Lattice Polygons

Yiwang Chen

Math Club Talk
January 30th, 2020
Math Club Talk on Lattice Polygons

The URL for the Beamer Presentation: “Pick’s Theorem and beyond” is available at
http://www-personal.umich.edu/~yiwchen/index.html
Table of Contents

1. Area Formula about Polygons
2. Pick’s Theorem
3. Generalized Pick’s Theorem
4. Higher Dimensional Analogs
Given any polygon, how do we find the area of the polygons?
If what we are having is a triangle, we have:

- The formula:
  \[ A = \frac{1}{2}ah \]
Triangle Area Formula

If what we are having is a triangle, we have:

- The formula:
  \[ A = \frac{1}{2}ah \]

- Surveyor’s formula, if we have the coordinates of vertices:
  \[ A = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3| \]
Triangle Area Formula

If what we are having is a triangle, we have:

- The formula:
  \[ A = \frac{1}{2} ah \]

- Surveyor’s formula, if we have the coordinates of vertices:
  \[ A = \frac{1}{2} \left| x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3 \right| \]

- Heron’s formula
  \[ A = \sqrt{s(s - a)(s - b)(s - c)} \]
Triangle Area Formula

If what we are having is a triangle, we have:

- The formula:
  \[ A = \frac{1}{2} ah \]

- Surveyor’s formula, if we have the coordinates of vertices:
  \[ A = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3| \]

- Heron’s formula
  \[ A = \sqrt{s(s - a)(s - b)(s - c)} \]

- ...
Convex quadrilateral Area Formula

If it is a convex quadrilateral, we are lucky to have:

- The formula:
  \[ A = \frac{1}{2}pq \sin \theta \]

  \((p, q): \text{length of the diagonals. } \theta: \text{the angle between them.}\)
Convex quadrilateral Area Formula

If it is a convex quadrilateral, we are lucky to have:

- The formula:
  \[ A = \frac{1}{2}pq \sin \theta \]
  
  \((p, q): \text{length of the diagonals. } \theta: \text{the angle between them.})

- Surveyor’s formula, if we have the coordinates of vertices:
  \[ A = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4) \]
Convex quadrilateral Area Formula

If it is a convex quadrilateral, we are lucky to have:

- The formula:
  \[ A = \frac{1}{2}pq \sin \theta \]
  
  \((p, q): \text{length of the diagonals. } \theta: \text{the angle between them.}\)

- Surveyor’s formula, if we have the coordinates of vertices:
  \[ A = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4) \]

- Bretschneider’s formula (Generalized Heron’s formula):
  \[ K = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd \cdot \cos^2\left(\frac{\alpha + \gamma}{2}\right)}, \]
  
  with \(\alpha\) and \(\gamma\) two opposite angles.
Convex quadrilateral Area Formula

If it is a convex quadrilateral, we are lucky to have:

- The formula:
  $$A = \frac{1}{2}pq \sin \theta$$
  ($p, q$: length of the diagonals. $\theta$: the angle between them.)

- Surveyor’s formula, if we have the coordinates of vertices:
  $$A = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4)$$

- Bretschneider’s formula (Generalized Heron’s formula):
  $$K = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd \cdot \cos^2\left(\frac{\alpha + \gamma}{2}\right)},$$
  with $\alpha$ and $\gamma$ two opposite angles.

...
As we are in the case of general polygons (not necessarily convex), the formulas mentioned above will get very complicated. However, there is a magic formula that allows us to compute the area of a certain type of polygon just by counting!
Pick’s Theorem about Lattice Polygons

Definition

A polygon is a set of line segments plus the region they enclose. A polygon is simple if the boundary of the polygon is a simply closed curve. A polygon is a lattice polygon if the coordinates of its vertices are integers.

Theorem (Pick’s Theorem)
The area of a simple lattice polygon $P$ is given by:

$$A(P) = \frac{i}{2} + \frac{1}{2}b - 1 = \frac{l - 1}{2}$$

where $i$, $b$, and $l$ are, respectively, the number of interior lattice points, the number of boundary points, and the total number of lattice points of the polygon $P$. 

Yiwang Chen
Lattice Polygons 8/25
A polygon is a set of line segments plus the region they enclose.
Pick’s Theorem about Lattice Polygons

**Definition**

- A *polygon* is a set of line segments plus the region they enclose.
- A polygon is *simple* if boundary of polygon is simply closed curve.
Pick’s Theorem about Lattice Polygons

**Definition**

- A *polygon* is a set of line segments plus the region they enclose.
- A polygon is *simple* if boundary of polygon is simply closed curve.
- A polygon is *lattice polygon* if coordinates of vertices are integers.

**Theorem (Pick’s Theorem)**
The area of a simple lattice polygon $P$ is given by

$$A(P) = \frac{i}{2} + 1 - \frac{b - 1}{2} = l - 1 - \frac{b - 1}{2}$$

where $i$, $b$, and $l$ are, respectively, the number of interior lattice points, the number of boundary points, and the total number of lattice points of the polygon $P$. 
Pick’s Theorem about Lattice Polygons

**Definition**

- A *polygon* is a set of line segments plus the region they enclose.
- A polygon is *simple* if boundary of polygon is simply closed curve.
- A polygon is *lattice polygon* if coordinates of vertices are integers.

**Theorem (Pick’s Theorem)**

The area of a simple lattice polygon $P$ is given by

$$A(P) = i + \frac{1}{2}b - 1 = l - \frac{1}{2}b - 1$$

where $i$, $b$ and $l$ are, respectively, the number of interior lattice points, the number of the boundary points, and the total number of the lattice points of the polygon $P$. 
Who is Georg Pick?

The Austrian Georg Pick completed his thesis at the University of Vienna under Königsberger and Weyr. Except for visiting Felix Klein in Leipzig in 1884, he worked his whole career at the Charles Ferdinand University in Prague. He returned to Vienna upon retirement in 1927. He died in the Theresienstadt Concentration Camp in 1942.

Pick wrote papers in differential geometry and complex analysis. He headed the committee to appoint Albert Einstein to the chair of mathematical physics in 1911. He introduced Einstein to the recent work by Ricci-Curbastro and Levi-Civita in curved manifolds, without which Einstein couldn't have formulated his theory of General Relativity of curved spacetimes.
Who is Georg Pick?

The Austrian Georg Pick completed his thesis at the University of Vienna under Königsberger and Weyr. Except for visiting Felix Klein in Leipzig in 1884, he worked his whole career at the Charles Ferdinand University in Prague. He returned to Vienna upon retirement in 1927. He died in the Theresienstadt Concentration Camp in 1942.

Pick wrote papers in differential geometry and complex analysis. He headed the committee to appoint Albert Einstein to the chair of mathematical physics in 1911. He introduced Einstein to the recent work by Ricci-Curbastro and Levi-Civita in curved manifolds, without which Einstein couldn't have formulated his theory of General Relativity of curved spacetimes.
Who is Georg Pick?

The Austrian Georg Pick completed his thesis at the University of Vienna under Königsberger and Weyr. Except for visiting Felix Klein in Leipzig in 1884, he worked his whole career at the Charles Ferdinand University in Prague. He returned to Vienna upon retirement in 1927. He died in the Theresienstadt Concentration Camp in 1942. Pick wrote papers in differential geometry and complex analysis. He headed the committee to appoint Albert Einstein to the chair of mathematical physics in 1911. He introduced Einstein to the recent work by Ricci-Curbastro and Levi-Civita in curved manifolds, without which Einstein couldn’t have formulated his theory of General Relativity of curved spacetimes.
Examples: Pick’s Theorem
Examples: Pick’s Theorem

Here $I = 8$, $B = 12$, $A = I + \frac{1}{2}B - 1 = 8 + 6 - 1 = 13$. 
Example: Geometric Computation of Area

\[ A = 30 - 5 - 4 - 1 - 3 - 4 = 13 \]
Example: Geometric Computation of Area

\[ A = 30 - 5 - 4 - 1 - 3 - 4 = 13 \]
In the case where the polygon is not simple

\[ I = 0, \quad B = 19, \quad A = I + 1 - 1 = 0 + 19 / 2 - 1 = 8.5. \]

But
\[ A = 1 \times 5 \times 2 / 2 + 1 \times 2 \times 2 / 2 = 7. \]
In the case where the polygon is not simple

Here \( I = 0, \ B = 19, \ A = I + \frac{1}{2}B – 1 = 0 + 19/2 – 1 = 8.5. \)
In the case where the polygon is not simple

Here $I = 0$, $B = 19$, $A = I + \frac{1}{2}B - 1 = 0 + 19/2 - 1 = 8.5$.

But

$$A = 1 \times 5 \times 2/2 + 1 \times 2 \times 2/2 = 7.$$
Proof of the Pick’s Theorem: Weight function

We will follow the proof of Varberg.

**Definition**

Given any polygon $P$, we can assign every single lattice point $L_k$ a weight by $w_k = \frac{\theta_k}{2\pi}$ where $\theta_k$ is the “visibility” angle for $L_k$ sees into $P$.

We can then define the total weight $W(P) = \sum_{L_k \in P} w_k(L_k)$.
Proof of the Pick’s Theorem: Weight function

We will follow the proof of Varberg.

**Definition**

Given any polygon $P$, we can assign every single lattice point $L_k$ a weight by $w_k = \frac{\theta_k}{2\pi}$ where $\theta_k$ is the “visibility” angle for $L_k$ sees into $P$. We can then define the total weight $W(P) = \sum_{L_k \in P} w_k(L_k)$.

**Example**

For interior lattice $L_i$, $w_k = 1$.
For the boundary lattice that is not a vertex, $w_k = 1/2$.
For the vertices which is a right angle corner point, $w_k = 1/4$. 
Proof: Total weight equals to area.

Lemma

\[ W(P) = A(P). \]
Proof: Total weight equals to area.

Lemma

\[ W(P) = A(P). \]

Proof of the lemma.
Proof: Total weight equals to area.

**Lemma**

\[ W(P) = A(P). \]

**Proof of the lemma.**
Proof: Total weight equals to area.

Lemma

\[ W(P) = A(P). \]

Proof of the lemma.
Proof: Total weight equals to area.

**Lemma**

\[ W(P) = A(P). \]

**Proof of the lemma.**

Firstly, \( W \) is additive, as the visibility angles of \( S_1 \) and \( S_2 \) adds up at the common lattice point to give the visibility angle of \( P \) as illustrated below.

Then, note that we can decompose any lattice polygon into a union of lattice triangles, thus if the lemma holds for lattice triangles, we are done.

Thus, we just need to prove that the lemma holds for lattice triangles.
Proof of the lemma: Triangle case.

Continued.

We will prove the lattice triangle case successively. Firstly, we will show that for the lattice rectangles that are having the sides parallel to lattices. This is obvious. Secondly, the case where a triangle having a right angle. This is done by using the first case divide by 2. Thirdly, for any general lattice triangle, we can always find the embedded rectangle and subdivide the rectangle into right triangles along with our initial triangle, and then we use the additivity of both $W$ and $A$. 

Yiwang Chen
Proof of Pick’s Theorem

Theorem (Pick’s Theorem)

The area of a simple lattice polygon $P$ is given by

$$A(P) = i + \frac{1}{2}b - 1 = l - \frac{1}{2}b - 1$$
Proof of Pick’s Theorem

**Theorem (Pick’s Theorem)**

The area of a simple lattice polygon $P$ is given by

$$A(P) = i + \frac{1}{2}b - 1 = l - \frac{1}{2}b - 1$$

**Proof.**

A simple polygon will have the outer angle sum $2\pi$, thus the sum of visible angles along the boundary is $b\pi - 2\pi$.

Therefore,

$$A(P) = W(P) = \sum_{L_k \in I} w_k(L_k) + \sum_{L_k \in B} w_k(L_k)$$

$$= i + \frac{(b - 2)\pi}{2\pi} = i + \frac{1}{2}b - 1 = l = \frac{1}{2}b - 1.$$
**Definition (Euler Characteristic)**

Euler Characteristic $\chi$ of a polygon is defined by

$$\chi = v - e + f,$$

Where $v$ is the total number of the vertices, $e$ is the total number of edges, and $f$ is the total number of the faces, of the triangulation of $P$, heuristically, if the polygon has $h$ holes, then

$$\chi = 1 - h.$$
Definition (Euler Characteristic)

Euler Characteristic $\chi$ of a polygon is defined by

$$\chi = v - e + f,$$

Where $v$ is the total number of the vertices, $e$ is the total number of edges, and $f$ is the total number of the faces, of the triangulation of $P$, heuristically, if the polygon has $h$ holes, then

$$\chi = 1 - h.$$

Theorem (General Pick’s Theorem)

The area of a lattice polygon $P$ (not necessarily simple) is given by

$$A(P) = v - \frac{1}{2}e_h - \chi,$$

where $v$ is total number of lattice points, $e_h$ is the number of edges on the boundary of $P$ and that $\chi$ is the Euler characteristic of $P$. 
Example revisited: Generalized Pick’s Theorem

Here I = 0, B = 19, eh = 22, \( \chi(P) = 1 \). Therefore we have

\[
A = 19 - \frac{1}{2} \times 22 - 1 = 19 - 22/2 - 1 = 7.
\]

And recall our calculation,

\[
A = 1 \times 5 \times 2/2 + 1 \times 2 \times 2/2 = 7.
\]
Example revisited: Generalized Pick’s Theorem

Here $I = 0$, $B = 19$, $e_h = 22$, $\chi(P) = 1$. Therefore we have

$$A = 19 - \frac{1}{2} \times 22 - 1 = 19 - 22/2 - 1 = 7.$$ 

And recall our calculation,

$$A = 1 \times 5 \times 2/2 + 1 \times 2 \times 2/2 = 7.$$
Proof of the Generalized Pick’s Theorem

We will prove it assuming primary triangulation of polygons (so we can decompose polygons into the triangles with area 1/2). We denote \( v, e \) and \( f \) as the number of vertices, edges and faces of the decomposition, as noted above. Since each triangle has 3 edges and each edge is shared by two triangles, we have \( 3f = 2e - e_h \).

Therefore, we have \( f = 2e - 2f - e_h = 2v - e_h - 2\chi \), and

\[
A(S) = \frac{1}{2}f = v - \frac{1}{2}e_h - \chi
\]
Reeve has given a formula for volume of three dimensional polyhedra involving also the Euler-Poincaré characteristic from algebraic topology. However, no formula involving only counts of lattice points on faces of $P$ can exist. In fact, no such formula exists for lattice tetrahedra (the convex hull of four lattice points in three dimensions).
Theorem

For three dimensional lattice tetrahedra $P$, there is no volume formula for $P$ of the form

$$a_1 I(P) + a_2 F(P) + a_3 E(P) + a_4 W(P) + a_5 = V(P).$$

where $I(P)$ is the number of interior lattice points, $F(P)$ is the number of lattice points on the interior of the faces, $E(P)$ the number of lattice points on the edges excluding the vertices and $W(P)$ the number of vertices.
Proof.

Consider the tetrahedra with given vertices.

<table>
<thead>
<tr>
<th>Vertices</th>
<th>I</th>
<th>F</th>
<th>E</th>
<th>W</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$ (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>1/6</td>
</tr>
<tr>
<td>$T_2$ (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>2/6</td>
</tr>
<tr>
<td>$T_3$ (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>3/6</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$T_r$ (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>$r/6$</td>
</tr>
</tbody>
</table>

Then it results in the system of equations, for all $r$ which is an integer.

$$4a_4 = r/6.$$ 

Therefore the system of equations are inconsistent, so there is no solution for $a_1, a_2, a_3, a_4, a_5$.

Note that these tetrahedra are called Reeve tetrahedra.
Visualization: Reeve Tetrahedron

(a)  (b)  (c)  (d)
References

Thanks!

Thanks!