Math 116 — Practice for Exam 2

Generated October 10, 2018

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 8 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2012	2	8		12	
Winter 2013	2	4		13	
Winter 2016	2	9		12	
Winter 2012	2	7	rod	9	
Winter 2018	2	8		7	
Fall 2012	2	7		13	
Winter 2013	2	6		11	
Fall 2015	2	9		12	
Total				89	

Recommended time (based on points): 80 minutes

8. [12 points] Determine if the following integrals converge or diverge. Justify your answer. If you use the comparison test, be sure to show all your work.

a. [3 points] $\displaystyle\int_1^\infty \frac{1}{x+e^x}dx.$

Solution: Using the comparison

$$\frac{1}{x+e^x} \le \frac{1}{e^x},$$

we get convergence, since

$$\int_1^\infty \frac{1}{e^x}dx$$

converges.

b. [4 points] $\displaystyle\int_1^e \frac{1}{x(\ln x)^2}dx.$

Solution: This is improper because $\ln 1 = 0$, so there is an asymptote at $x = 1$. Here we use the substitution $u = \ln x$, so $du = \frac{1}{x}dx$, and we get

$$\int_1^e \frac{1}{x(\ln x)^2}dx = \int_0^1 \frac{1}{u^2}du.$$

The right hand side diverges by the p-test ($p = 2 > 1$).

c. [5 points] $\displaystyle\int_{2\pi}^\infty \frac{x\cos^2 x + 1}{x^3}dx.$

Solution: Break this up into two integrals:

$$\int_{2\pi}^\infty \frac{x\cos^2 x + 1}{x^3}dx = \int_{2\pi}^\infty \frac{x\cos^2 x}{x^3}dx + \int_{2\pi}^\infty \frac{1}{x^3}dx$$

The second integral converges by the p-test. For the first, we need to use another comparison:

$$\frac{x\cos^2 x}{x^3} \le \frac{1}{x^2}$$

so by comparison, the first integral also converges. The sum of two convergent improper integrals converges, so this integral converges.

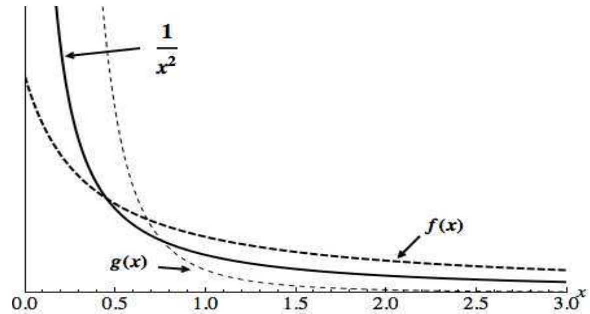
4. [13 points]

a. [8 points] Consider the functions $f(x)$ and $g(x)$ where

$$\frac{1}{x^2} \le g(x) \qquad \text{for} \qquad 0 < x < \frac{1}{2}.$$

$$g(x) \le \frac{1}{x^2} \qquad \text{for} \qquad 1 < x$$

$$\frac{1}{x^2} \le f(x) \qquad \text{for} \qquad 1 < x.$$



Using the information about $f(x)$ and $g(x)$ provided above, determine which of the following integrals is convergent or divergent. Circle your answers. If there is not enough information given to determine the convergence or divergence of the integral circle NI.

i) $\int_1^\infty f(x)dx$ CONVERGENT DIVERGENT **NI**

ii) $\int_1^\infty g(x)dx$ **CONVERGENT** DIVERGENT NI

iii) $\int_0^1 f(x)dx$ **CONVERGENT** DIVERGENT NI

iv) $\int_0^1 g(x)dx$ CONVERGENT **DIVERGENT** NI

b. [5 points] Does $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}dx$ converge or diverge? If the integral converges, compute its value. Show all your work. Use u substitution.

Solution:

$$\int_e^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}dx$$

using $u = \ln x$ $$= \lim_{b\to\infty}\int_1^{\ln b} \frac{1}{u^2}dx = \lim_{b\to\infty} -\frac{1}{u}\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{\ln b} = 1$$ converges.

9. [12 points]

a. [6 points] Show that the following integral diverges. Give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use.

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt$$

Solution: $t \geq 1 \Rightarrow \frac{1}{t} \leq 1 \Rightarrow \cos(\frac{1}{t}) \geq \cos(1)$ because the function $F(x) = \cos x$ is decreasing in the interval $[0, 1]$. Therefore,

$$\frac{\cos(\frac{1}{t})}{\sqrt{t}} \geq \frac{\cos(1)}{\sqrt{t}}$$

The improper integral

$$\int_1^\infty \frac{\cos(1)}{\sqrt{t}}\, dt = \cos(1) \int_1^\infty \frac{1}{\sqrt{t}}\, dt$$

diverges by the p-test since $p = \frac{1}{2} \leq 1$. So the integral

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt$$

diverges by the comparison test (notice that $\cos(1) > 0$).

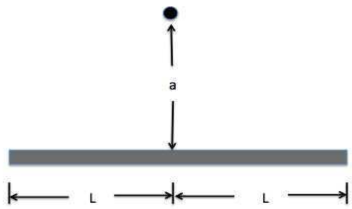
b. [6 points] Find the limit

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt}{\sqrt{x}}$$

Solution: Notice that by (a), this is $\frac{\infty}{\infty}$. We use L'Hopital's rule along with the 2nd Fundamental Theorem in the numerator:

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2\cos\left(\frac{1}{x}\right) = 2\cos(0) = 2$$

7. [9 points] A particle of mass m is positioned at a perpendicular distance a from the center of a rod of length $2L$ and constant mass density δ as shown below



The force of gravitational attraction F between the rod and the particle is given by

$$F = Gm\delta a \int_{-L}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx.$$

a. [5 points] Does the force of gravitational attraction F approach infinity as the length of the rod goes to infinity? Justify your answer using the comparison test.

Solution:

$$F = Gm\delta a \int_{-L}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx = 2Gm\delta a \int_{0}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx$$

$$\begin{aligned}
\lim_{L\to\infty} F &= \lim_{L\to\infty} 2Gm\delta a \int_{0}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
&= \lim_{L\to\infty} 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + 2Gm\delta a \int_{1}^{\infty} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
&\le 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + \lim_{L\to\infty} 2Gm\delta a \int_{1}^{L} \frac{1}{(x^2)^{\frac{3}{2}}} dx \\
&= 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + 2Gm\delta a \int_{1}^{\infty} \frac{1}{x^3} dx.
\end{aligned}$$

The last integral converges since $\int_1^\infty \frac{1}{x^p} dx$ converges for $p = 3 > 1$.

b. [4 points] Consider the integral

$$I = \int_{1}^{\infty} \frac{1}{(a^2+x^2)^p} dx$$

i. Give a power function which, if integrated over $[1, \infty)$, will have the same convergence or divergence behavior as I.

Solution: $\frac{1}{(x^2)^p} = \frac{1}{x^{2p}}$

ii. For which values of p would you predict I is convergent? For which would I be divergent?

Solution: Convergent: Need $2p > 1$, hence $p > \frac{1}{2}$.

Divergent: Need $2p \le 1$, hence $p \le \frac{1}{2}$.

8. [7 points] Consider the integral

$$\int_1^\infty \frac{e^{rx}}{x}\,dx,$$

where r is a constant.

a. [3 points] Show that this integral converges for $r < 0$. **Show all work and indicate any convergence tests used.**

Solution: We know that $\dfrac{e^{rx}}{x} \le e^{rx}$ for all $x \ge 1$.

Further, when $r < 0$, we know that $\displaystyle\int_1^\infty e^{rx}\,dx$ converges by exponential decay test.

Therefore, by (direct) comparison test, $\displaystyle\int_1^\infty \frac{e^{rx}}{x}\,dx$ converges.

b. [4 points] Show that the integral diverges for $r \ge 0$. **Show all work and indicate any convergence tests used.**

Solution: Now, when $r \ge 1$, we know that $e^{rx} \ge 1$ for all $x \ge 1$, so $\dfrac{e^{rx}}{x} \ge \dfrac{1}{x}$.

$\displaystyle\int_1^\infty \frac{dx}{x}$ diverges by p-test with $p = 1$.

Therefore, by comparison, $\displaystyle\int_1^\infty \frac{e^{rx}}{x}\,dx$ diverges.

Alternative solution:

For $r > 0$, $\displaystyle\lim_{x\to\infty} \frac{e^{rx}}{x} = \infty$. Since the integrand approaches infinity, the integral diverges.

This still leaves the $r = 0$ case. In this case, $\dfrac{e^{rx}}{x} = \dfrac{1}{x}$, so the integral diverges by p-test with $p = 1$.

7. [13 points] Consider the following improper integrals. Show all your work to receive full credit.

a. [5 points] Determine the convergence or divergence of the following improper integral. If the integral converges, compute its value.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx$$

Solution: The integral is improper at $x = 0$ since $\sin 0 = 0$. Changing to a limit of proper integrals and using the substitution $u = \sin x$:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx &= \lim_{a \to 0^+} \int_a^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx \\ &= \lim_{a \to 0^+} \int_{\sin a}^{1} u^{-1/2} du \\ &= \lim_{a \to 0^+} 2u^{1/2}\Big|_{\sin a}^{1} \\ &= \lim_{a \to 0^+} 2 - 2\sqrt{\sin a} \\ &= 2.\end{aligned}$$

Determine the convergence or divergence of the following improper integrals. Circle your answers.

b. [4 points] $\displaystyle\int_2^{\infty} \frac{5 - 3\sin(2x)}{x^2} dx$ $\boxed{\text{Converges}}$ Diverges

Solution: Since $0 \le 5 - 3\sin(2x) \le 8$,

$$\int_2^{\infty} \frac{5 - 3\sin(2x)}{x^2} dx \le 8 \int_2^{\infty} \frac{1}{x^2} dx,$$

which converges by the p-test with $p = 2$.

c. [4 points] $\displaystyle\int_1^{\infty} \frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}}\, dx$, where a is a positive constant.

Converges $\boxed{\text{Diverges}}$

Solution: Since $a^2 + \frac{1}{\sqrt{x}} \ge a^2$ for $x > 0$,

$$\frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}} \ge \frac{a}{x},$$

and so

$$\int_1^{\infty} \frac{1}{x}\sqrt{a^2 + \frac{1}{\sqrt{x}}} dx \ge a \int_1^{\infty} \frac{1}{x} dx,$$

which diverges by the p-test, with $p = 1$.

6. [11 points]

a. [8 points] Use the **comparison method** to determine the convergence or divergence of the following improper integrals. Justify your answers. Make sure to properly cite any results of convergence or divergence of integrals that you use.

i) $\displaystyle\int_1^\infty \frac{3+\sin(4x)}{\sqrt[3]{x}}dx.$

Solution: We compare the integrand with the function $\frac{1}{x^{1/3}}$. Because $3+\sin(4x)\geq 2$, we know that

$$\frac{3+\sin(4x)}{x^{1/3}}\geq\frac{2}{x^{1/3}}.$$

By the p-test with $p=1/3$, we know that $\int_1^\infty \frac{1}{x^{1/3}}dx$ diverges. Therefore, by the comparison method, we know that this integral diverges, too.

ii) $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}+x^2}dx.$

Solution: We compare the integrand with the function $\frac{1}{x^2}$. Because $\sqrt{x}\geq 0$, we know that

$$\frac{1}{\sqrt{x}+x^2}\leq\frac{1}{x^2}.$$

By the p-test with $p=2$, we know that $\int_4^\infty \frac{1}{x^2}dx$ converges. Therefore, by the comparison test, $\int_4^\infty \frac{1}{\sqrt{x}+x^2}$ converges, too.

b. [3 points] For which values of p does the following integral converges?

$$\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx.$$

No justification is required.

Solution: If $p\leq 2$, the function $\displaystyle\frac{x^2-1}{x^p+4x^2+2}dx$ behaves as the function $\displaystyle\frac{1}{4}$ for large values of x. Hence the integral $\displaystyle\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx.$ diverges.

If $p>2$, then the function $\displaystyle\frac{x^2-1}{x^p+4x^2+2}dx$ behaves as the function $\displaystyle\frac{x^2}{x^p}=\frac{1}{x^{p-2}}$ for large values of x. Then $\displaystyle\int_2^\infty \frac{1}{x^{p-2}}dx$ converges if $p>3$ $(p-2>1)$. Therefore the integral $\displaystyle\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx$ converges for $p>3$.

9. [12 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges". If the integral diverges, circle "diverges". In either case, **you must show all your work and indicate any theorems you use**. You do not need to calculate the value of the integral if it converges.

a. [6 points] $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x+1}}\,dx$

Converges $\boxed{\textbf{Diverges}}$

Solution: $\dfrac{2+\sin x}{\sqrt{x+1}} \geq \dfrac{1}{\sqrt{2x}}$, so $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x+1}}\,dx \geq \int_1^\infty \frac{1}{\sqrt{2x}}\,dx$.
But $\displaystyle\int_1^\infty \frac{1}{\sqrt{2x}}\,dx$ diverges by the p-test, $p = \frac{1}{2} \leq 1$, so $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x+1}}\,dx$ also diverges by the direct comparison test.

b. [6 points] $\displaystyle\int_1^\infty \frac{\theta}{\sqrt{\theta^5+1}}\,d\theta$

$\boxed{\textbf{Converges}}$ Diverges

Solution: $\dfrac{\theta}{\sqrt{\theta^5+1}} \leq \dfrac{\theta}{\sqrt{\theta^5}} = \dfrac{1}{\theta^{3/2}}$, so $\displaystyle\int_1^\infty \frac{\theta}{\sqrt{\theta^5+1}}\,d\theta \leq \int_1^\infty \frac{1}{\theta^{3/2}}\,d\theta$.
But $\displaystyle\int_1^\infty \frac{1}{\theta^{3/2}}\,d\theta$ converges by the p-test, $p = \frac{3}{2} > 1$, so $\displaystyle\int_1^\infty \frac{\theta}{\sqrt{\theta^5+1}}\,d\theta$ also converges by the direct comparison test.