The Method of Partial Fractions

Math 116

What's the idea behind the method of partial fractions?

The method of partial fractions is a technique of algebra. It allows you to re-write complicated fractions using simpler pieces. Recall that a **rational function** is a function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

For example:

- Some rational functions are: $\dfrac{1}{x} \qquad \dfrac{x+3}{x^2-\pi} \qquad \dfrac{x^3+7x+3}{x^5-9x^3-2} \qquad x$
- These are not rational functions (WHY?): $\quad e^x \qquad \dfrac{x^2+7}{\ln(x)} \qquad \sin(x) \qquad \dfrac{x^7-9}{x^x}$

We have been able to integrate some rational functions using integration by substitution and by parts. For instance, we can integrate $\int \frac{1}{(1-x)}\,dx$ using the substitution $u = 1-x$:

$$\int \frac{1}{(1-x)}\,dx = \int \frac{-1}{(u)}\,du = -\ln|u| + C = -\ln|1-x| + C$$

Some rational functions, however, cannot be directly integrated by parts or by substitution. Consider the integral

$$\int \frac{1}{x^2-1}\,dx.$$

Let's look at this integral from a different point of view. Notice that the denominator of the integrand (x^2-1) factors into $(x^2-1) = (x-1)(x+1)$. The *key point* of the method of partial fractions is to rewrite the integrand as a sum of two simpler rational functions by "breaking apart" the denominator.

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}. \tag{1}$$

Here A and B are unknown constants, but we will solve for the values of A and B momentarily. The process of rewriting a rational expression by decomposing it into simpler rational expressions that add or subtract to get the original rational expression is called **partial fraction decomposition**.

Note that the denominator of the original function is a common denominator for the two fractions on the right. Using this common denominator for the right side, we find

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}.$$

Focusing only on the resulting numerators (or "clearing denominators"), we find the following equation involving A and B:

$$1 = A(x+1) + B(x-1). \tag{2}$$

We want to find values of A and B so that Equation (2) holds for all values of x. Two approaches to finding such values of A and B are described below.

Stop! Before you turn to the next page of this handout, think carefully about what number(s) you could substitute for the variable x in Equation (2) that would result in some nice cancellation.

Approach 1: Choosing values of x

Consider what happens to Equation (2) when we use the substitutions $x = 1$ and $x = -1$.

Substitute $x = 1$	**Substitute** $x = -1$
$1 = A(1+1) + B(1-1)$ $1 = A \cdot 2 + B \cdot 0$ $A = \frac{1}{2}$	$1 = A(-1+1) + B(-1-1)$ $1 = A \cdot 0 + B(-2)$ $B = -\frac{1}{2}$

We have found values for the constants in our partial fraction decomposition: $A = 1/2$ and $B = -1/2$. (Check that Equation (1) does indeed hold with these values of A and B.)

If we go back to our original integral, we now have:

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\,dx = \int \frac{1}{2(x-1)}\,dx + \int \frac{-1}{2(x+1)}\,dx \\ &= \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2} + C.\end{aligned}$$

Approach 2: Equating coefficients

Note that this approach is also demonstrated and explained in section 7.4 of your textbook.

Starting from Equation 2, we can distribute and then collect the terms on the right side involving x to find

$$1 = A(x+1) + B(x-1) = Ax + A + Bx - B = (A+B)x + (A-B).$$

That is,

$$0x + 1 = (A+B)x + (A-B).$$

Then, equating coefficients gives the two equations:

$$\begin{aligned} 0 &= A + B \qquad \text{(from the degree 1 coefficients)} \\ 1 &= A - B \qquad \text{(from the constant terms)} \end{aligned}$$

Solving this pair of equations gives $A = 1/2$ and $B = -1/2$ (the same values as in Approach 1).

Note: Either Approach 1 or Approach 2 as described above can be used in many cases. In fact, sometimes combining the two approaches is convenient.

Summary of method demonstrated above

For this course, we will focus on using partial fractions when the denominator has two distinct linear factors, and when the numerator has degree less than 2. Here is a recap of the method.

(I) Starting with a rational function, factor the denominator into two distinct linear factors. Suppose the denominator factors as $(x-p)(x-q)$ (where p and q are numbers and $p \neq q$).

(II) Re-write the original rational function as a sum of two terms. This gives an equation of the form

$$\text{Original rational function} = \frac{A}{x-p} + \frac{B}{x-q}.$$

(III) Find a common denominator (or clear denominators) to find an equation of linear functions.

(IV) Substitute chosen values of x and/or equate coefficients (as in Approaches 1 and 2 above) to solve for the unknowns A and B.

(V) Re-write the original rational function in its new, decomposed form. Recall that we are want to apply this method for integration, so be prepared to integrate the decomposed form!

Partial Fractions Problems and Examples

Practice problems.

Remember that we will only cover partial fraction decompositions where the denominator factors into two distinct linear factors and where the numerator is linear or constant. Some of these practice problems have been started for you.

1. $\dfrac{3x+1}{x^2+x} = \dfrac{A}{(x+1)} + \dfrac{B}{x}$
2. $\dfrac{2-x}{x^2+3x+2} = \dfrac{C}{(x+1)} + \dfrac{D}{(x+2)}$
3. $\dfrac{3x+11}{x^2-x-6}$
4. Use the partial fraction decompositions you found in problems 1-3 above to find a formula for the general antiderivative for each of these functions. (Do this without using a calculator or table of integrals.)

Where can I find more examples?

There are more examples in the textbook. Examples 1 and 2 in Section 7.4 are similar to the ones here. You will also be asked to use partial fractions in web homework. Examples 3 and 4 in Section 7.4 illustrate more complicated partial fractions decompositions as do some of exercises 1-17 in that section.

If you are interested, there are several other facets to doing partial fractions that we have not discussed here. You may be wondering:

- What if there is a repeated linear factor in the denominator? For example, how do we integrate $\frac{1}{(x-1)^2(x+8)}$?
- What about quadratic factors in the denominator that do not factor into linear factors? For example, how do we integrate $\frac{1}{x^2+x+1}$?
- What if the numerator has higher degree than the denominator? (You can get past this issue by using long division!)

The answers to these questions can be found in Section 7.4, but these additional topics will not appear on Math 116 exams this semester.

Answers (and some solutions) to the practice problems

Note that either of the approaches described on page 2 above could be used in each of 1-3 below. The example solutions shown here happen to use "choosing values of x" for problem 1 and "equating coefficients" for problem 2.

1. For this problem, the factors of the denominator are provided to you. Writing things down with common denominators shows us that $3x+1 = Ax + B(x+1)$. Solving for the unknowns gives $B = 1$ and $A = 2$. In all,
$$\frac{3x+1}{x^2+x} = \frac{2}{x+1} + \frac{1}{x}.$$

2. Again, we are given the factors for the denominator. By finding a common denominator, we see that $2 - x = C(x+2) + D(x+1)$. Using the approach of equating coefficients, we see that $2 - x = (2c + D) + (C + D)x$ so
$$2 = 2C + D \qquad \text{and} \qquad -1 = C + D.$$
Using the first equation, we find that $D = 2 - 2C$. Substituting this into the second equation, we see that $-1 = C + (2 - 2C)$ which implies that $C = 3$. We then see that $D = -4$. The resulting partial fractions decomposition is
$$\frac{2-x}{x^2+3x+2} = \frac{3}{x+1} + \frac{-4}{x+2}.$$

3. For this problem, we are not given the factors of the denominator, so we have to factor it ourselves. $(x^2 - x - 6) = (x-3)(x+2)$. So our partial fraction decomposition will look like:
$$\frac{3x+11}{x^2-x-6} = \frac{E}{x-3} + \frac{F}{x+2}.$$
After we find a common denominator, we see that $3x + 11 = E(x+2) + F(x-3)$. Solving for E and F (using either approach), we find that $E = 4$ and $F = -1$.

   So, we have found that
$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} + \frac{-1}{x+2}.$$

4. Use substitution (or "guess-and-check") to find antiderivatives of each of the functions in the partial fraction decompositions found above.

   1. $$\int \frac{3x+1}{x^2+x}\,dx = 2\ln|x+1| + \ln|x| + C.$$

   2. $$\int \frac{2-x}{x^2+3x+2}\,dx = 3\ln|x+1| - 4\ln|x+2| + C.$$

   3. $$\int \frac{3x+11}{x^2-x-6}\,dx = 4\ln|x-1| - \ln|x+2| + C.$$