Math 116 — Practice for Exam 1

Generated February 8, 2021

Name: SOLUTIONS

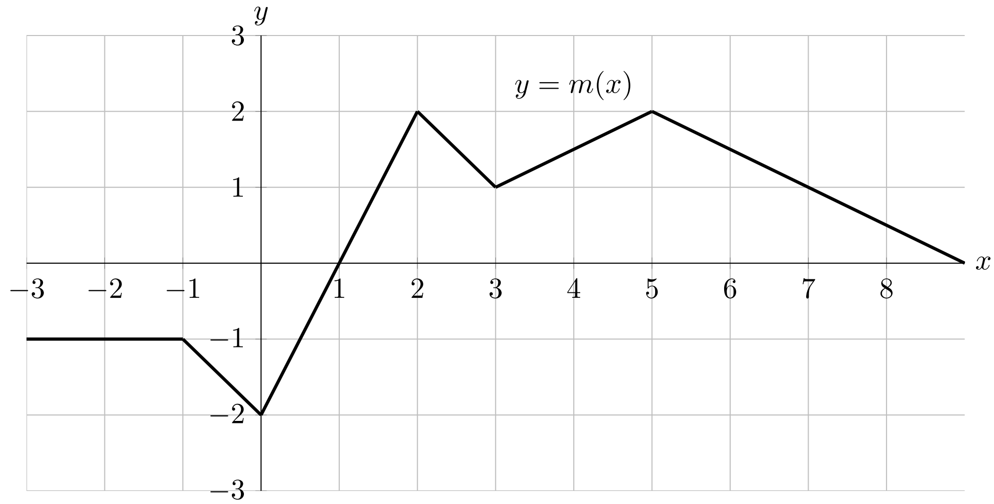
Instructor: ______________________ Section Number: __________

1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2019	1	2		14	
Winter 2020	1	5	heart rate	10	
Winter 2019	1	5		9	
Fall 2016	1	4		15	
Winter 2019	1	9		6	
Winter 2018	1	8		14	
Total				68	

Recommended time (based on points): 61 minutes

2. [14 points] Part of the graph of a continuous, piecewise-linear function $m(x)$ is given below. The domain of $m(x)$ is all real numbers.



Let:

- $F(x) = \int_1^x m(t)\,dt$
- $G(x) = \int_2^{x/2} m(t)\,dt$
- $H(x)$ is an antiderivative of $m(x)$ with $H(2) = 8$.

You do not need to show work for this problem.

a. [11 points] Find the following values. If it is not possible to do so based on the information provided, write "NI". If the value does not exist, write "DNE".

(i) $F(1) =$ 0

(ii) $F(3) =$ 2.5

(iii) $F(-2) =$ 3.5

(iv) $F'(4) =$ 1.5

(v) $G(2) =$ -1

(vi) $G(6) =$ 1.5

(vii) $G'(8) =$ 0.75

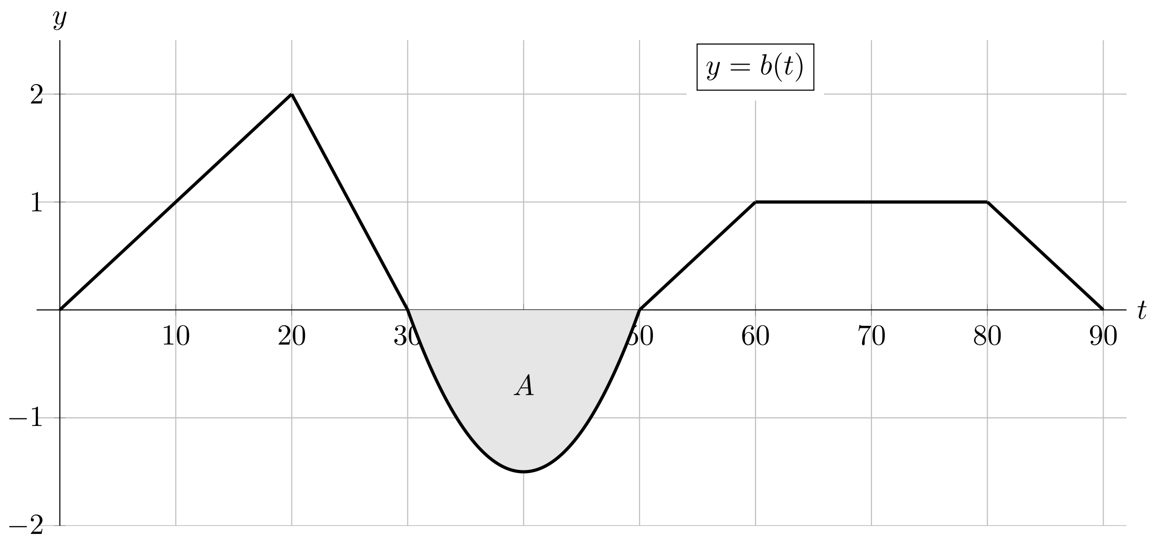
(viii) $H(3) =$ 9.5

(ix) $H(10) - F(10) =$ 7

b. [3 points] On which of the following intervals is $H(x)$ concave up on the entire given interval? Circle all correct answers.

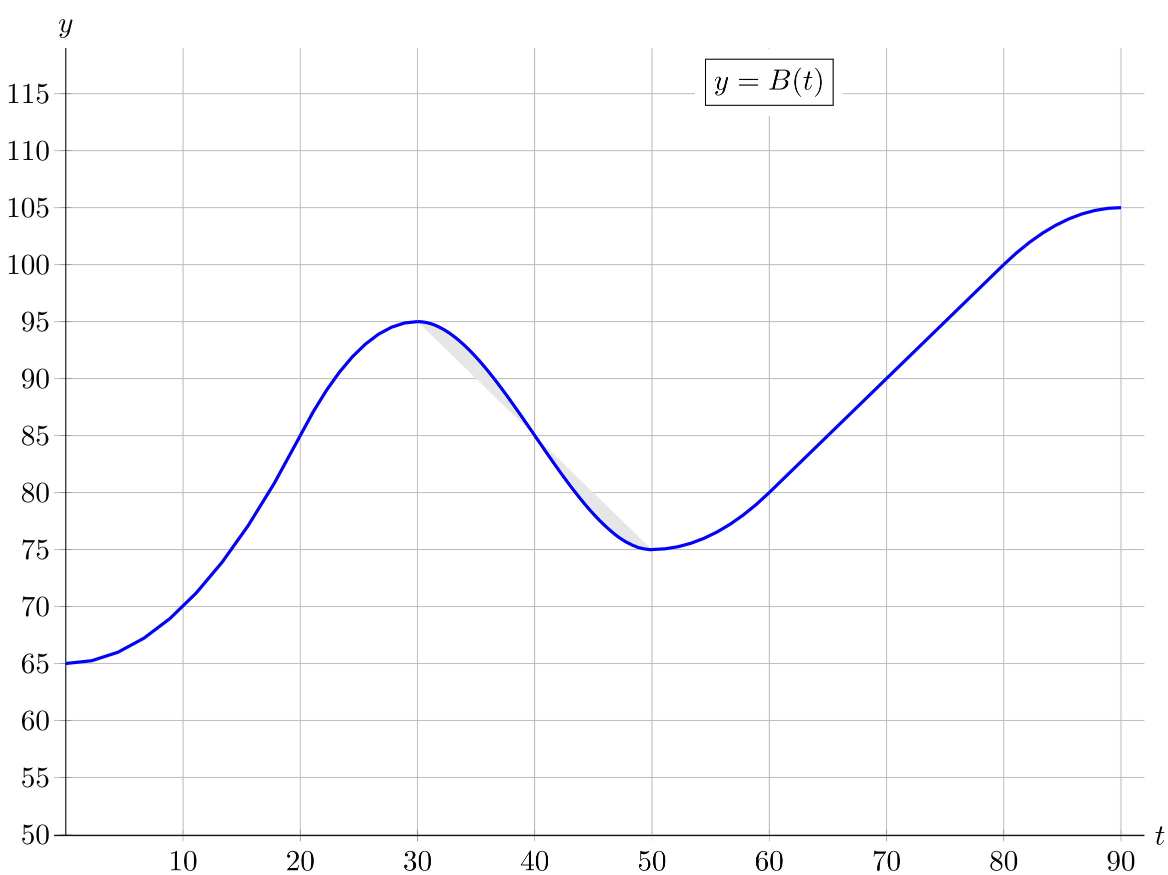
$\boxed{(0,2)}$ $(1,3)$ $(2,5)$ $\boxed{(3,5)}$ NONE OF THESE

5. [10 points] After a long day filming a space walk scene for a movie, an actress attends a loud rock concert. Her fancy new watch monitors her stress levels during the concert. Among other data, it outputs the rate of change in her heart rate. Let $b(t)$ be the rate of change in her heart rate (in beats/minute^2) t minutes after the concert begins.
A graph of $b(t)$ is given below. The area of the shaded region A is 20.



Let $B(t)$ be the actress's heart beat (in beats per minute) t minutes after the concert begins. Suppose that her heart rate 60 minutes into the concert is 80 beats/minute.
Sketch a detailed graph of $B(t)$ for $0 \le t \le 90$. Pay careful attention to where your graph is differentiable, increasing/decreasing, and concave up/concave down.

Solution:



5. [9 points] Before calculators existed, it was a difficult task for scientists and engineers to compute values of special functions like logarithm. In this problem, we will use simple arithmetic operations to approximate the value of $\ln 2$.
Note that for $x > 0$,

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

a. [3 points] Approximate the integral $\int_1^2 \frac{1}{t}\,dt$ using LEFT(4). Write out each term in your sum.

Solution:

$$\begin{aligned}\text{LEFT}(4) &= \frac{2-1}{4}\left(\frac{1}{1} + \frac{1}{1+1/4} + \frac{1}{1+2/4} + \frac{1}{1+3/4}\right)\\ &= \frac{1}{4}(1 + \frac{4}{5} + \frac{4}{6} + \frac{4}{7})\\ &= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}.\end{aligned}$$

Answer: $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

b. [3 points] Which of the following are equal to the LEFT(n) approximations for $\int_1^2 \frac{1}{t}\,dt$? Circle the one best answer.

i. $\frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{i}$

ii. $\frac{1}{n}\sum_{i=0}^{n} \frac{1}{1+i/n}$

iii. $\boxed{\frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{1+i/n}}$

iv. $\frac{1}{n}\sum_{i=1}^{n} \frac{1}{i}$

v. $\frac{1}{n}\sum_{i=n}^{n} \frac{1}{1+i/n}$

vi. $\frac{1}{n}\sum_{i=0}^{n} \frac{1}{1+1/n}$

c. [3 points] How many subintervals would be needed so that a scientist who lived before calculators were invented would be certain that the resulting left-hand Riemann sum approximates $\ln(2)$ to within 0.01? Justify your answer.

Solution:

$$|\text{LEFT}(n) - \text{RIGHT}(n)| \le \left|\frac{1}{1} - \frac{1}{2}\right| \cdot \frac{2-1}{n} = \frac{1}{2}\cdot\frac{1}{n} = \frac{1}{2n}$$

Since $1/2n \le 0.01$ if and only if $n \ge 50$, at least 50 subintervals would be needed.

Answer: **(at least)** 50

4. [15 points] For this problem, m is a differentiable function with $m'(x) > 0$ for all x. The following table gives some values of m.

x	0	1	2	3	4	5	6	7	8
$m(x)$	0	2	3	4	6	9	10	11	12

a. [3 points] What is the average value of $m'(x)$ on $[1, 7]$?

Solution: The average value is

$$\frac{1}{6}(m(7) - m(1)) = \frac{3}{2}.$$

b. [3 points] Use a left Riemann sum with 3 subdivisions to estimate $\int_2^8 m(x)\,dx$. Write out each term of your sum. Is this an overestimate or underestimate?

Solution: The left sum $2(3 + 6 + 10) = 38$ is an underestimate.

c. [3 points] Use a midpoint sum with 3 subdivisions to estimate $\int_0^{12} m^{-1}(y)\,dy$. Write out each term of your sum.

Solution: The correct sum is $4(1 + 4 + 6) = 44$.

d. [6 points] Consider the region bounded by the y-axis, the line $y = 12$ and the curve $y = m(x)$. Write an integral that gives the volume of the solid obtained by rotating this region about the y-axis. Use a right Riemann sum with 2 subdivisions to estimate your integral. Write out each term of your sum.

Solution: There are several possibilities. The shell method gives the volume as

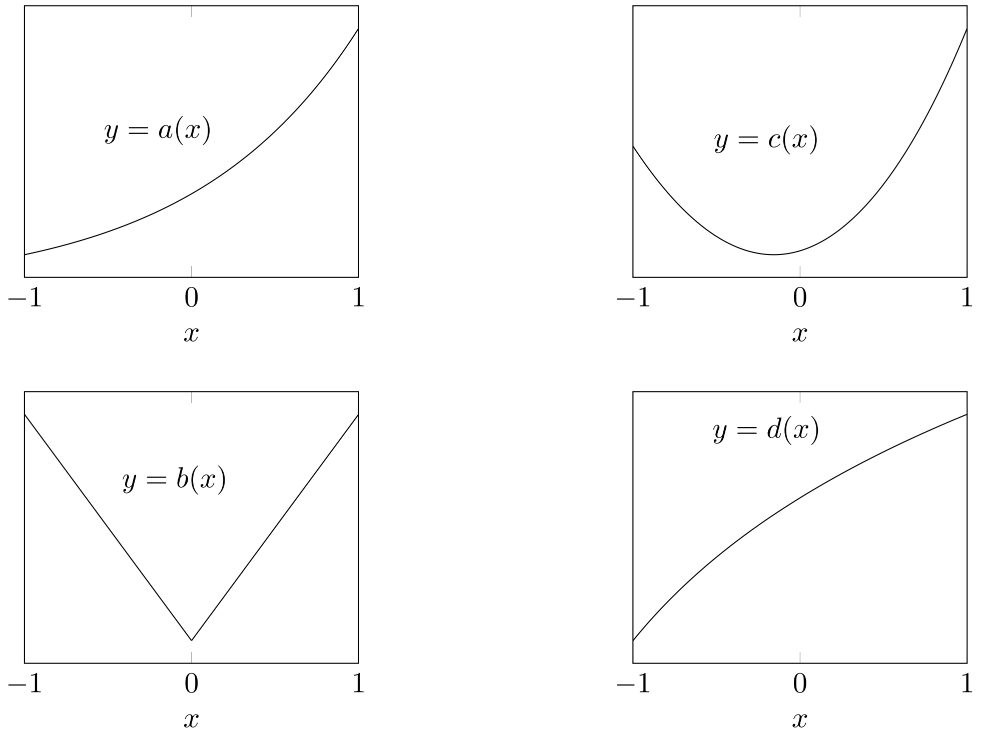
$$2\pi \int_0^8 x(12 - m(x))\,dx,$$

where the associated right sum is $8\pi(4(12-6)+8(12-12)) = 192\pi$. The washer method gives the volume as

$$\pi \int_0^{12} (m^{-1}(y))^2\,dy,$$

and the associated right sum is $6\pi(4^2 + 8^2) = 480\pi$.

9. [6 points] Below are the graphs of four functions. Note that the vertical scales are not given and may not be the same.



We have calculated LEFT(6), RIGHT(6), TRAP(6), AND MID(6) for the definite integral of each of three of these functions on the interval $[-1, 1]$. These estimates are listed below. For each one, circle the corresponding function.

(i)

LEFT(6)	1.21875
RIGHT(6)	1.58496
MID(6)	1.40185
TRAP(6)	1.29890

Function: $a(x)$ $c(x)$ $b(x)$ $\boxed{d(x)}$

(ii)

LEFT(6)	5.1552
RIGHT(6)	8.0374
MID(6)	5.0882
TRAP(6)	6.5963

Function: $a(x)$ $\boxed{c(x)}$ $b(x)$ $d(x)$

(iii)

LEFT(6)	42.00
RIGHT(6)	42.00
MID(6)	42.00
TRAP(6)	42.00

Function: $a(x)$ $c(x)$ $\boxed{b(x)}$ $d(x)$

8. [14 points] Let $g(x)$ be a differentiable function with domain $(-1, 10)$ where some values of $g(x)$ and $g'(x)$ are given in the table below. Assume that all local extrema and critical points of $g(x)$ occur at points given in the table.

x	0	1	2	3	4	5	6	7	8
$g(x)$	2.0	3.3	5.7	6.8	6.0	4.3	2.4	0.2	-4.9
$g'(x)$	2.8	2.5	2.0	0.0	-1.4	-1.9	-1.6	-3.0	-8.1

a. [3 points] Estimate $\int_0^8 g(x)\,dx$ using RIGHT(4). Write out each term in your sum.

Solution: With 4 rectangles the width of each is $\Delta x = \frac{8-0}{4} = 2$. Then

$$\begin{aligned}\text{RIGHT}(4) &= g(2)\Delta x + g(4)\Delta x + g(6)\Delta x + g(8)\Delta x \\ &= (5.7 + 6.0 + 2.4 - 4.9) \cdot 2 \\ &= 18.4\end{aligned}$$

Answer: **18.4**

b. [4 points] Approximate the area of the region between $g(x)$ and the function $f(x) = x + 2$ for $0 \le x \le 4$, using MID(n) to estimate any integrals you use. Use the greatest number of subintervals possible, and write out each term in your sums.

Solution: The function $g(x)$ is concave down on $[0, 4]$, so $g(x)$ is greater than or equal to the linear function $f(x)$ on this interval. The integral to compute this area is

$$\int_0^4 g(x) - f(x)\,dx = \int_0^4 g(x)\,dx - \int_0^4 x + 2\,dx.$$

Since $f(x)$ is linear, we get the same answer whether we use MID to approximate $\int_0^4 g(x) - f(x)\,dx$ or just $\int_0^4 g(x)\,dx$ and compute $\int_0^4 f(x)\,dx$ exactly. In either case, we can use at most 2 subintervals and $\Delta x = 2$.

If we compute MID(2) for $\int_0^4 g(x) - f(x)\,dx$, we get

$$\begin{aligned}\text{MID}(2) &= (g(1) - f(1))\Delta x + (g(3) - f(3))\Delta x \\ &= ((3.3 - 3) + (6.8 - 5))2 \\ &= (.3 + 1.8)2 \\ &= 4.2.\end{aligned}$$

If we compute $\int_0^4 f(x)\,dx = 16$ and then compute MID(2) for $\int_0^4 g(x)\,dx$ we get

$$\begin{aligned}\text{MID}(2) &= g(1)\Delta x + g(3)\Delta x \\ &= (3.3 + 6.8)2 \\ &= 20.2.\end{aligned}$$

Then we get $20.2 - 16 = 4.2$ for the total area.

Answer: **4.2**

c. [3 points] Is your answer to **b.** an overestimate, an underestimate, or is there not enough information to tell? Briefly justify your answer.

Solution: Since we are only given a table and not told that the concavity does not change between points, we technically **do not have enough information** to answer this question.
Had it been the case that $g'(x)$ has no critical points aside from those in the table, it would follow that $g(x)$ is concave down, because $g'(x)$ would be decreasing on the given interval. Since $f(x)$ is linear, the concavity of $g(x) - f(x)$ would also be concave down. In that case, MID(2) would be an **overestimate.**
Credit was awarded for both of these answers.

d. [4 points] Write an integral giving the arc length of $y = g(x)$ between $x = 2$ and $x = 8$. Estimate this integral using TRAP(2). Write out each term in your sum.

Answer: Integral: $\int_2^8 \sqrt{1+g'(x)^2}\,dx$

Solution: The arc length is given by the integral

$$\text{Arc length} = \int_2^8 \sqrt{1+g'(x)^2}\,dx.$$

The width of our trapezoids is $\Delta x = \frac{8-2}{2} = 3$.
If we compute the areas of the trapezoids directly we get

$$\begin{aligned}\text{TRAP}(2) &= \left(\frac{\sqrt{1+g'(2)^2}+\sqrt{1+g'(5)^2}}{2}\right)\Delta x + \left(\frac{\sqrt{1+g'(5)^2}+\sqrt{1+g'(8)^2}}{2}\right)\Delta x\\ &\approx (2.1915795 + 5.1542930)3\\ &\approx 22.0376175.\end{aligned}$$

If we compute LEFT(2) and RIGHT(2) first and then take an average we get

$$\begin{aligned}\text{LEFT}(2) &= \sqrt{1+g'(2)^2}\Delta x + \sqrt{1+g'(5)^2}\Delta x\\ &\approx (4.3831590)3\\ &\approx 13.1494770.\end{aligned}$$

$$\begin{aligned}\text{RIGHT}(2) &= \sqrt{1+g'(5)^2}\Delta x + \sqrt{1+g'(8)^2}\Delta x\\ &\approx (10.3085860)3\\ &\approx 30.9257580.\end{aligned}$$

Then

$$\text{TRAP}(2) = \frac{1}{2}(\text{LEFT}(2) + \text{RIGHT}(2)) \approx 22.0376175.$$

Answer: TRAP(2)= **22.0376175**