## A Big Note for Calculus

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Note for Calculus
This note is intended to be finished one day as a note that I can teach my daughter, Clementine. Therefore, I will keep updating it overtime. If there is any bugs/typos, please email me at yiwchen@umich.edu, your help is really appreciated.

## 1 What is Calculus?

What is calculus?
Calculus as a subject, studies what its name suggests. Historically, calculus is a Latin word, means 'small pebble". Therefore, one should realize that this is a subject that study the infinity: infinitely small and infinitely huge.
When we talk about any specific function in life, we care about many question related to an infinite property of it. For example, the infinitely small change at some moment, or the sum of infinitely many pieces of area. We will figure out the meaning of calculus throughout the study of it.

Example 1.1. As the first few examples in the note, I will introduce some real life problem that triggers the study of calculus.
Think about Evan is driving a car on the road, getting Clementine to daycare. Note that Daddy's driving and the weather is bad. Therefore, the speed of the car changes dramastically.
Now comes a question, how do we measure the speed?
An intuitive way to answer it will be to count the travel distance of the car in a fixed time. Clementine start by fixing one minute. However, Clementine figure out that, during the fixed one minute, the travel distance might be the same, but sometimes Evan drives really fast and really slow, and this averages out within the fixed one minute! Clementine feels like this will not be a good measure of speed. Therefore, she changed the fixing time to be 1 second. This does not feels safe, Clementine thinks. The same thing will still happen if Evan's driving is too bad!
This triggers an interesting idea in Clementine's mind. Maybe, having a stopwatch with infinitely small time and measuring the travel distance within that time might work! However, this has an issue. The travel distance will then be infinitely small too, and this will result in a ' $0 / 0$ " type division which we have no idea how to work on! Fortunately, this issue will be resolved by Calculus nicely and will be well define. Working on this will be perfectly fine in the Calculus scheme.

## Example 1.2.

Example 1.3. Take a piece of woodstick. Cut it in half, then we know that we have $1 / 2+1 / 2=1$ which can recover the original length of the stick. Cut one of the small pieces again, then we have $1 / 4+1 / 4+1 / 2=1$ which recovers the original length of the stick. Do the same procedure again, then we will have $1 / 8+1 / 8+1 / 4+1 / 2=1$ which recovers the oriinal length of the stick.
Now, think about we keep doing the same procedure for infinitely many times.
This will result in evaluation of the sum

$$
\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\frac{1}{2^{1}}+\frac{1}{2^{2}}+\frac{1}{2^{3}}+\ldots
$$

Following the previous principle, we knows that if ever we stop at any finite stage, then we will have the sum to be 1 . Therefore, one should be able to be convinced that the infinite sum will give you the value 1 .

Note for Calculus
However, up to now, we still have no enough information to justify the value of any infinite sum. This will be study in the later chapter of this note, but should serve as an intuitive idea on what kind of object what we are working on.

## 2 General Functions

Definition 2.1. A function is a rule that takes certain numbers as inputs and assigns to each a definite output number.
The set of all input numbers is called the domain of the function and the set of resulting output numbers is called the range of the function.
The input is called the independent variable and the output is called the dependent variable.

Definition 2.2. The Rule of Four: Tables, Graphs, Formulas, and Words
Definition 2.3. We denote the graph of a function to be a set of points $(x, y)$ with an extra property $y=f(x)$.

Definition 2.4. A function f is increasing if the values of $f(x)$ increase as x increases. A function f is decreasing if the values of $f(x)$ decrease as x increases. Moreover, A function $\mathrm{f}(\mathrm{x})$ is monotonic if it increases for all x or decreases for all x .

Remark. The graph of an increasing function climbs as we move from left to right. The graph of a decreasing function falls as we move from left to right.

Definition 2.5. The graph of function is Concave up if it bends upwards as we move left to right ("Smile Face").
The graph of function is Concave down if it bends downwards as we move left to right ("Sad Face").

Definition 2.6. Composition of function is basically combining of functions so that the output from one function becomes the input for the next function. We usually denote $f(g(x))$ as " $f$ composed with $g$ of $x$ ".

Definition 2.7. Let $f: A \rightarrow B$ be a function. If for any $b \in f(A)$, there is a unique $a \in A$ such that $b=f(a)$. Then we see that given any $b$ we can track back to see what $a$ it corresponds to. Thus we have another function $f^{-1}: B \rightarrow A$ such that $f^{-1}(b)=a$. Therefore,

$$
f^{-1}(b)=a \Longleftrightarrow f(a)=b
$$

and we have $f\left(f^{-1}(x)\right)=f^{-1}(f(x))=x$.
A function $f$ is invertible if whenever $f\left(a_{1}\right)=f\left(a_{2}\right)$, it necessarily follows that $a_{1}=a_{2}$. (This is so-called the horizontal test.

Definition 2.8. A function $f(x)$ is said to be periodic with period $p$ if $f(x)=f(x+n p)$ for any $n$ which is an integer. (We denote the least $p$ to be the period.)

Interpretation: Period is the smallest time needed for the function to run a complete cycle.

Definition 2.9. A periodic function come with three property, Amplitute, Midline, Period, where Amplitute A is defined as

$$
A=(\max -\min ) / 2
$$

, Midline $y=m$ is defined as

$$
y=m=(\max +\min ) / 2
$$

Example 2.1. $f(t)=\sin t$ is of period $2 \pi$, amplitute $A=1$, and midline is $y=0$. $f(t)=\cos t$ is of period $2 \pi$, amplitute $A=1$, and midline is $y=0$. $f(t)=\tan t$ is of period $\pi$.

Definition 2.10. A function $f$ is defined on an interval around $c$ (not necessarily defined on $c$ ). The limit of the function $f(x)$ as $x$ approaches $c$, written $\lim _{x \rightarrow c} f(x)$, to be a number $L$ such that $f(x)$ is as close to $L$ as we want whenever $x$ is sufficiently close to $c$ (but $x \neq c)$. If $L$ exists, we write $\lim _{x \rightarrow c} f(x)=L$

Remark. Informally, we write $\lim _{x \rightarrow c} f(x)=L$ if the values of $f(x)$ approach $L$ as $x$ approaches $c$.
Remark. Assuming all the limits on the right hand side exist:

1. If b is a constant, then $\lim _{x \rightarrow c}(b f(x))=b \lim _{x \rightarrow c} f(x)$.
2. $\lim _{x \rightarrow c}(f(x)+g(x))=\lim _{x \rightarrow c} f(x)+\lim _{x \rightarrow c} g(x)$
3. $\lim _{x \rightarrow c}(f(x) g(x))=\lim _{x \rightarrow c} f(x) \lim _{x \rightarrow c} g(x)$
4. $\lim _{x \rightarrow c} \frac{f(x)}{g(x)}=\frac{\lim _{x \rightarrow c} f(x)}{\lim _{x \rightarrow c} g(x)}$
5. $\lim _{x \rightarrow c} k=k$, for $k$ a constant
6. $\lim _{x \rightarrow c} x=c$

Definition 2.11. Formal definition for Asymptotes.
If the graph of $y=f(x)$ approaches a horizontal line $y=L$ as $x \rightarrow \infty$ or $x \rightarrow-\infty$, then the line $y=L$ is called a horizontal asymptote, i.e. The horizontal asymptotes are

$$
y=\lim _{x \rightarrow \infty} f(x)
$$

, and

$$
y=\lim _{x \rightarrow-\infty} f(x)
$$

if they exist.
If the graph of $y=f(x)$ approaches a vertical line $x=K$ as $x \rightarrow K^{+}$or $x \rightarrow K^{-}$, then the line $x=K$ is called a vertical asymptote, i.e. The vertical asymptotes are all the $x=K$ such that either of the following holds.

$$
\lim _{x \rightarrow K^{+}} f(x)=\infty \text { or } \lim _{x \rightarrow K^{+}} f(x)=-\infty \text { or } \lim _{x \rightarrow K^{-}} f(x)=\infty \text { or } \lim _{x \rightarrow K^{-}} f(x)=-\infty
$$

Example 2.2.

$$
\frac{b_{t} x^{t}+\ldots+b_{1} x+b_{0}}{c_{r} x^{r}+\ldots+c_{1} x+c_{0}} \text { has HA: } \begin{cases}y=\frac{b_{t}}{c_{r}} & r=t \\ y=0 & r>t \\ \text { None } & \text { else }\end{cases}
$$

and has VA at $x=K$, if $K$ is a root of $a_{n} x^{n}+\ldots+a_{1} x+a_{0}$.

## 3 Continuity

The function $f$ is continuous at $x=c$ if f is defined at $x=c$ and if $\lim _{x \rightarrow c} f(x)=f(c)$.
The function is continuous on an interval $[a, b]$ if it is continuous at every point in the interval.
Suppose that $f$ and $g$ are continuous on an interval and that $b$ is a constant. Then, on that same interval,

1. $b f(x)$ is continuous.
2. $f(x)+g(x)$ is continuous.
3. $f(x) g(x)$ is continuous.
4. $f(x) / g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

## 4 Shifting and Stretches

Given function $y=f(x)$. There are six transformation onto the function.

## Vertical Transformation.

1. Vertical Shift.

To shift the function $y=f(x)$ vertically by $k$, we have the new function $y=$ $f(x)+k$. If $k>0$, we shift the graph upwards and if $k<0$, we shift the graphs downwards.
2. Vertical compression or stretch.

To vertically compress/stretch the function $y=f(x)$ by a factor of $a$, we have the new function as $y=a f(x)$. Here, a is always positive. If $0<a<1$, we are compressing the graph vertically by $a$. If $a>1$ we are stretching the graph vertically by $a$.
3. Reflection along x -axis.

We reflect a function along x -axis by multiplying a -1 outside our function.

## Horizontal Transformation.

1. Horizontal shift.

To shift the function $y=f(x)$ horizontally by $t$, we have the new function as $y=f(x-k)$. Note we are subtracting $t$ here. If $t>0$, we are shifting the graph towards right and if $k<0$, we are shifting the graphs towards left.
2. Horizontal compression/stretch.

To horizontally compress/stretch the function $y=f(x)$ by a factor of $r$, we have the new function as $y=f\left(\frac{x}{r}\right)$. Note that we are dividing $r$ here.
if $r>1$ we are horizontally compressing the graph horizontally, and if $0<r<1$ we are horizontally stretching the graph horizontally.
3. Reflection along y-axis.

We reflect a function along y-axis by multiplying a -1 inside our function.

## 5 Linear Functions

Definition 5.1. A linear function has the form $y=f(x)=b+m x$. Its graph is a line such that

- $m$ is the slope, or rate of change of $y$ with respect to $x . b$ is the vertical intercept, or value of $y$ when $x$ is zero.

Remark. Two different points on the graph determine a unique linear function that has the function graph through them.

Definition 5.2. $y$ is proportional to x if there is a nonzero constant $k$ such that $y=k x$. This $k$ is called the constant of proportionality

## 6 Exponential Functions and Logarithm

Definition 6.1. A Exponential Functions has the form $P(t)=P_{0} \cdot a^{t}=e^{k t}=(1+r)^{t}$.
Definition 6.2. The logarithm of $x$ to base $b$, denoted $\log _{b}(x)$, is the unique real number $y$ such that $b^{y}=x$. Therefore, logarithmic function is the inverse function for exponential function.
In the case where $b=e$, we denote it as $\ln (x)$, i.e. $\log _{e}(x)=\ln (x)$.
In the case where $b=10$, we denote it as $\log (x)$, i.e. $\log _{10}(x)=\log (x)$
Remark. Properties of Log Function:
(1) $\log _{a}\left(a^{x}\right)=x$; (2) $a^{\log _{a}(y)}=y$;
(3) $\log _{a}(x y)=\log _{a}(x)+\log _{a}(y)$; (4) $\log _{a}(1)=0$;
(5) $\log _{a}(1 / x)=\log _{a}(x) ;(6) \log _{a}(x)=\ln (x) / \ln (a)$.

Definition 6.3. The half-life of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
The doubling time of an exponentially increasing quantity is the time required for the quan tity to double.

## 7 Trigonometric Function

Definition 7.1. Unit circle Consider the set of points $\left\{(x, y): x^{2}+y^{2}=1\right\}$. It is called the unit circle centered at the origin. (Warning: this is not a graph of a function since it fails the vertical line test.)

Definition 7.2. Angle $\theta$ is defined as the arc length on the unit circle measured counterclockwise from the x -axis along the arc of the unit circle.
Definition 7.3. Thus angle is a real number and it is of the unit (radian). The angle in radian and the angle in degree is connected in the following way,

$$
360^{\circ}=2 \pi(\text { radian })
$$

Here I bracket the radian since radian is considered as unit-free.

Remark. arc formula connecting the arc length with the radius and angle, i.e.

$$
\text { Arc length }=l=r \theta
$$

Definition 7.4. cos and $\sin$ is the horizontal coordinate of the arc endpoint and the endpoint coordinate of the arc endpoint. Moreover, we define tangent of the angle $\theta$ as $\tan \theta=\frac{\sin \theta}{\cos \theta}$, and secant of the angle $\theta$ as $\sec (\theta)=1 / \cos (\theta)$.
Remark. We have those identity coming from the definition.

1. $\cos ^{2}(\theta)+\sin ^{2}(\theta)=1$,
2. $1+\tan ^{2}(\theta)=\sec ^{2}(\theta)$,
3. $\cos (x)=\cos (x+2 \pi)$,
4. $\sin (x)=\sin (x+2 \pi)$,
5. $\tan (x)=\tan (x+\pi)$,
6. $\cos (t+\pi)=-\cos (t)$,
7. $\sin (t+\pi)=-\sin (t)$,
8. $\sin (x+\pi / 2)=\cos (x)$,
9. $\cos (x-\pi / 2)=\sin (x)$,
10. $\cos (x)=\cos (-x)$,
11. $\sin (x)=-\sin (-x)$,
12. $\tan (x)=-\tan (-x)$

Remark. Since the function graph does not pass the vertical line test, we cannot define an inverese trigonometry function in general. However, restricted the function to some certain domain, we can define the inverse of the trigonometry function.

## 8 Power Function, Polynomials, and Rational Function

Definition 8.1. A function $f$ is called a Power functions if we can write $f(x)=k x^{p}$ where $k, p$ are fixed constants.

Definition 8.2. A Polynomials is a function that is in the form

$$
f(x)=a_{n} x^{n}+a_{n 1} x^{n 1}+\ldots+a_{1} x+a_{0}
$$

where $n \geq 0$ is a fixed integer and $a_{n}, a_{n 1}, \ldots, a_{1}, a_{0}$ are fixed constants. Moreover, we call $a_{n} x^{n}$ as its leading coefficient.

Definition 8.3. A Rational functions. is a function $f(x)=\frac{p(x)}{q(x)}$ for all $x$ in its domain, with $p(x)$ and $q(x)$ are both fixed polynomials.

Theorem 8.1. Fundamental Theorem of Algebra
Any polynomial with degree n can be factored as $f(x)=a_{n}\left(x-r_{1}\right) \ldots\left(x-r_{N}\right)\left(x^{2}+\right.$ $\left.b_{1} x+c_{1}\right) \ldots\left(x^{2}+b_{M} x+c_{M}\right)$ with $n=N+2 M$ and each $x^{2}+b_{j} x+c_{j}$ has no real roots.

## 9 Derivative

## Definition 9.1.

Average rate of change of f over the interval from a to $(\mathrm{a}+\mathrm{h})=\frac{f(a+h)-f(a)}{h}$
Definition 9.2. The derivative of $f$ at a is defined as

$$
\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
$$

If the limit exists, then f is said to be differentiable at a.
Remark. Differentiable function is continuous function, while continuous function fails to be differentiable function easily!

Definition 9.3. The derivative function
For any function f , we dene the derivative function, $f^{\prime}$, by

$$
f^{\prime}(x)=\text { Rate of change of } \mathrm{f} \text { at } \mathrm{x}=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

For every x -value for which this limit exists, we say f is differentiable at that x -value. If the limit exists for all $x$ in the domain of $f$, we say $f$ is differentiable everywhere.

Remark. If $f>0$ on an interval, then $f$ is increasing over that interval. If $f<0$ on an interval, then $f$ is decreasing over that interval.

Theorem 9.1. Power Function. If $f(x)=x^{n}$, then $f(x)=n x^{n-1}$.
Interpretation of derivatives

- NEVER use mathematical terminologies to write down a sentence interpretations.
- The statement $y=f^{\prime}(x)$ should be interpreted as if $x$ is increased by $*(1,0.1,0.01, \ldots$, whatever is comparably small in the context), the resulting increase of $f$ is approximately $* x y$. (With units)
- The statement $y=\left(f^{-1}\right)^{\prime}(x)$ should be interpreted as if $x$ is increased by ${ }^{*}(1,0.1,0.01, \ldots$, whatever is comparably small in the context), the resulting increase of $f^{-1}$ is approximately $* \times y$. (With units) You need to find out the meaning of $f^{-1}$ as well in this case.

Definition 9.4. The second derivative: For a function $f$, the derivative of its derivative is called the second derivative, and written $f^{\prime \prime}$ (read $f$ double-prime). If $y=f(x)$, the second derivative can also be written as $\frac{d^{2} y}{d x^{2}}$, which means $\frac{d}{d x}\left(\frac{d y}{d x}\right)$, the derivative of $\frac{d x}{d y}$.

Proposition 9.1. Property of second derivative:
If $f^{\prime \prime}>0$ on an interval, then $f^{\prime}$ is increasing, so the graph of $f$ is concave up there.
If $f^{\prime \prime}<0$ on an interval, then $f^{\prime}$ is decreasing, so the graph of $f$ is concave down there.
If the graph of f is concave up on an interval, then $f^{\prime \prime} \geq 0$ there.
If the graph of f is concave down on an interval, then $f^{\prime \prime} \leq 0$ there.
Remark. Important: a straight line is neither concave up nor concave down, and has second derivative zero.

## Remark. Derivative Rules:

1. General rules:
(a) Derivative of a constant times a function

$$
\frac{d}{d x}[c f(x)]=c f^{\prime}(x)
$$

(b) Derivative of Sum and Difference

$$
\begin{aligned}
\frac{d}{d x}[f(x)+g(x)] & =f^{\prime}(x)+g^{\prime}(x) \\
\frac{d}{d x}[f(x)-g(x)] & =f^{\prime}(x)-g^{\prime}(x)
\end{aligned}
$$

(c) Product Rule and Quotient Rule

Product Rule:

$$
(f \cdot g)^{\prime}(x)=f^{\prime}(x) \cdot g(x)+f(x) \cdot g^{\prime}(x) .
$$

or equivalently,

$$
\frac{d}{d x}(f(x) g(x))=f(x) \frac{d}{d x} g(x)+\frac{d}{d x} f(x) g(x)
$$

Quotient Rule:

$$
\left(\frac{f}{g}\right)^{\prime}(x)=\frac{f^{\prime}(x) g(x)-g^{\prime}(x) f(x)}{g^{2}(x)}
$$

or equivalently,

$$
\frac{d}{d x}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{d x} f(x) g(x)-f(x) \frac{d}{d x} g(x)}{g^{2}(x)}
$$

(d) Chain rule:

$$
\frac{d y}{d x}=\frac{d y}{d z} \frac{d z}{d x}
$$

or equivalently,

$$
\frac{d}{d x}(f(g(x)))=f^{\prime}(g(x)) \cdot g^{\prime}(x)
$$

(e) Derivative of general inverse function

$$
\frac{d}{d x}\left(f^{-1}(x)\right)=\frac{1}{f^{\prime}\left(f^{-1}(x)\right)}
$$

2. Rules for specific type functions:
(a) Power Rule

$$
\frac{d}{d x}\left(x^{n}\right)=n x^{n-1}
$$

(b) Exponential Rule

$$
\frac{d}{d x}\left(a^{x}\right)=\ln a a^{x}
$$

(c) Trignometric Rules

$$
\begin{aligned}
\frac{d}{d x}(\sin x) & =\cos x \\
\frac{d}{d x}(\cos x) & =-\sin x \\
\frac{d}{d x}(\tan x) & =\frac{1}{\cos ^{2} x}
\end{aligned}
$$

(d) Logarithmetic Rules

$$
\frac{d}{d x}(\ln x)=\frac{1}{x}
$$

(e) Anti-Trig Rules

$$
\begin{aligned}
\frac{d}{d x}(\arcsin x) & =\frac{1}{\sqrt{1-x^{2}}} \\
\frac{d}{d x}(\arccos x)- & =\frac{1}{\sqrt{1-x^{2}}} \\
\frac{d}{d x}(\arctan x) & =\frac{1}{1+x^{2}}
\end{aligned}
$$

3. Implicit function derivative.

## 10 Definite Integral

Definition 10.1. A definite integral of $f$ from $a$ to $b$ is definted as

$$
\int_{a}^{b} f(x) d x=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} f\left(x_{i}\right) \Delta x(\text { Limit of Right-hand sum })
$$

or

$$
\int_{a}^{b} f(x) d x=\lim _{n \rightarrow \infty} \sum_{i=0}^{n-1} f\left(x_{i}\right) \Delta x(\text { Limit of Left-hand sum })
$$

Here, Left-hand sum and Right-hand sum are equal after taking limits and it the the so-called Riemann Sum.

There are two more type of Riemann sum I would like to discuss in the future, which is the Mid sum and the Trapezoidal sum.
I will only give definition here.

$$
\int_{a}^{b} f(x) d x \approx \sum_{i=0}^{n-1} \frac{x_{i}+x_{i+1}}{2} \Delta x(\text { Mid sum })
$$

and

$$
\int_{a}^{b} f(x) d x \approx \sum_{i=0}^{n-1} \frac{f\left(x_{i}\right)+f\left(x_{i+1}\right)}{2} \Delta x(\text { Trapezoid sum })
$$

In all these Riemann sum we discussed, we are assuming $\Delta(x)=\frac{b-a}{n}$, thus as $n \rightarrow \infty$, $\Delta x \rightarrow 0$.
Note that

$$
\begin{gathered}
\frac{\operatorname{LEFT}(n)+\operatorname{RIGHT}(n)}{2}=\operatorname{TRAP}(n) \\
M I D(n) \neq \operatorname{TRAP}(n)
\end{gathered}
$$

Remark. Properties of definite integral:
1.

$$
\int_{b}^{a} f(x) d x=-\int_{a}^{b} f(x) d x
$$

2. 

$$
\int_{b}^{a} f(x) d x+\int_{c}^{b} f(x) d x=\int_{c}^{a} f(x) d x
$$

3. 

$$
\int_{b}^{a}(f(x) \pm g(x)) d x=\int_{b}^{a} f(x) d x \pm \int_{b}^{a} g(x) d x
$$

4. 

$$
\int_{b}^{a} c f(x) d x=c \int_{b}^{a} f(x) d x
$$

5. Symmetry due to the oddity of the function.

Remark. Interpretation of Define Integral as Area under graph of $f$ between $x=a$ and $x=b$, counting positivity.
Important cases wanted to discuss on Friday's course:

1. Compute

$$
\int_{-1}^{1} \sqrt{1-x^{2}} d x
$$

2. How about

$$
\int_{-1}^{1}\left(\sqrt{1-x^{2}}-1\right) d x ?
$$

3. Maybe try

$$
\int_{0.5}^{0.5} \tan (x) d x
$$

Find out the answer yourself only geometrically even you know more techniques! More Importantly there are two major topics I want to mention here and maybe discuss:

- When is the estimation done by Riemann sum a underestimate/overestimate?


## It is also covered in 7.5, check it out and try problem 2.

- Error estimation

Think about the case where you know that $f(x)$ lies between any pair of $\operatorname{LEFT}(n)$ and $\operatorname{RIGHT}(n)$, then we see that $|\operatorname{LEFT}(n)-f(x)|<|\operatorname{LEFT}(n)-\operatorname{RIGHT}(n)|=$ $(f(b)-f(a)) \Delta x$. This usually gives a bound for $n$.

Theorem 10.1. The Fundamental Theorem of Calculus is basically the theorem defined below.
If $f$ is continuous on interval $[a, b]$ and $f(t)=F^{\prime}(t)$, then

$$
\int_{a}^{b} f(t) d t=F(b)-F(a) .
$$

## Application of the Fundamental Theorem of Calculus

1. Average value of function $f(x)$ in $[a, b]$ is

$$
\frac{1}{b-a} \int_{a}^{b} f(x) d x
$$

Here, think about the integration is analogue to summation in the discrete world, then this average value is the analogue of

$$
x_{\text {average }}=\frac{x_{1}+x_{2}+\ldots+x_{n}}{1+1+1+\ldots}=\frac{x_{1}+x_{2}+\ldots+x_{n}}{n} .
$$

where in the integration case is actually

$$
x_{\text {average }}=\frac{\lim _{n \rightarrow \infty} \sum_{i=0}^{n-1} f\left(x_{i}\right) \Delta x}{\lim _{n \rightarrow \infty} \sum_{i=0}^{n-1} 1 \times \Delta x}=\frac{\int_{a}^{b} f(x) d x}{\int_{a}^{b} 1 d x}=\frac{1}{b-a} \int_{a}^{b} f(x) d x
$$

Now, try to work on the rest of problems in the exercise and email me if you have any question, if you think the problems are too hard, I can give you some basic problems to work on first.
Problem 5 is going to be discussed in Monday so we have more time.

## 11 Chapter 6 Anti-derivatives

## Anti-derivative of usual functions

1. Try to find the antiderivatives by the graphs
2. Compute an antiderivative using definite integrals.

Suggested Problems: § $6.13,7,9,13,17,29,31,33$
Construct antiderivative analytically

Note for Calculus
Definition 11.1. We define the general antiderivative family as indefinite integral.
Remark.

$$
\begin{gathered}
\int C d x=0 \\
\int k d x=k x+C \\
\int x^{n} d x=\frac{x^{n+1}}{n+1}+C,(n \neq-1) \\
\int \frac{1}{x} d x=\ln |x|+C \\
\int e^{x} d x=e^{x}+C \\
\int \cos x d x=\sin x+C \\
\int \sin x d x=-\cos x+C
\end{gathered}
$$

Properties of antiderivatives:
1.

$$
\int(f(x) \pm g(x)) d x=\int f(x) d x \pm \int g(x) d x
$$

2. 

$$
\int c f(x) d x=c \int f(x) d x
$$

Suggested Problems: § 6.2 51-59, 65,71,75

## Second FTC (Construction theorem for Antiderivatives)

Theorem 11.1. If $f$ is a continuous function on an interval, and if $a$ is any number in that interval then the function $F$ defined on the interval as follows is an antiderivative of $f$ :

$$
F(x)=\int_{a}^{x} f(t) d t
$$

Suggested Problems: § 6.4 5,7,9,11,17,27,31-34

## 12 Integration Techniques

In this section, we will focus more on how to find the family of antiderivatives that is determined by the function we provided, since if we know such family, then given an initial value of $F(x)$, we can recover the specific antiderivative that we need.

### 12.1 Guess and Check method

Frankly speaking, this is gambling based on experience.
Although it is one of the fundamental method we use to solve problems, for example, it appears in also cryptography with the name Bruce force attack, and it also appears in computer science with the name Brute force search. However, as you probably know, this is usually not the most systematic or the most efficient way to the solution, and it usually requires either experience on the work or machine that do not rest.

### 12.2 Integration using Substitution

Exercise as motivation: $\S 6.4$ 35,37.
Similarly, the chain rule of derivative is also extremely useful in finding integral.
Theoretically, integration using substitution is using the philosophy that given an integration as $\int f(g(x)) g^{\prime}(x) d x$, we will have $F(g(x))+C$ as its derivative, which is due to $\frac{d}{d x}(F(g(x)))=f(g(x)) g^{\prime}(x)$, and doing integration on both side will give us the "substitution rule" we want.
Technically, you want to recognize the " $g(x)$ " part in your function, then substitude $g(x)$ as $w$, with the rule $d w=w^{\prime}(x) d x=\frac{d w}{d x} d x$ (naively, we do have $\frac{d w}{d x} d x=d w$ ! However, this does not make much sense here which if you are particular interesting, you can ask me in person.)
Now let us consider an example, to see how is it actually applies. Personally, my favorite almost trivial example is

$$
\int_{0}^{\infty} e^{-x} d x=-\int_{0}^{-\infty} e^{w}-d w=\int_{-\infty}^{0} e^{w} d w=1 .(\text { Here I let } w=-x)
$$

I will explain why is this one interesting in the next subsection, but here there is a warning you may see, Check the up and low bound for your integration! Let us see another example to see this clearer.

$$
\int_{0}^{\pi / 4} \frac{\tan ^{3} \theta}{\cos ^{2} \theta} d \theta=\int_{0}^{1} w^{3} d w=\frac{1}{4},(\text { Here I take } w=\cos \theta .)
$$

Fast question: Can I do the following integrals? Why?

$$
\text { (1) } \int_{0}^{\pi / 2} \frac{\tan ^{3} \theta}{\cos ^{2} \theta} d \theta \text { or (2) } \int_{-\pi / 4}^{\pi / 4} \frac{\tan ^{3} \theta}{\cos ^{2} \theta} d \theta
$$

Suggested Problems: $\S 7.1$ some ex, $81,89,90-96,97,99,109,111,114,125,141,145,147$

### 12.3 Integration using partial fractions

Small break after the introduction to substitution, we have this partial fractions.
Using the method of the substitution, we can already compute many weird integration like $\int \frac{2}{x-10} d x$.
Now, how about $\int \frac{1}{x^{2}-1} d x$ ? We are out of knowledge about how to do it right now, but there is an important method we ususally can apply, which is the partial fractions.
The idea here is to break $\frac{1}{x^{2}-1}$ as a sum:

$$
\frac{1}{x^{2}-1}=\frac{1}{2(x-1)}+\left(-\frac{1}{2(x+1)}\right)
$$

But then now we can compute the integration in the question, since we can definitely compute $\int \frac{1}{2(x-1)} d x$ and $\int \frac{1}{2(x+1)} d x$.
In general, we can do this partial fraction to

$$
\frac{p(x)}{\left(x-c_{1}\right)\left(x-c_{2}\right) \ldots\left(x-c_{n}\right)}=\frac{A_{1}}{x_{1}-c_{1}}+\frac{A_{2}}{x_{2}-c_{2}}+\ldots+\frac{A_{n}}{x_{n}-c_{n}}
$$

### 12.4 Integration by parts

If we call the integration using substitution the masterpiece established on the Chain Rule, then I have to say integration by parts is the same thing established on product rule.
Similar to the substitution, we reverse our thoughts here.
What is exactly happen in the integration by parts is that we are just undoing the product rules. Therefore, the key part is recognize your product rule carefully,
Consider the product rule:

$$
(u v)^{\prime}=u^{\prime} v+u v^{\prime}=u^{\prime} v+u v^{\prime}
$$

By some twisted, we will have the following formula for $u v^{\prime}$ which we have not seen the advantages yet.

$$
u v^{\prime}=(u v)^{\prime} u^{\prime} v
$$

But now, integrate both sides will give us:

$$
\int u d v=\int(u v) \int v d u=u v-\int v d u
$$

, now if you remember what we talked about in the substitution chapter, you will immediately recognize that this is what we want.
Traditional Example:

$$
\begin{aligned}
& \int \ln (x) d x \\
& \int \cos ^{2} x
\end{aligned}
$$

Suggested Problems: § 7.2 49,53,55,65,69,73

## 13 Integration Approximation

Estimate the integral is based on the fact that after taking limits of Left/Right/Mid/Trapezoidal Riemann Sum, they are gonna equals and will give the actual integration value, thus if we just compute the Riemann sums in finite many block, we will have a value that is pretty much close to the actual integration, and we can refine our calculation by taking more blocks.
Reminder: we have then four ways of estimating an integral using a Riemann Sum:

1. LEFT(n)
2. RIGHT(n)
3. $\operatorname{MID}(\mathrm{n})$
4. $\operatorname{TRAP}(\mathrm{n})$

Think about what they are? Think about Pictures!
Is your estimation over/under?
Again, Pictures!(It probably will never make sense without picture.)

1. If the graph of $f$ is increasing on $[a, b]$, then

$$
\operatorname{LEFT}(n) \leq \int_{a}^{b} f(x) d x \leq R I G H T(n)
$$

2. If the graph of $f$ is decreasing on $[a, b]$, then

$$
R I G H T(n) \leq \int_{a}^{b} f(x) d x \leq L E F T(n)
$$

3. If the graph of $f$ is concave up on $[a, b]$, then

$$
\operatorname{MID}(n) \leq \int_{a}^{b} f(x) d x \leq T R A P(n)
$$

4. If the graph of $f$ is concave down on $[a, b]$, then

$$
T R A P(n) \leq \int_{a}^{b} f(x) d x \leq M I D(n)
$$

## 14 Find Area/Volumes by slicing

- Compute the area of triangle;
- Compute the area of (semi)circle;
- Compute the volume of a sphere;
- Compute the volume of a cone with base radius 5 and height 5;
- Volume of revolution: $y=e^{-x}$ from 0 to 1 around $x$-axis.

Suggested Problems: $\S 8.1$ 1-4,10,12,14,16,18,19,21,29,31,37

## 15 Volumes of Solids of Revolution

There are two methods to find such a volume. One way is using the slicing as we just discussed above, which is officially called the Disk Method.

### 15.1 Disk Method

Check above.

### 15.2 Shell Method

This is not so intuitive that why will we want to consider other method to integrating any function since we already have this disk method. However, consider the following example:
Determine the volume of the solid obtained by rotating the region bounded by $y=$ $(x-1)(x-3)^{2}$ and the $x$-axis about the $y$-axis.
Try to graph it and see why it is bad.
Thus, the shell method arises naturally as a different approach, as it integrates along an axis perpendicular to the axis of revolution.

## 16 Arc length

$$
\text { Arc Length }=\int_{a}^{b} \sqrt{1+\left(f^{\prime}(x)\right)^{2}} d x
$$

The function we are integrating might not see as intuitive, but when we think about the picture of "small increment" as in class, you will see how it is making sense.
Suggested Problems: $\S 8.2$ 25,27,35,47,49,57,63,65

## 17 Relation of Integration and Physics

Think about what integration is doing. Integral is defined as a limit of the Riemann sum, thus what it is doing is exactly the same as what the Riemann sum is doing, furthermore, you can think definite integral as just infinite block Riemann sum.
What we had done in the last section is to find Volumes by integrating, which is the first time we figure out what this mysterious " $\mathrm{f}(\mathrm{x})$ " in the integration really is, i.e. if it is an area, then $f(x) \Delta x$ is actually a volume, and we are summing many slices/shells of volumes to get the exact amount of volume of the object.
Here, the most essential formula we are using is actually $V=A h$, where $V$ is the volume of prism/cylinder, $A$ is the area, and $h$ is the height. In the most settings above, we will see $S$ as function of $h$ (which is exactly disk method), or sometimes $h$ is a function of $A$ ( what we only deal are the good cases, where $S$ is a function of $r$ and $h$ is a function of $r$, since $d S$ is not making sense yet).

We will see in this section that all our important example here comes from a product formula, and what we are going to do is merely same things, integrate the factor parts, get the information of the product.

### 17.1 Mass/Center of Mass

### 17.1.1 Mass

Here, the basic formula we are doing is:

1. One dimensional: $M=\delta l$ where $M$ is the total mass, $\delta$ is the density, $l$ is line.
2. Two dimensional: $M=\delta A$ where $M$ is the total mass, $\delta$ is the density, $A$ is Area.
3. Three dimensional (real world): $M=\delta V$ where $M$ is the total mass, $\delta$ is the density, $V$ is Volume.

It is confusing that when should one use which, but if you think about what the unit of the density is, it will make sense of itself. (Actually, if you think area as the density of Volume, i.e. think of unit of it as $m^{2}=m^{3} / m$, then you will see integration is basically integrating the density to get the total, this is actually the essential picture you should have.)

### 17.1.2 Center of Mass

This is making more sense if you have any idea on vectors, since then it will naturally be the average of the vectors, with each vector "values" differently. Otherwise, you can think of this as average of weighted points, which is the same thing.
The finite version we are doing:

$$
\bar{x}=\frac{\sum_{i=1}^{n} m_{i} x_{i}}{M}=\frac{\sum_{i=1}^{n} m_{i} x_{i}}{\sum_{i=1}^{n} m_{i}}
$$

Thus the integration version will be

$$
\bar{x}=\frac{\int x \delta d x}{\int \delta d x}
$$

Generalization to three dimension, we have

$$
\bar{x}=\frac{\int x \delta A_{x} d x}{\text { Mass }}, \bar{y}=\frac{\int y \delta A_{y} d y}{\text { Mass }}, \bar{z}=\frac{\int z \delta A_{z} d z}{\text { Mass }}
$$

, where $A_{x}(x), A_{y}(y), A_{z}(z)$ are the the area of a slice perpendicular to the $x$ (respectively, $y, z$ )-axis at $x$ (respectively, $y, z$ ).
Here you can really think of $\delta A_{x}(x)$ as a kind of general density when you really smash the whole thing to $x$ axis, so to speak.
Suggested Problems: $\S 8.42,9,14,15,25,27,33$

### 17.2 Work

Key formula we are using:

$$
\text { Work done }=\text { Force } \cdot \text { Distance }
$$

or

$$
W=F \cdot d
$$

Integration version:

$$
W=\int_{a}^{b} F(x) d x
$$

(Question: why usually not $x=x(F)$ ?)

### 17.3 Pressure

Key formula we are using:

$$
\text { Pressure }=\text { Mass density } \cdot g \cdot \text { Depth. }
$$

or

$$
p=\delta g h
$$

(where $g$ is the acceleration due to gravity)

### 17.4 Force

Key formula we are using:

$$
\text { Force }=\text { Pressure } \cdot \text { Area }
$$

## 18 Parametric Equations and Polar Coordinate

### 18.1 Parametric Equations

To represent the motion of a particle in the $x y$-plane we use two equations, $x=f(t)$ and $y=g(t)$, then at the time $t$ the particle is at the location $(f(t), g(t)$. In this case, we call the equations for $x$ and $y$ the parametric equations, with parametrization $t$.
Remember that, in parametric equation, for the same line, the parametrization is not unique, and the different parametrization encodes two information:

1. Speed of the particle.
2. Direction of the motion.

### 18.1.1 Special Parametric Equations

- Parametric Equations for a Straight Line

An object moving along a line through the point $\left(x_{0}, y_{0}\right)$, with $d x / d t=a$ and $d y / d t=b$, has parametric equations $x=x_{0}+a t, y=y_{0}+b t$. The slope of the line is $m=b / a$.

- Parametric Equations for a circle with radius $k$

An object moving along a circle of radius $k$ counterclockwise has parametric equations $x=k \cos (t), y=k \sin (t)$.

### 18.1.2 Slope and concavity of the curve

As we discussed in class, we can think of this as a result due to chain rule if we have that $y=F(x)$ as well.
But to summarize, we have the slope of the parametrized curve to be

$$
\frac{d y}{d x}=\frac{d y / d t}{d x / d t}
$$

and the concavity of the parametrized curve to be

$$
\frac{d^{2} y}{d x^{2}}=\frac{(d y / d x) / d t}{d x / d t}
$$

### 18.1.3 Speed and distance

The instantaneous speed of a moving object is defined to be

$$
v=\sqrt{(d x / d t)^{2}+(d y / d t)^{2}}=\sqrt{\left(v_{x}\right)^{2}+\left(v_{y}\right)^{2}}
$$

Note for Calculus
. The quantity $v_{x}=d x / d t$ is the instantaneous velocity in the $x$-direction; $v_{y}=d y / d t$ is the instantaneous velocity in the $y$-direction. And we call that $\left(v_{x}, v_{y}\right)$ to be the velocity vector.
Moreover, the distance traveled from time $a$ to $b$ is

$$
\int_{a}^{b} v(t) d t=\int_{b}^{a} \sqrt{(d x / d t)^{2}+(d y / d t)^{2}} d t
$$

### 18.2 Polar Coordinate

Polar coordinates is the coordinates determined by specifying the distance of the point to origin and the angle measured counterclockwise from positive $x$-axis to the line joining the line connecting the point and the origin.

### 18.2.1 Relation between Cartesian and Polar

Cartesian to Polar:

$$
\left.(x, y) \rightarrow\left(r=\sqrt{x^{2}+y^{2}}, \theta\right) \text { (Here we have that } \tan \theta=y / x\right)
$$

Polar to Cartesian:

$$
(r, \theta) \rightarrow(x=r \cos \theta, y=r \sin \theta)
$$

### 18.2.2 Slope, Arc length and Area in Polar Coordinates

By the relation $x=r \cos \theta, y=r \sin \theta$, given a curve $r=f(\theta)$, we have that $x=$ $f(\theta) \cos \theta, y=f(\theta) \sin \theta$, and thus are parametrized equations of parameter $\theta$. Therefore we have that the slope of to be

$$
\frac{d y}{d x}=\frac{d y / d \theta}{d x / d \theta}
$$

The arc length from angle $a$ to $b$ is

$$
\int_{b}^{a} \sqrt{(d x / d \theta)^{2}+(d y / d \theta)^{2}} d \theta
$$

Moreover, due to the fact that the area of the sector is $1 / 2 r^{2} \theta$, we have that for a curve $r=f(\theta)$, with $f(\theta) \geq 0$, the area of the region enclosed is

$$
\frac{1}{2} \int_{a}^{b} f(\theta)^{2} d \theta
$$

## 19 Differential Equation

Differential equation is the equations that is in the form

$$
\frac{d y}{d x}=f(x, y)
$$

Note that the solution of the differential equations is usually a family of solution, and is usually not unique.

### 19.1 Slope Field

To approximate the solution numerically, we need the tool of the slope field. A slope field is the normal Cartesian coordinates for $(x, y)$ with the little slope defined at the point $(x, y)$ drawn at the point. Then we can approximately see the solutions from the slope field already!
Moreover,

- Slope field includes the information of ( $x, y, d y / d x$ ) at a point.
- Follow the slope field, we can recover the graph approximately.
- In the slope field, we can clearly see that there are several equilibrium lines, they can be categorized as stable/unstable, see definition in 11.5.


### 19.2 Euler's method.

- Approximately, we can approximate the original curve's data using dy/dx at that point.
- More specifically,

$$
y\left(x_{1}\right) \approx y\left(x_{0}\right)+\left.\frac{d y}{d x}\right|_{x_{0}}\left(x_{1}-x_{0}\right)
$$

- Looking at the grid defined by the lines where $\frac{d y}{d x}=0$, we can tells how the differential equation looks like.
- Beware, function of two variables might appears, be sure about its meaning.
- Euler's Method leads to an underestimate when the curve is concave up, just as it will lead to an overestimate when the curve is concave down.


### 19.3 Separation of variables

Given $d y / d x=g(x) f(y)$, we then will have that $1 / f(y) d y / d x=g(x)$, and thus

$$
\int 1 / f(y) d y=\int g(x) d x
$$

, then we can solve $y=h(x)$ which as the solution from here.
19.3.1 General solution to $d y / d x=k y$

Note that here we have a special case, the general solution to $d y / d x=k y$ is $y=B e^{k x}$ for any constant $B$.

### 19.4 Equilibrium

An equilibrium solution is constant for all values of the independent variable. The graph is a horizontal line.
An equilibrium is stable if a small change in the initial conditions gives a solution which tends toward the equilibrium as the independent variable tends to positive infinity.
An equilibrium is unstable if a small change in the initial conditions gives a solution curve which veers away from the equilibrium as the independent variable tends to positive infinity.

Example 19.1. $y(t)=C e^{t}$ has a stable equilibrium $y(t)=0$.
Example 19.2. $y(t)=C e^{-t}$ has a stable equilibrium $y(t)=0$.

### 19.5 How to write down a differential equation?

Follow the steps:

- Begin from the rate, i.e. $d y / d x$.
- Find out what is each single part that contributes to the rate at a moment, with sign.
Here a usual strategy will be to split the rate as two parts: how much is in and how much is out.
- Now write down formally.


## 20 L'Hopital's rule

LHopitals rule: If $f$ and $g$ are differentiable and (below $a$ can be $\pm \infty$ )
i) $f(a)=g(a)=0$ for finite $a$,

Or ii) $\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a} g(x)= \pm \infty$,
Or iii) $\lim _{x \rightarrow \infty} f(x)=\lim _{x \rightarrow \infty} g(x)=0$ then

$$
\lim _{x \rightarrow a} \frac{f(x)}{g(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{g^{\prime}(x)}
$$

### 20.1 Dominance

We say that $g$ dominates $f$ as $x \rightarrow \infty$ if $\lim _{x \rightarrow \infty} f(x) / g(x)=0$.

### 20.2 How to determine some bad limit?

There are several types of the limits that is "bad" which requires L'Hopital's rule to calculate: $0 / 0, \infty / \infty, \infty \cdot 0, \infty-\infty, 1^{\infty}, 0^{0}, \infty^{0}$. Although the first two cases we can use L'Hopital's rule to calculate, the others we cannot use it directly.
Read the book, and there are several things that we can consider.

- Consider adding the fractions.
- Consider taking log.
- Consider $1 / f(x)$ so that we can transform $\infty$ to ' $1 / 0$ ' or 0 to ' $1 / \infty$ '.


## 21 Improper integral

Formal definition of the improper integral I will let you read the book carefully, they are in the box. However, informally, there are two types of improper integral which we just interpret them as a limit.

- The first case is where we have the limit of the integration goes to infinity, i.e. $\lim _{b \rightarrow \infty} \int_{a}^{b} f(x) d x$.
- The integrand goes to infinity as $x \rightarrow a$.


### 21.1 Converges or diverges?

The basic question that one want to know about the improper integral is basically is it well defined?
This turns to ask if an improper integral converges or not.
There are four ways people ususally use to check this fact.

1. Check by definition, this means check the limit directly.
2. $p$-test.

|  | $\boldsymbol{p}<\mathbf{1}$ | $\boldsymbol{p}=\mathbf{1}$ | $\boldsymbol{p}>\mathbf{1}$ |
| :---: | :---: | :---: | :---: |
| Type I : $\int_{a}^{\infty} \frac{1}{x^{p}} d x$ | diverges | $=\left.\ln x\right\|_{a} ^{\infty} \Rightarrow$ diverges | converges |
| Type II : $\int_{0}^{a} \frac{1}{x^{p}} d x$ | converges | $=\left.\ln x\right\|_{0} ^{a} \Rightarrow$ diverges | diverges |

3. 

$$
\int_{0}^{\infty} e^{-a x} d x
$$

converges for $a>0$.
4. Comparison test.

If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty]$ then,

- If $\int_{a}^{\infty} f(x) d x$ converges then so does $\int_{a}^{\infty} g(x) d x$.
- If $\int_{a}^{\infty} g(x) d x$ diverges then so does $\int_{a}^{\infty} f(x) d x$.

5. Limit Comparison theorem.

Limit Comparison Test. If $f(x)$ and $g(x)$ are both positive on the interval $[a, b)$ where $b$ could be a real number or infinity. and

$$
\lim _{x \rightarrow b} \frac{f(x)}{g(x)}=C
$$

such that $0<C<\infty$ then the improper integrals $\int_{a}^{b} f(x) d x$ and $\int_{a}^{b} g(x) d x$ are either both convergent or both divergent.

## 22 Sequences and Series

### 22.1 Sequence

Definition 22.1. A sequence is an enumerated collection of objects in which repetitions are allowed. We denote the sequence $a_{1}, a_{2}, \ldots, a_{n} \ldots$ as $\left(a_{n}\right)$.

Note that for sequence, there are two things that we will usually concern. The first one is the convergence of the sequence itself, which is defined as

Definition 22.2. The sequence $s_{1}, s_{2}, s_{3}, \ldots, s_{n}, \ldots$ has a limit $L$, written $\lim _{n \rightarrow \infty} s_{n}=$ $L$, if $s_{n}$ is as close to $L$ as we please whenever $n$ is sufficiently large. If a limit, $L$, exists, we say the sequence converges to its limit $L$. If no limit exists, we say the sequence diverges.

If we think about the situation more clearly, we will see that, in the definition it actually encodes an information: A convergent sequence is bounded. Is the converse true here? Unfortunately, it is not true that a bounded sequence is convergent. However, by the following theorem, we knows when will the bounded sequence becomes convergent.

Theorem 22.1. If a sequence $s_{n}$ is bounded and monotone, it converges.

### 22.2 Series

There is another thing that we will usually concern.
Consider the partial sum of sequence $s_{n}$, i.e., $S_{n}=\sum_{i=1}^{n} s_{i}$, then we will see that the partial sum forms a sequence as well. Therefore there is a natural question to ask here, when will the sequence $S_{n}$ of partial sums converges?

Definition 22.3. The associated series for a sequence $\left(a_{n}\right)$ is defined as the ordered $\operatorname{sum} \sum_{n=1}^{\infty} a_{n}$.

Definition 22.4. If the sequence $S_{n}$ of partial sums converges to $S$, so $\lim _{n \rightarrow \infty} S_{n}=S$, then we say the series $\sum_{n=1}^{\infty} a_{n}$ converges and that its sum is $S$. We write $\sum_{n=1}^{\infty} a_{n}=S$. If $\lim _{n \rightarrow \infty} S_{n}$ does not exist, we say that the series diverges.

There are several properties for convergent series, which is super useful, summarized as below.

Theorem 22.2. Convergence Properties of Series

1. If $\sum_{n=1}^{\infty} a_{n}$ and $\sum_{n=1}^{\infty} b_{n}$ converge and if $k$ is a constant, then
$\sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)$ converges to $\sum_{n=1}^{\infty} a_{n}+\sum_{n=1}^{\infty} b_{n}$.
$\sum_{n=1}^{\infty} k a_{n}$ converges to $k \sum_{n=1}^{\infty} a_{n}$
2. Changing a finite number of terms in a series does not change whether or not it converges,

Note for Calculus
3. If $\lim _{n \rightarrow \infty} a_{n} \neq 0$ or $\lim _{n \rightarrow \infty} a_{n}$ does not exist, then $\sum_{n=1}^{\infty} a_{n}$ diverges. (Remember this!)
4. If $\sum_{n=1}^{\infty} a_{n}$ diverges, then $\sum_{n=1}^{\infty} a_{n}$ diverges if $k \neq 0$.

Moreover, there are several test to determine if a series is convergent, detailed discussion about those is in class.

## 1. The Integral Test

Suppose $a_{n}=f(n)$, where $f(x)$ is decreasing and positive.
a. If $\int_{1}^{\infty} f(x) d x$ converges, then $\sum_{n=1}^{\infty} a_{n}$ an converges.
b. If $\int_{1}^{\infty} f(x) d x$ diverges, then $\sum_{n=1}^{\infty} a_{n}$ an diverges.
2. p-test

The $p$-series $\sum_{n=1}^{\infty} 1 / n^{p}$ converges if $p>1$ and diverges if $p \leq 1$.
3. Comparison Test

Suppose $0 \leq a_{n} \leq b_{n}$ for all $n$ beyond a certain value.
a. If $\sum b_{n}$ converges, then $\sum a_{n}$ converges.
b. If $\sum a_{n}$ diverges, then $\sum b_{n}$ diverges.

## 4. Limit Comparison Test

Suppose $a_{n}>0$ and $b_{n}>0$ for all $n$. If $\lim _{n \rightarrow \infty} a_{n} / b_{n}=c$ where $c>0$, then the two series $\sum a_{n}$ and $\sum b_{n}$ either both converge or both diverge.
5. Convergence of Absolute Values Implies Convergence If $\sum\left|a_{n}\right|$ converges, then so does $\sum a_{n}$.
6. The Ratio Test For a series $\sum a_{n}$, suppose the sequence of ratios $\left|a_{n+1}\right| /\left|a_{n}\right|$ has a limit: $\lim _{n \rightarrow \infty}\left|a_{n+1}\right| /\left|a_{n}\right|=L$, then

- If $L<1$, then $\sum a_{n}$ converges.
- If $L>1$, or if $L$ is infinite, then $\sum a_{n}$ diverges.
- If $L=1$, the test does not tell us anything about the convergence of $\sum a_{n}$ (Important!).

7. Alternating Series Test A series of the form $\sum_{n=1}^{\infty}(-1)^{n-1} a_{n}=a_{1}-a_{2}+a_{3}-$ $a_{4}+\ldots+(-1)^{n-1} a_{n}+\ldots$ converges if $0<a_{n+1}<a_{n}$ for all $n$ and $\lim _{n \rightarrow \infty} a_{n}=0$. Moreover, let $S=\lim _{n \rightarrow \infty} S_{n}$, then we will have $\left|S-S_{n}\right|<a_{n+1}$.

Notably, We say that the series $\sum a_{n}$ is

- absolutely convergent if $\sum a_{n}$ and $\sum\left|a_{n}\right|$ both converge.
- conditionally convergent if $\sum a_{n}$ converges but $\sum\left|a_{n}\right|$ diverges.

Test we consider for proving convergence:

1. The integral test
2. p-test
3. Comparison test
4. Limit comparison test
5. Check the absolute convergence of the series
6. Ratio Test
7. Alternating Series Test

Test we consider for proving divergence:

1. The integral test
2. p-test
3. Comparison test
4. Limit comparison test
5. Ratio Test
6. Check $\lim _{n \rightarrow \infty} \neq 0$ or $\lim _{n \rightarrow \infty}$ does not exist.

### 22.3 Geometric Series

There is a special series that we learn about, which is the Geometric Series, notice that the formula on the right hand side is what we called closed form. A finite geometric series has the form

$$
a+a x+a x^{2}+\cdots+a x^{n 2}+a x^{n 1}=\frac{a\left(1-x^{n}\right)}{1-x} \text { For } x \neq 1
$$

An infinite geometric series has the form

$$
a+a x+a x^{2}+\cdots+a x^{n 2}+a x^{n 1}+a x^{n}+\cdots=\frac{a}{1-x} \text { For }|x|<1
$$

## 23 Power Series and Taylor Series

### 23.1 Power Series

Definition 23.1. A power series about $x=a$ is a sum of constants times powers of $(x-a): C_{0}+C_{1}(x-a)+C_{2}(x-a)^{2}+\ldots+C_{n}(x-a)^{n}+\ldots=\sum_{n=0}^{\infty} C_{n}(x-a)^{n}$.

If we fix a specific value of $x$, we can just consider plugging x with the value we have, and convergence here makes sense.

Definition 23.2. For a fixed value of $x$, if this sequence of partial sums converges to a limit $L$, that is, if $\lim _{n \rightarrow \infty} S_{n}(x)=L$, then we say that the power series converges to $L$ for this value of $x$.

Based on the discussion we will see that, The interval of convergence for a power series is usually centered at a point $x=a$, and extends the same length to both side, thus we denote this length as radius of convergence.

Moreover, each power series falls into one of the three following cases, characterized by its radius of convergence, $R$.

- The series converges only for $x=a$; the radius of convergence is defined to be $R=0$.
- The series converges for all values of $x$; the radius of convergence is defined to be $R=\infty$.
- There is a positive number $R$, called the radius of convergence, such that the series converges for $|x-a|<R$ and diverges for $|x-a|>R$.

The interval of convergence is the interval between $a-R$ and $a+R$, including any endpoint where the series converges.

Then there is a question arises, how to find this radius of convergence then?

This question can be determined by considering using ratio test on the series, assuming $x \neq a$. The details are included in Chapter 9.5 in the book.

### 23.2 Taylor Polynomial and Taylor Series

### 23.2.1 Taylor Polynomial

If we try to approximate the function locally using a polynomial, there is one thing we want to acquire, i.e. we want the polynomial $P(x)$ with the property that $P^{(n)}(a)=$ $f^{(n)}(a)$ if we approximate the function at the point $x=a$. Considering merely the situation about $x=0$, recall what we did in the class, we will have the following.

Taylor Polynomial of Degree $n$ Approximating $f(x)$ for $x$ near 0 is

$$
f(x) \approx P_{n}(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\frac{f^{\prime \prime \prime}(0)}{3!} x^{3}+\frac{f^{(4)}(0)}{4!} x^{4}+\ldots+\frac{f^{(n)}(0)}{n!} x^{n}
$$

We call $P_{n}(x)$ the Taylor polynomial of degree $n$ centered at $x=0$, or the Taylor poly nomial about $x=0$.

More generally, Taylor Polynomial of Degree $n$ Approximating $f(x)$ for $x$ near $a$ is
$f(x) \approx P_{n}(x)=f(a)+f^{\prime}(a)(x-a)+\frac{f^{\prime \prime}(a)}{2!}(x-a)^{2}+\frac{f^{\prime \prime \prime}(a)}{3!}(x-a)^{3}+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^{n}$
We call $P_{n}(x)$ the Taylor polynomial of degree $n$ centered at $x=a$, or the Taylor poly nomial about $x=a$.

Notice that Taylor Polynomial of Degree $n$ Approximating $f(x)$ for $x$ near $a$ will have the property that $P_{n}^{(m)}(a)=f^{(m)}(a)$ for $0 \leq m \leq n$.

### 23.2.2 Taylor Series

Notice that in the Taylor polynomial, if we let $n$ here goes to infinity, we will get a series $P(x)$ with $P^{(m)}(a)=f^{(m)}(a)$ for $0 \leq m<\infty$ and thus we will expect that the series gives a good approximation about $f(x)$ around $a$, and actually when it converges, it is exactly the value you will get in $f(x)$, and this is called the Taylor Series.
Taylor Series for $f(x)$ about $x=0$ is

$$
f(x)=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(0)}{2!} x^{2}+\frac{f^{\prime \prime \prime}(0)}{3!} x^{3}+\frac{f^{(4)}(0)}{4!} x^{4}+\ldots+\frac{f^{(n)}(0)}{n!} x^{n}+\ldots
$$

We call $P_{n}(x)$ the Taylor polynomial of degree $n$ centered at $x=0$, or the Taylor poly nomial about $x=0$.

More generally, Taylor Series for $f(x)$ about $x=a$ is
$f(x)=f(a)+f^{\prime}(a)(x-a)+\frac{f^{\prime \prime}(a)}{2!}(x-a)^{2}+\frac{f^{\prime \prime \prime}(a)}{3!}(x-a)^{3}+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^{n}+\ldots$
We call $P_{n}(x)$ the Taylor polynomial of degree $n$ centered at $x=a$, or the Taylor poly nomial about $x=a$.

Moreover, there are several important cases that we consider, each of them is an Taylor expansion of a function about $x=0$ :

$$
e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\frac{x^{5}}{5!}+\frac{x^{6}}{6!}+\frac{x^{7}}{7!}+\frac{x^{8}}{8!}+\cdots \text { converges for all } x
$$

$$
\sin (x)=\sum_{n=0}^{\infty} \frac{x^{2 n+1}}{(2 n+1)!} \cdot(-1)^{n}=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots \text { converges for all } x
$$

.

$$
\cos (x)=\sum_{n=0}^{\infty} \frac{x^{2 n}}{(2 n)!} \cdot(-1)^{n}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\ldots \text { converges for all } x
$$

.

$$
\begin{gathered}
(1+x)^{p}=\sum_{k=0}^{\infty}\binom{p}{k} x^{k}=\sum_{k=0}^{\infty} \frac{p!}{k!(p-k)!} x^{k}= \\
1+p x+\frac{p(p-1)}{2!} x^{2}+\frac{p(p-1)(p-2)}{3!} x^{3}+\cdots \text { converges for }-1<x<1 .
\end{gathered}
$$

$$
\ln (1+x)=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{n+1}}{n+1}=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+\cdots
$$

Moreover, we can definitely find Taylor Series based on the existing series using four methods:

- Substitude

Example: Taylor Series about $x=0$ for $f(x)=e^{-x^{2}}$

- Differentiate

Example: Taylor Series about $x=0$ for $f(x)=\frac{1}{(1-x)^{2}}$

- Integrate

Example: Taylor Series about $x=0$ for $f(x)=\arctan x$ (Hint: What is $\frac{d}{d x}(\arctan x)$ ?)

- Multiply

Example: Taylor Series about $x=0$ for $f(x)=x^{2} \sin x$
Example: Taylor Series about $x=0$ for $f(x)=\sin x \cos x$
Example: Taylor Series about $x=0$ for $f(x)=e^{\sin x}$

