

MATH 116 — PRACTICE FOR EXAM 3

Generated April 9, 2021

NAME: SOLUTIONS

INSTRUCTOR: _____ SECTION NUMBER: _____

1. This exam has 5 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2014	3	2		6	
Fall 2014	3	12	chicken coop	8	
Fall 2015	3	2		8	
Winter 2017	3	10		8	
Fall 2017	3	6		8	
Total				38	

Recommended time (based on points): 46 minutes

2. [6 points] Let $f(x) = xe^{-x^2}$.

- a. [4 points] Find the Taylor series of $f(x)$ centered at $x = 0$. Be sure to include the first 3 nonzero terms and the general term.

Solution: We can use the Taylor series of e^y to find the Taylor series for e^{-x^2} by substituting $y = -x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$

Therefore the Taylor series of xe^{-x^2} is

$$xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} = x - x^3 + \frac{x^5}{2!} + \cdots + \frac{(-1)^n x^{2n+1}}{n!} + \cdots$$

- b. [2 points] Find $f^{(15)}(0)$.

Solution: We know that $\frac{f^{(15)}(0)}{15!}$ will appear as the coefficient of the degree 15 term of the Taylor series. Using part (a), we see that the degree 15 term has coefficient $\frac{-1}{7!}$. Therefore

$$f^{(15)}(0) = \frac{-15!}{7!} = -259,459,200$$

3. [3 points] Determine the exact value of the infinite series

$$1 - \frac{2}{1!} + \frac{4}{2!} - \frac{8}{3!} + \cdots + \frac{(-1)^n 2^n}{n!} + \cdots$$

Solution: Notice that this is the Taylor series for e^y applied to $y = -2$. Therefore, the series has exact value e^{-2} .

12. [8 points] Franklin, your friendly new neighbor, is building a large chicken sanctuary. You decide to help Franklin build a special chicken coop with volume (in cubic km) given by the integral

$$\int_0^1 x \sqrt{1 - \cos(x^2)} dx.$$

This integral is difficult to evaluate precisely, so you decide to use the methods you've learned this semester to help out Franklin. Your friend and president-elect, Kazilla, stops by to give you a hand. She suggests finding the 4th degree Taylor polynomial, $P_4(x)$, for the function $1 - \cos(x^2)$ near $x = 0$.

- a. [4 points] Find $P_4(x)$.

Solution: We can use the Taylor series expansion for $\cos(x^2)$

$$\cos(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} = 1 - \frac{y^2}{2!} + \dots + \frac{(-1)^n y^{2n}}{(2n)!} + \dots$$

so

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = 1 - \frac{(x^2)^2}{2!} + \dots + \frac{(-1)^n (x^2)^{2n}}{(2n)!} + \dots$$

Therefore

$$P_4 = 1 - \left(1 - \frac{x^4}{2!}\right) = \frac{x^4}{2!}$$

- b. [4 points] Substitute $P_4(x)$ for $1 - \cos(x^2)$ in the integral and compute the resulting integral by hand, showing all of your work.

Solution:

$$\begin{aligned} \int_0^1 x \sqrt{P_4(x)} dx &= \int_0^1 x \sqrt{\frac{x^4}{2}} \\ &= \int_0^1 \frac{x^3}{\sqrt{2}} \\ &= \frac{x^4}{4\sqrt{2}} \Big|_{x=0}^1 \\ &= \frac{1}{4\sqrt{2}} \end{aligned}$$

2. [8 points] Let $f(x) = x^{2x}$. The first two derivatives of f are given below.

$$f'(x) = 2(1 + \ln x)x^{2x}$$
$$f''(x) = 2x^{2x-1} + 4(1 + \ln x)^2 x^{2x}$$

- a. [4 points] Find the 2nd degree Taylor polynomial $P_2(x)$ of f centered at $x = 1$.

Solution: Using the formula for Taylor polynomials,

$$P_2(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$= 1 + 2(x-1) + 3(x-1)^2$$

$$P_2(x) = \underline{\hspace{10em} 1 + 2(x-1) + 3(x-1)^2 \hspace{10em}}$$

- b. [4 points] Find

$$\lim_{x \rightarrow 1} \frac{x^{2x} - 1}{3x - 3}.$$

Clearly show your reasoning. Your answer from part (a) may be helpful.

Solution:

$$\lim_{x \rightarrow 1} \frac{x^{2x} - 1}{3x - 3} = \lim_{x \rightarrow 1} \frac{1 + 2(x-1) + 3(x-1)^2 - 1}{3(x-1)}$$
$$= \lim_{x \rightarrow 1} \frac{2 + 3(x-1)}{3}$$
$$= \frac{2}{3}$$

10. [8 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for all x and is given below.

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}$$

- a. [3 points] What is the value of $F^{(101)}(0)$?
 Make sure your answer is exact. You do not need to simplify.

$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n$ by Taylor's Theorem. So

The x^{101} term is $\frac{F^{(101)}(0)}{101!} x^{101}$. When you plug $n=25$ into the series above you get $-\frac{x^{101}}{50!(101)}$

So $F^{(101)}(0) = \frac{-101!}{50!(101)}$ Answer: $F^{(101)}(0) = \boxed{-\frac{100!}{50!}}$

- b. [3 points] Find $P_9(x)$, the 9th degree Taylor polynomial that approximates $F(x)$ near $x = 0$.

$$P_9(x) = \frac{x^{0+1}}{0!(0+1)} - \frac{x^{4+1}}{2!(4+1)} + \frac{x^{8+1}}{4!(8+1)}$$

$$= \boxed{x - \frac{1}{10}x^5 + \frac{1}{216}x^9}$$

- c. [2 points] Use your Taylor polynomial from part b. to compute

$$\lim_{x \rightarrow 0} \frac{F(x) - x}{2x^5} = \lim_{x \rightarrow 0} \frac{\cancel{x} - \frac{1}{10}x^5 + \frac{1}{216}x^9 - \cancel{x}}{2x^5}$$

$$= \lim_{x \rightarrow 0} -\frac{1}{20} + \frac{1}{432}x^4 = \boxed{-\frac{1}{20}}$$

6. [8 points]

Values of a function f and some of its derivatives are given in the table on the right. Use this information to answer the questions that follow.

x	0	π
$f(x)$	-6	2π
$f'(x)$	6	2
$f''(x)$	1	-3
$f'''(x)$	-1	0
$f^{(4)}(x)$	5	$-9/2$

a. [4 points] Find a formula for the Taylor polynomial of degree 4 for f about $x = \pi$.

$$\begin{aligned}
 P_4(x) &= \sum_{n=0}^4 \frac{f^{(n)}(\pi)}{n!} (x-\pi)^n \\
 &= f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!} (x-\pi)^2 + \frac{f'''(\pi)}{3!} (x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!} (x-\pi)^4 \\
 &= 2\pi + 2(x-\pi) + \frac{-3}{2} (x-\pi)^2 + \frac{0}{6} (x-\pi)^3 + \frac{-9/2}{24} (x-\pi)^4 \\
 &= \boxed{2\pi + 2(x-\pi) - \frac{3}{2} (x-\pi)^2 - \frac{3}{16} (x-\pi)^4}
 \end{aligned}$$

b. [4 points] Find the first three nonzero terms of the Taylor series for $\int_0^x f(t^2) dt$ about $x = 0$.

$$\begin{aligned}
 \text{Near } 0, f(x) &\approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!} (x-0)^2 \\
 &= -6 + 6x + \frac{1}{2} x^2 \\
 \text{So } \int_0^x f(t^2) dt &\approx \int_0^x -6 + 6t^2 + \frac{1}{2} t^4 dt \\
 &= -6t + 2t^3 + \frac{1}{10} t^5 \Big|_0^x \\
 &= \boxed{-6x + 2x^3 - \frac{1}{10} x^5}
 \end{aligned}$$