located y meters **below** the top of the tank. The density of the oil is 880 kg/m^3. Don't forget to include units.

Solution: Using similar triangles, the side length of a horizontal slice of thickness Δy located y meters below the top of the tank is $\frac{2y}{5}$ meters. Thus the mass of the slice is $(880)\left(\frac{2y}{5}\right)^2 \Delta y$ kg.

b. [5 points] Write a definite integral that represents the total amount of work required to pump all of the oil to the top of the tank. Do not evaluate the integral. Don't forget to include units.

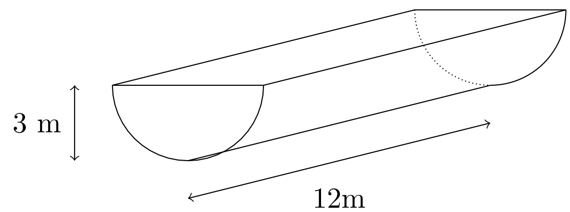
Solution: The total amount of work is

$$\int_4^{10} (880)\left(\frac{2y}{5}\right)^2 (9.8)(y)\,dy \text{ J}.$$

5. [11 points] Robber baron and philanthropist Calvin Currency is making a large cash donation in \$100 bills. Before making the donation, he decides to fill an empty pool with the money. The pool is a half cylinder with radius 3 meters and length 12 meters as shown below. After an afternoon of diving into his pool of money and swimming around, the distribution of bills in the pool becomes nonuniform and so the density of money in the pool is given by

$$\delta(y) = 30{,}000\sqrt{\frac{10}{\pi}}e^{-y^2},$$

measured in bills per m^3, where y is height in meters measured from the bottom of the pool. Recall the gravitational constant is $g = 9.8\ \mathrm{m/s}^2$



a. [5 points] Write a definite integral which gives the volume of the pool.

Solution: The volume of the pool is $\displaystyle\int_0^3 12(2\sqrt{9-(y-3)^2})dy$.

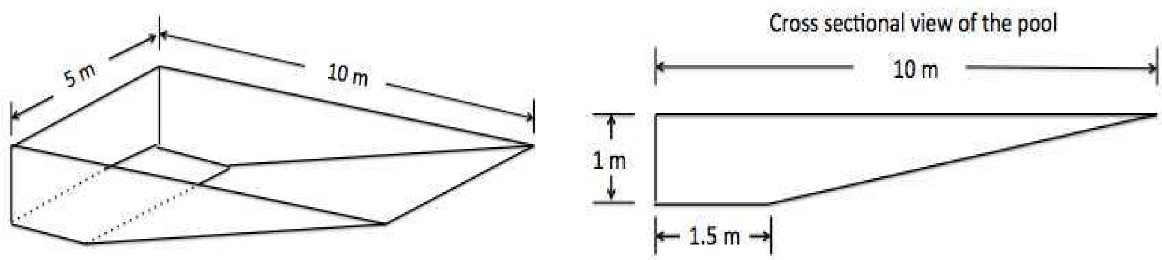
b. [2 points] Write a definite integral which gives the value of the money in the pool, in dollars.

Solution: The value of money in the pool is given by $\displaystyle 100\int_0^3 \delta(y)(12)(2\sqrt{9-(y-3)^2})dy$.

c. [4 points] Write a definite integral which gives the amount of work done in lifting the money out of the pool if each bill has mass 0.001 kg.

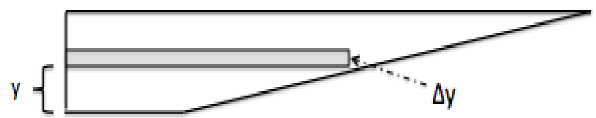
Solution: The work done in lifting the money out of the pool is given by
$\displaystyle\int_0^3 (0.001)\delta(y)(g)(12)(2\sqrt{9-(y-3)^2})(3-y)dy$.

6. [11 points] A swimming pool 10 m long and 5 m wide has varying depth. Its maximum depth is 1 m as shown in the picture below



The swimming pool has water up to a level of maximum depth of 0.6 m. The density of water is 1000 kg per m^3. Use $g = 9.8$ $\mathrm{m/s}^2$ for the acceleration due to gravity.

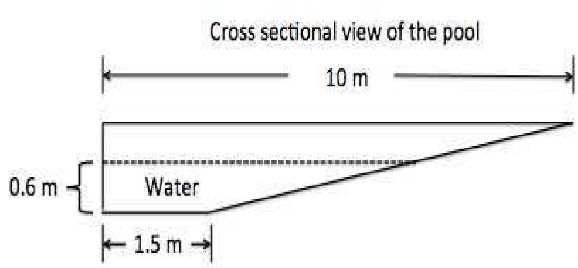
a. [9 points] Write an expression that approximates the work done in lifting a horizontal slice of water with thickness Δy meters, that is at a distance of y meters above the bottom, to the top of the swimming pool.



Solution: First we must find a formula for the length of the swimming pool at depth for a given height above the bottom. Let's call this function $l(y)$. We know that $l(0) = 1.5$ and $l(1) = 10$. Since $l(y)$ is a linear function, this tells us that $l(y) = 8.5y + 1.5$.
The volume of such a slice is $\Delta y(8.5y+1.5)\cdot 5$. Multiplying by 1000 $\mathrm{kg/m}^3$ and 9.8 $\mathrm{m/s}^2$ gives us the weight of the water in Newtowns. The amount the water needs to be lifted is $(1 - y)$. We therefore get:

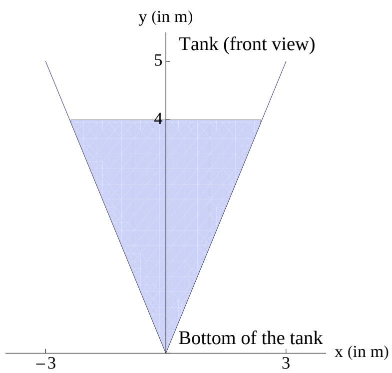
$$W_{slice} \approx 1000 \cdot 9.8 \cdot (8.5y + 1.5) \cdot 5 \cdot (1 - y)\Delta y.$$

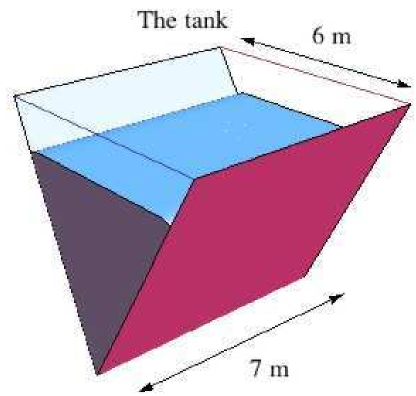
b. [2 points] Write a definite integral that computes the work required to pump all the water to the top of the pool.



Solution: Work $= \displaystyle\int_0^{0.6} 1000 \cdot 9.8 \cdot 5(8.5y + 1.5)(1 - y)dy$ Joules.

6. [11 points] The Math Department has recently acquired a triangular storage tank 6 m wide, 5 m tall and 7 m long, which it will use to store coffee for its graduate students. The tank currently contains a special coffee blend, with a mass density 1033 kg per m^3, up to a depth of 4 m.





a. [8 points] Write an expression that approximates the work done in lifting a horizontal slice of the liquid in the tank that is y meters above the bottom of the tank, with thickness Δy, to the top of the tank. Use $g = 9.8$ m per s^2 for the acceleration due to gravity.

Solution:

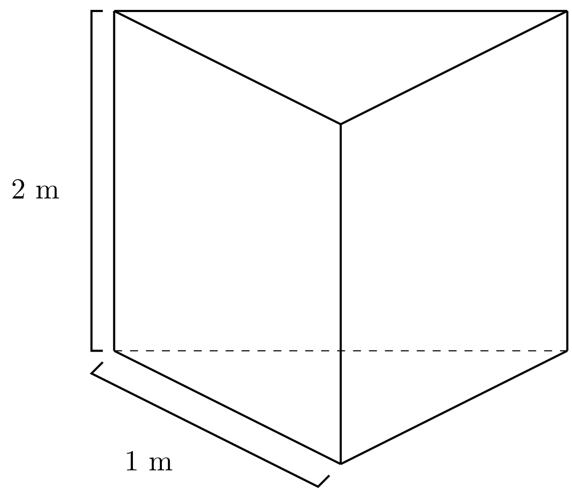
$$\Delta W = W_{slice} = \tfrac{6}{5}y \cdot 7 \cdot 9.8 \cdot 1033 \cdot (5 - y)\Delta y \text{ Joules.}$$

b. [3 points] While grading this exam, the grad students will need coffee. Find a definite integral that computes the work required to pump all the coffee to the top of the tank. Give the units of this integral. You do not need to evaluate it.

Solution:

$$W = \int_0^4 \tfrac{6}{5}y \cdot 7 \cdot 9.8 \cdot 1033 \cdot (5 - y)\,dy \text{ Joules.}$$

9. [9 points] The tank pictured below has height 2 meters, and the top and bottom are equilateral triangles with sides of length 1 meter. It is filled **halfway** with hot chocolate. The hot chocolate has uniform density 1325 kg/m^3. The acceleration due to gravity is 9.8 m/s^2. Calculate the work needed to pump all the chocolate to the top of the tank. Show all your work. Give an **exact** answer. Include **units**.



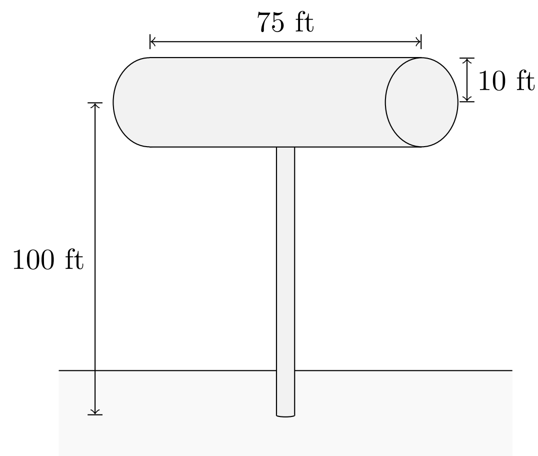
Solution: We take a horizontal slice at height y meters from the bottom of the tank. It has mass $1325 \cdot \frac{\sqrt{3}}{4} 1^2 \Delta y$. We need to move it $2 - y$ meters up. Thus, the work needed to pump all the chocolate to the top is

$$\int_0^1 1325 \cdot \frac{\sqrt{3}}{4} 1^2 \cdot 9.8 \cdot (2-y)\, dy = \frac{1325 \cdot 9.8\sqrt{3}}{4} \left[2y - \frac{y^2}{2} \right]_{y=0}^{y=1} = \frac{1325 \cdot 9.8\sqrt{3}}{4} \cdot \frac{3}{2} \text{ Joules}$$

9. [9 points]

De'von Baptiste is a shrewd industrialist. When energy costs are low, De'von pumps purified muck (which he gets *for free* from the city) into very tall tanks. In this way he stores cheap potential energy. Someday, when energy prices soar, Mr. Batiste will convert it all back into useful kinetic energy at a great profit.

His tanks are cylinders 75 ft long with radius 10 ft. The center of a tank is 100 ft above the ground. Purified muck has a density of 800 pounds/ft^3.



a. [3 points] What is the area, in square feet, of a cross-section parallel to the ground taken y feet above the **center** of the tank?

Solution: The cross-sections are rectangles with a length of 75 ft and a width $w(y)$ which depends on y. Using the Pythagorean Theorem we find that

$$10^2 = y^2 + (w(y)/2)^2 \longrightarrow w(y) = 2\sqrt{100 - y^2} = \sqrt{400 - 4y^2}.$$

Hence the area of a cross-section is

$$\text{Area of cross-section} = 75w(y) = 75 \cdot 2\sqrt{100 - y^2} = 150\sqrt{100 - y^2}.$$

Answer: $150\sqrt{100 - y^2}$

b. [6 points] Write an integral which represents the total work (in foot-pounds) required to fill one of De'von Batiste's tanks with purified muck. **Do not evaluate this integral.**

Solution: If we consider the tank after its filled, we can compute the work required to get each slice of muck y feet above the center of the tank from the ground to its height at $100 + y$ ft above the ground. If $A(y) = 300\sqrt{100 - y^2}$ is the area of a cross-section y feet above the center of the tank, then the total work is

$$\begin{aligned}
\text{Total work} &= \int_{-10}^{10} (\text{density})(\text{distance})(\text{slice volume}) \\
&= \int_{-10}^{10} 800(100 + y)A(y)\,dy \\
&= \int_{-10}^{10} 800(100 + y)150\sqrt{100 - y^2}\,dy
\end{aligned}$$