MIRROR SYMMETRIC CIRCULAR PLANAR ELECTRICAL NETWORKS AND TYPE B ELECTRICAL LIE THEORY

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Abstract. We introduce **mirror symmetric circular planar electrical networks** as the mirror symmetric subset of circular planar electrical networks studied by Curtis-Ingerman-Morrow [CIM] and Colin de Verdière-Gitler-Vertigan [dVGV]. These mirror symmetric networks can be viewed as type B generalization of circular planar electrical networks. We show that most of the properties of circular planar electrical networks are well inherited by these mirror symmetric electrical networks. Inspired by Lam [Lam], the space of mirror symmetric circular planar electrical networks can be compactified using **mirror symmetric cactus networks**, which admit a stratification indexed by mirror symmetric matchings. The partial order on the mirror symmetric matchings emerging from mirror symmetric electrical networks is dual to a subposet of affine Bruhat order of type C. We conjecture that this partial order is the closure partial order of the stratification of mirror symmetric cactus networks.

1. Introduction

A **circular planar electrical network** Γ is an undirected weighted planar graph which is bounded inside a disk. The weights can be thought as the conductances of electrical networks. The vertices on the boundary are called boundary vertices, say there are n of them. When voltages are put on the boundary veritces, there will be current flowing in the edges. This transformation

$$\Lambda(\Gamma) : \mathbb{R}^{|n|} \longrightarrow \mathbb{R}^{|n|}$$

from voltages on the boundary vertices to current flowing in or out of the boundary vertices is linear, and called the **response matrix** of Γ. If two electrical networks are **electrically-equivalent**, then their response matrices are the same. Curtis-Ingerman-Morrow [CIM] and Colin de Verdière-Gitler-Vertigan [dVGV] classified the response matrices of circular planar electrical networks, which form a space that can be decomposed into disjoint union of $\mathbb{R}^{d_i}_{>0}$.

The study of circular planar electical networks can be seen as of type A, since generators of such networks, **adjoining a boundary spike** and **adjoining a boundary edge**, are viewed by Lam-Pylyavskyy as one-parameter subgroups of the **electrical Lie group of type** A [LP].

In [KW], Kenyon and Wilson studied **grove measurements** $L_\sigma(\Gamma)$ of Γ, which is determined by all spanning subforrest given that the roots on the boundary disk of each subtree follow a fixed non-crossing partition σ. They also connected grove measurements and response matrices.

In [Lam], Lam viewed the map $\Gamma \longrightarrow \mathcal{L}(\Gamma)$ as projective coordinates of circular planar electrical networks, where $\mathcal{L}(\Gamma) := (L_\sigma(\Gamma))_\sigma$ is in the projective space $\mathbb{P}^{\mathcal{NC}_n}$ indexed by non-crossing partitions. The Hausdorff closure E_n of the image of this map can be seen as the compactification of the space of circular planar electrical networks. The preimage of E_n is the space of **cactus networks**, which are obtained by contracting some of edges and identifying the corresponding boundary vertices. A cactus network can also be seen a union of circular networks, whose shape looks like a cactus. He showed that E_n admits a cell decomposition

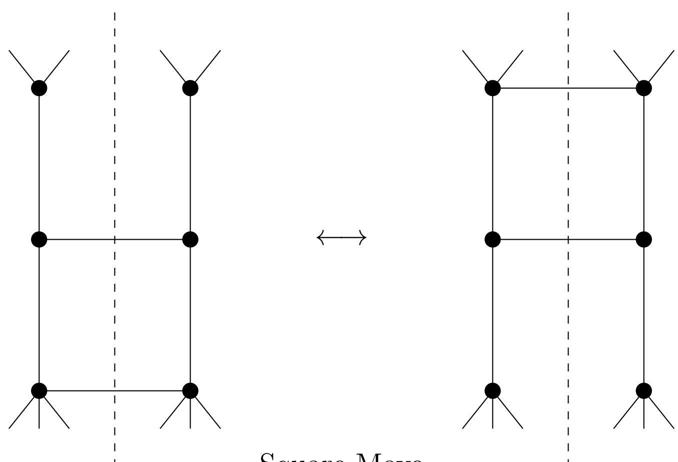
$$E_n = \bigsqcup_{\tau \subset \mathcal{P}_n} E_\tau$$

where $\mathcal{P}_n$ is the set of matchings of $\{1, 2, \ldots, 2n\}$. There is a graded poset structure on $\mathcal{P}_n$ which is dual to an induced subposet of the affine Bruhat order of type A. Lam [Lam] showed that this is exactly the closure partial order of the above cell decomposition, that is,

$$\overline{E}_\tau = \bigsqcup_{\tau' \leq \tau} E_{\tau'}$$

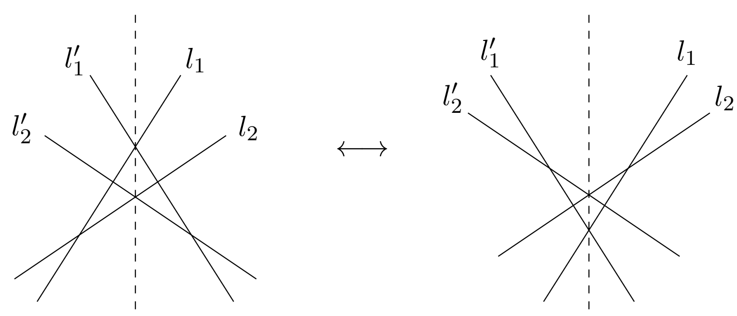
The main objective of this paper is to develop a similar theory in the case of Type B. There the combinatorial object is a **mirror symmetric circular planar electrical network**, which is a circular planar electrical networks on $2n$ boundary vertices, such that the network is mirror symmetric with respect to some mirror line.

We introduce a new electrically-equivalent transformation, the **square transformation** or **square move** as in the following picture. The square move cannot be decomposed into symmetric star-triangle transformations. Thus it can be seen as one of fundamental transformations in electrical equivalence for mirror symmetric circular planar electrical networks.



Square Move

On the level of **medial graphs**, the square move translates into the **crossing interchanging transformation**. Later we will see that crossing interchanging transformation can be seen as the type B version of Yang–Baxter transformation.



Crossing Interchanging Transformation

We show that the response matrices are preserved under (1) symmetric reductions: symmetric leaf and loop removal, symmetric series and parallel transformation, and double series and parallel transformation, and (2) symmetric transformations: symmetric star-triangle transformation and square transformation across the mirror line. Furthermore, every two **critical** mirror symmetric circular planar electrical networks with the same response matrix can be obtained from each other by symmetric transformations. The space of response matrices of mirror symmetric networks also admits a decomposition into cells isomorphic to $\mathbb{R}^{c_i}_{>0}$.

The generators of these networks, **adjoining two boundary spikes mirror symmetrically**, **adjoining two boundary edges mirror symmetrically**, and **adjoining a boundary edge across the mirror line** can be viewed as the one parameter subsemigroup of **electrical Lie group of type** B introduced in [LP], whose explicit structure is given in [Su]. Thus the study of these networks is a type B analogue of circular planar electrical networks. In particular, the operation of adjoining a boundary edge across the mirror line corresponds to the double edge in the Dynkin diagram of type B.

The notion of grove measurements is well inherited from ordinary circular networks. We view the map $\Gamma \longrightarrow \mathcal{L}(\Gamma)$ as projective coordinates of mirror circular planar electrical networks, where $\mathcal{L}(\Gamma)$ is in the subspace $\mathbb{P}^{\mathcal{SNC}_n}$ of $\mathbb{P}^{\mathcal{NC}_{2n}}$, such that if two partition σ and σ' are mirror symmetric, then $L_\sigma(\Gamma) = L_{\sigma'}(\Gamma)$. Similarly, the Hausdorff closure ME_n of the image of this map is the compactification of the space of mirror symmetric circular planar electrical networks. The preimage of ME_n are exactly **mirror symmetric cactus networks**. We prove that ME_n has a cell decomposition

$$ME_n = \bigsqcup_{\tau \subset \mathcal{MP}_n} ME_\tau$$

where $\mathcal{MP}_n$ is the mirror symmetric matchings $\{-2n+1, \ldots, 0, \ldots, 2n-1, 2n\}$. $\mathcal{MP}_n$ is a graded subposet of $\mathcal{P}_{2n}$. We show that $\mathcal{MP}_n$ is an induced subposet of the dual affine Bruhat order of type C, and conjecture that this partial order is the closure partial order of the cell decomposition for ME_n.

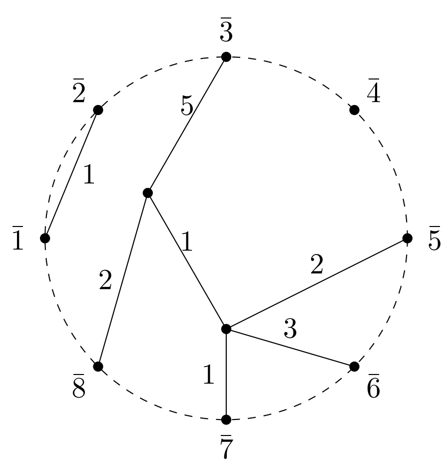
2. Circular Planar Electrical Network and Type A electrical Lie Theory

2.1. Circular Plannar Electrical Networks and Response Matrices.

This section is mainly attributed to [CIM] and [dVGV].

Definition 2.1. *Let $G = (V, V_B, E)$ be a planar undirected graph with the vertex set V, the boundary vertex set $V_B \subseteq V$ and the edge set E. Assume that V_B is labeled and nonempty. A* ***circular planar electrical network*** *$\Gamma = (G, \gamma)$ is a graph G together with a map $\gamma : E \to \mathbb{R}^{|E|}_{>0}$, called the* ***conductances*** *of G.*

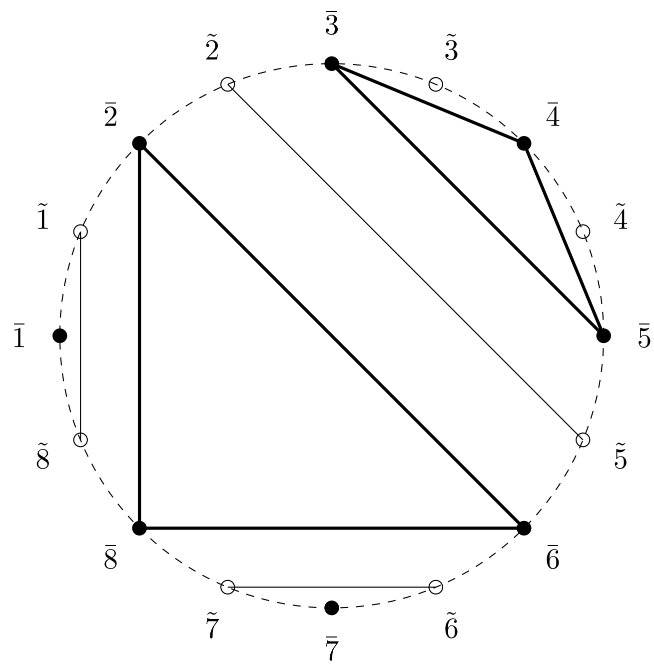
Example 2.2. The following is a circular planar electrical network with boundary vertices set $V_B = \{\bar{1}, \bar{2}, \ldots, \bar{8}\}$.



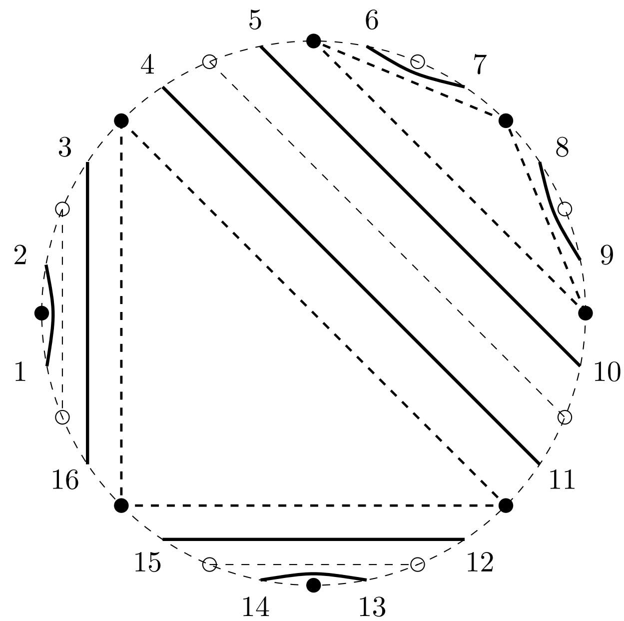
If we put voltages on each of the boundary vertices of Γ, by Ohm's Law and Kirchhoff's Law, electrical current will flow along edges. This electrical property is captured by the **response matrix** $\Lambda(\Gamma)$. We can interpret the response matrix $\Lambda(\Gamma)$ as a linear transformation in the following way: If one puts voltages $p = \{p(v_i)\}$ on each of the boundary vertices (think of p as column vector), then $\Lambda(\Gamma).p$ will be the current flowing into or out of each boundary vertex resulting from p. We say that two networks Γ and Γ' are **electrically-equivalent** if $\Lambda(\Gamma) = \Lambda(\Gamma')$.

2.2. **Groves of Circular Planar Electrical Networks.** The materials of this section are mainly from [KW] and [LP].

A **grove** of a circular planar electrical network Γ is a spanning subforrest F that uses all vertices of Γ, and every connected component F_i of F has to contain some boundary vertices. The **boundary partition** $\sigma(F)$ encodes boundary vertices that are in the same connected component. Note that $\sigma(F)$ is a **non-crossing partition**. Let $\mathcal{NC}_n$ denote the set of non-crossing partitions of $[\bar{n}]$. Each non-crossing partition has a dual non-crossing partition on $[\tilde{n}]$, where $\tilde{i}$ is placed between $\bar{i}$ and $\overline{i+1}$, and the numbers are modulo n. For example the partition $\{\{\bar{1}\}, \{\bar{2}, \bar{6}, \bar{8}\}, \{\bar{3}, \bar{4}, \bar{5}\}, \{\bar{7}\}\}$ is dual to $\{\{\tilde{1}, \tilde{8}\}, \{\tilde{2}, \tilde{5}\}, \{\tilde{3}\}, \{\tilde{4}\}, \{\tilde{6}, \tilde{7}\}\}$.



The set of non-crossing partition σ is in bijiection to the set of non-crossing matching $\tau(\sigma)$ of $[2n]$. We put $2i-1$ between the labeling $\widetilde{i-1}$ and $\bar{i}$, and $2i$ between $\tilde{i}$ and $\bar{i}$. Then the non-crossing partition $\sigma = \{\{\bar{1}\}, \{\bar{2}, \bar{6}, \bar{8}\}, \{\bar{3}, \bar{4}, \bar{5}\}, \{\bar{7}\}\}$ is corresponding to a non-crossing matching $\tau(\sigma) = \{(1,2), (3,16), (4,11), (5,10), (6,7), (8,9), (12,15), (13,14)\}$ as in the following picture:

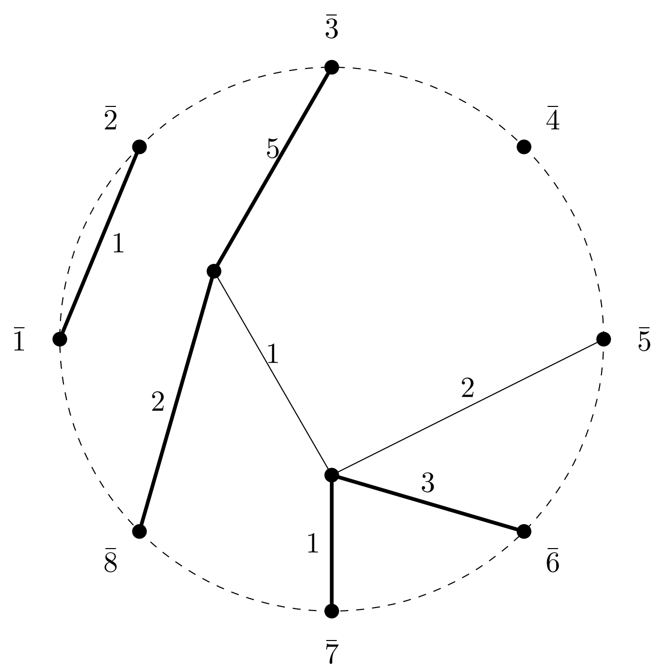


Theorem 2.3. [Lam] *The set of non-crossing partitions $\mathcal{NC}_n$ is in bijection with the set of non-crossing matchings on $[2n]$. Thus the number of non-crossing partitions is equal to the catalan number $\dfrac{1}{n+1}\dbinom{2n}{n}$.*

For a non-crossing partition σ, we can define

$$L_\sigma(\Gamma) = \sum_{\{F|\sigma(F)=\sigma\}} \mathrm{wt}(F),$$

where the summation is over all groves F such that the boundary partition $\sigma(F)$ is equal to the non-crossing partition σ, and $\mathrm{wt}(F)$ is the product of all weights of edges in F.



Grove F with boundary partition $\sigma(F) = \{\{\bar{1}, \bar{2}\}, \{\bar{3}, \bar{8}\}, \{\bar{4}\}, \{\bar{5}\}, \{\bar{6}, \bar{7}\}\}$, $\mathrm{wt}(F)$=30

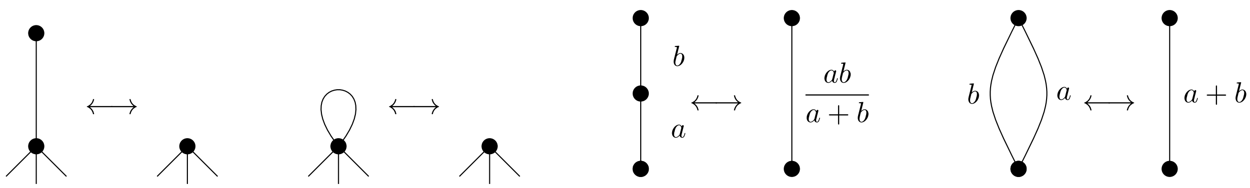
Let $\sigma_{i,j}$ be the partition in which each part contains a single number except the part $\{i, j\}$. Let $\sigma_{\text{singleton}}$ be the partition with each part being a singleton. The following theorem can be found in [KW].

Theorem 2.4. *We have the following identity:*

$$\Lambda_{ij}(\Gamma) = -\frac{L_{\sigma_{i,j}}(\Gamma)}{L_{\sigma_{singleton}}(\Gamma)}.$$

2.3. **Electrically-Equivalent Reductions and Transformations Networks.** The following propositions can be found in [CIM] and [dVGV]. Recall that two circular planar electrical networks are electrically-equivalent if they have the same response matrix. In this subsection, we are exploring reductions and transformations which do not change response matrices.

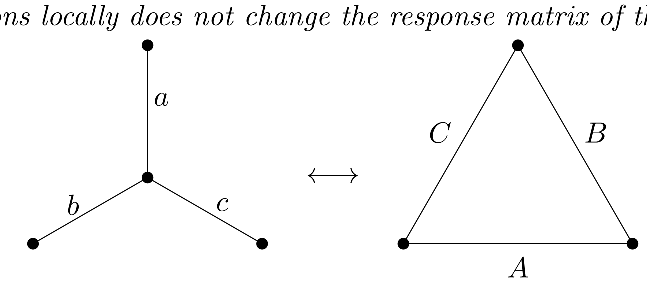
Proposition 2.5. *Removing interior vertices of degree 1, removing loops, and series and parallel transformation do not change the response matrix of an electrical network.*



Reductions

Note that the above operations reduce the number of resistors. We call these operations **reductions** of networks. We also have the following theorem due to [Ken].

Theorem 2.6 (Star-Triangle Transformation)**.** *In an electrical network, changing between the following two configurations locally does not change the response matrix of the network.*

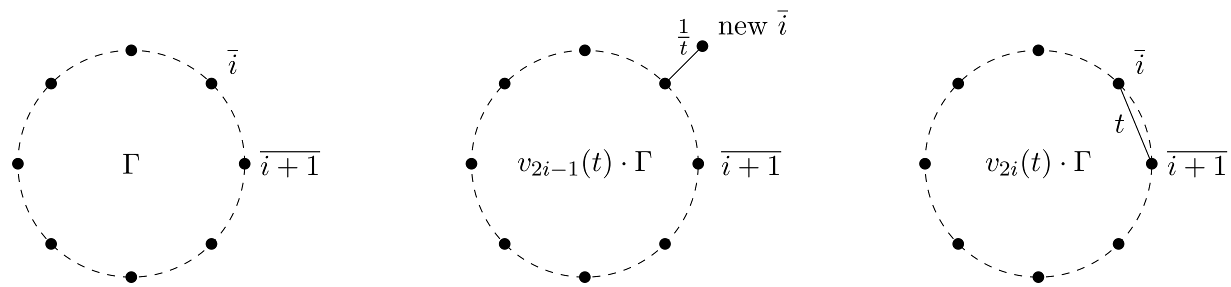


where a, b, c, and A, B, C are related by the following relations:

$$A = \frac{bc}{a+b+c}, \qquad B = \frac{ac}{a+b+c}, \qquad C = \frac{ab}{a+b+c},$$
$$a = \frac{AB+AC+BC}{A}, \quad b = \frac{AB+AC+BC}{B}, \quad c = \frac{AB+AC+BC}{C}.$$

2.4. **Generators of Circular Planar Electrical Networks and Electrical Lie Theory of Type** A**.** Curits-Ingerman-Morrow [CIM] and Colin de Verdière-Gitler-Vertigan [dVGV] studied the generators of the circular planar electrical networks with n boundary vertices (See Figure 2.1):

Adjoining a boundary spike: Define $v_{2i-1}(t) \cdot N$ to be the action on N by adding a vertex u into N, joining an edge of weight $1/t$ between this vertex and boundary vertex $\bar{i}$, and then treating u as the new boundary vertex $\bar{i}$, and old boundary vertex $\bar{i}$ as an interior vertex.



Adjoining a boundary spike Adjoining a boundary edge

FIGURE 2.1. Generators of Circular Planar Electrical Networks

Adjoining a boundary edge: Define $v_{2i}(t) \cdot N$ to be the action on N by adding an edge of weight t between boundary vertices $\bar{i}$ and $\overline{i+1}$.

These two operations can be seen as the generators of circular planar electrical networks. Next we introduce electrical Lie algebra of type A of even rank, denoted as $\mathfrak{e}_{A_{2n}}$ defined by Lam-Pylyavskyy. $\mathfrak{e}_{A_{2n}}$ is generated by $\{e_1, e_2, \ldots, e_{2n}\}$ under the relations:

$$[e_i, e_j] = 0 \quad \text{if } |i-j| \geq 2,$$
$$[e_i, [e_i, e_j]] = -2e_i \text{ if } |i-j| = 1.$$

Theorem 2.7 ([LP])**.** *$\mathfrak{e}_{A_{2n}}$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2n}$.*

Let Sp_{2n} be the electrical Lie group $E_{A_{2n}}$. Let $u_i(t) = \exp(te_i)$ for all i. Define the **nonnegative part** $(E_{A_{2n}})_{\geq 0}$ to be the Lie subsemigroup generated by all $u_i(t)$ for $t \geq 0$. The following theorem is due to [LP].

Theorem 2.8. *If $a, b, c > 0$, then the elements $u_i(a)$, $u_i(b)$, and $u_i(c)$ satisfy the relations:*

(1) $u_i(a)u_i(b) = u_i(a+b)$
(2) $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i-j| \geq 2$
(3) $u_i(a)u_j(b)u_i(c) = u_j\left(\frac{bc}{a+c+abc}\right)u_i(a+c+abc)u_j\left(\frac{ab}{a+c+abc}\right)$ *if* $|i-j| = 1$

Furthermore, these three relations are the only ones satisfied by $(E_{A_{2n}})_{\geq 0}$. Thus $(E_{A_{2n}})_{\geq 0}$ has a semigroup action on the set of response matrices of the circular planar electrical network Γ via $u_i(t) \cdot \Lambda(\Gamma) := \Lambda(v_i(t) \cdot \Gamma)$.

Remark 2.9. Relation (3) in Theorem 2.8 translates into to the star-triangle transformations on the boundary of electrical networks.

2.5. **Medial Graphs.** A circular planar electrical network Γ can be associated with a **medial graph** $\mathcal{G}(\Gamma)$. The construction of medial graphs go as follows: Say Γ has n boundary vertices $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. We put vertices $\{t_1, \ldots, t_{2n}\}$ on the boundary circle in the order $t_1 < \bar{1} < t_2 < t_3 < \bar{2} < t_4 < \ldots < t_{2n-1} < \bar{n} < t_{2n}$, and for each edge e of Γ, a vertex t_e of $\mathcal{G}(\Gamma)$ is placed in the middle of the edge e. Join t_e and $t_{e'}$ with an edge in $\mathcal{G}(\Gamma)$ if e and e' share a vertex in Γ and are incident to the same face. As for the boundary vertex t_{2i-1} or t_{2i}, join it with the "closest" vertex t_e where e is among the edges incident to $\bar{i}$. If $\bar{i}$ is an isolated vertex, then join t_{2i-1} with t_{2i}. Note that each interior vertex t_e of $\mathcal{G}(\Gamma)$ has degree 4, and each boundary vertex has degree 1. Note that $\mathcal{G}(\Gamma)$ only depends on the underlying graph of Γ.

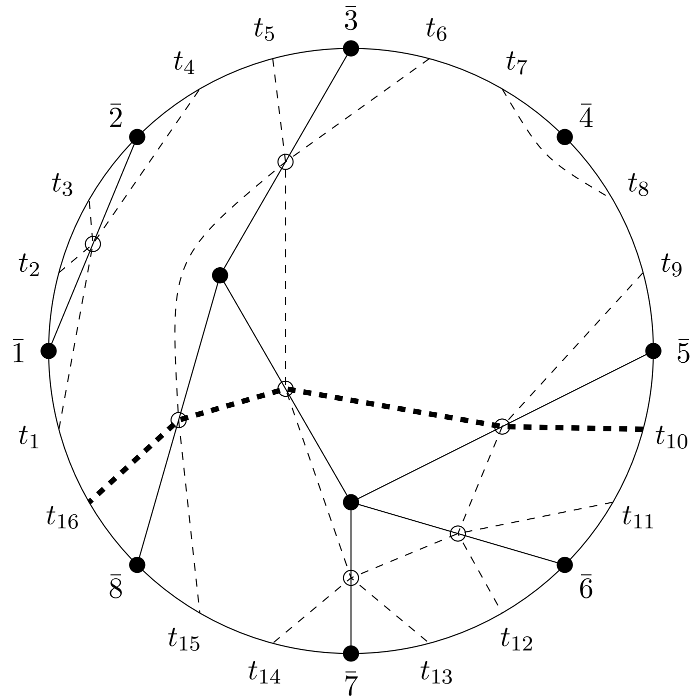


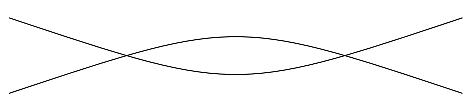
FIGURE 2.2. Medial Graph of a Circular Planar Electrical Network

A **strand** T of $\mathcal{G}(\Gamma)$ is a maximum sequence of connected edges such that it goes straight through any encountered 4-valent vertex. By definition, a strand either joins two boundary vertices t_i and t_j, or forms a loop. Hence, a medial graph contains a pairing on $[2n]$, which we call the **medial pairing** $\tau(\Gamma)$ (or $\tau(\mathcal{G}(\Gamma))$) of Γ. Medial pairings can also be regarded as matchings of $[2n]$.

The underlying graph of Γ can also be recovered from $\mathcal{G}(\Gamma)$ as follows: The edges of $\mathcal{G}(\Gamma)$ divides the interior of the boundary circle into regions. Color the regions into black and white so that the regions sharing an edge have different colors. Put a vertex in each of the white regions. By convention, the regions containing boundary vertices of $\mathcal{G}$ are colored white. When two white region share a vertex in $\mathcal{G}(\Gamma)$, join the corresponding vertices in Γ by an edge. The resulting graph is the underlying graph of Γ.

Example 2.10. Figure 2.2 is a network from Example 2.2 together with its medial graph in dashed lines. The medial pairing is $\{(1,3),(2,4),(5,13),(6,15),(7,8),(9,12),(10,16),(11,14)\}$.

A **lens** consists of two arcs intersecting with each other at two different vertices of the medial graph as in the following.



Lens

A medial graph $\mathcal{G}(\Gamma)$ is **lensless** if it does not contain lenses or loops, and every strand starts and ends on the boundary circle. Say Γ is **critical** or **reduced** if $\mathcal{G}(\Gamma)$ is lensless. Usually we talk about medial pairing only when Γ is critical. Let $c(\tau)$ be the number of crossings of the medial pairing τ. This number is independent of the choice of medial graph, as long as this medial graph is lensless.

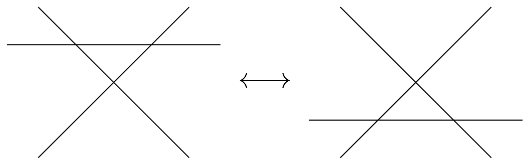


FIGURE 2.3. Yang-Baxter Move

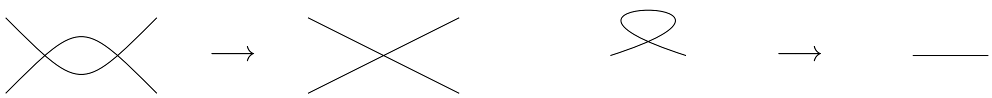


FIGURE 2.4. Lens and Loop Removal

Proposition 2.11. [CIM] *If two networks are related by relations in Proposition 2.5 then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by lens removal and loop removal in Figure 2.4. If Γ and Γ' are critical and related by star-triangle transformtion in Theorem 2.6, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are lensless and related by the Yang-Baxter move in Figure 2.3.*

Let $P = \{p_1, p_2, \ldots, p_k\}$, $Q = \{q_1, q_2, \ldots, q_k\}$ be two disjoint ordered subsets of $[2n]$. We say (P, Q) is a **circular pair** if $p_1, p_2 \ldots, p_k, q_k, \ldots, q_2, q_1$ is in circular order. The minor $\Lambda(P, Q)$ is said to be a circular minor of $\Lambda = \Lambda(\Gamma)$ if (P, Q) is a circular pair. The following theorem of circular planar electrical networks can be found in [CIM] and [dVGV].

Theorem 2.12.

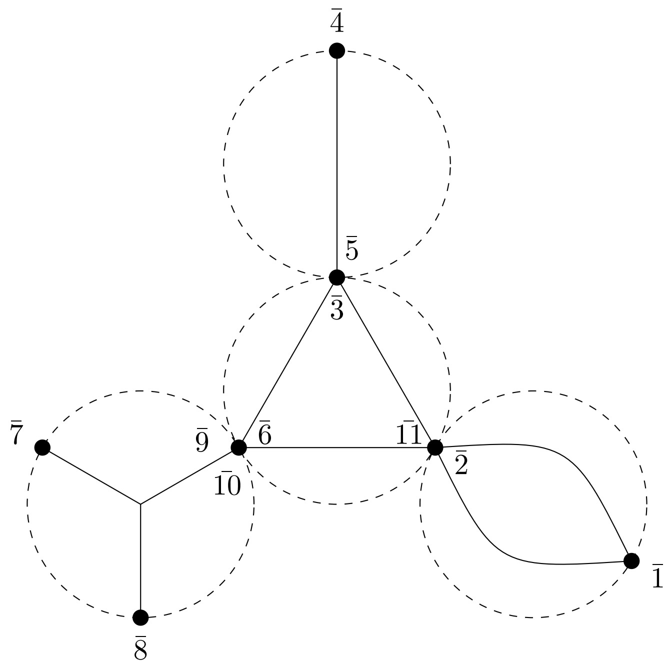
(1) *Every circular planar electrical network is electrically-equivalent to some critical network.*
(2) *The set of response matrices of all circular planar electrical networks consists of matrices M such that $(-1)^k M(P, Q) \geq 0$ for all k and all circular pairs (P, Q) such that $|P| = |Q| = k$.*
(3) *If two circular planar electrical networks Γ and Γ' have the same response matrix, then they can be connected by leaf removal, loop removal, series-parallel transformations (in Proposition 2.5), and star-triangle transformations (in Theorem 2.6). Furthermore, if both Γ and Γ' are critical, only the star-triangle transformations are required.*
(4) *The conductances of a critical circular planar electrical network can be recovered uniquely from its response matrix.*
(5) *The spaces E'_n of response matrices of circular planar electrical networks with n boundary vertices has a stratification $E'_n = \sqcup_i D_i$, where $D_i \cong \mathbb{R}_{>0}^{d_i}$ can be obtained as the response matrices of some fixed critical circular planar electrical networks with its conductances varying.*

3. Compactification of the Space of Circular Planar Electrical Networks

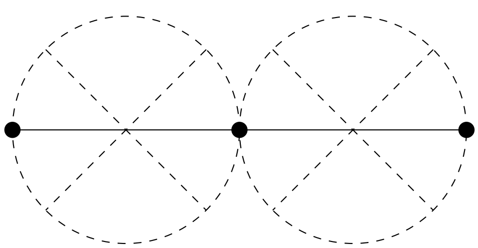
This section is attributed to [Lam]. From Subsection 2.2, we note that not every partition of $[2n]$ into pairs can be obtained as a medial pairing of some circular planar electrical network. We would like to generalize the notion of circular planar electrical networks to resolve this issue.

3.1. **Cactus Networks.** Let σ be a non-crossing partition on $[\bar{n}]$. Let S be a circle with vertices $\{\bar{1}, \ldots, \bar{n}\}$. A **hollow cactus** S_σ is obtained from S by gluing boundary vertices according to parts of σ. S_σ can be seen as a union of circles glued together by the identified point according to σ. The interior of a hollow cactus is the union of the open disk bounded by the circles. A **cactus** is the hollow cactus together with its interior. A **cactus network** is a planar electrical network embedded in a cactus, which can also be seen as a union of circular planar networks. σ is called the **shape** of a cactus network. One can think of a cactus network as a circular planar electrical network where the conductance between any two identified boundary points goes to infinity.

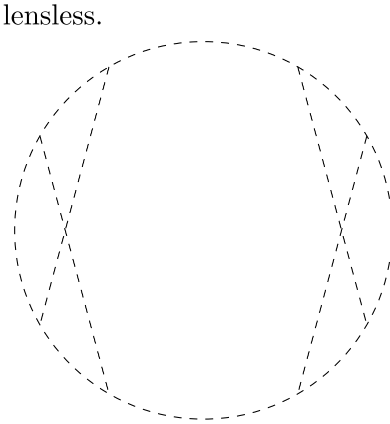
Example 3.1. The following is a cactus network with all conductances equal to 1 and shape $\sigma = \{\{\bar{1}\}, \{\bar{2}, \bar{11}\}, \{\bar{3}, \bar{5}\}, \{\bar{4}\}, \{\bar{6}, \bar{9}, \bar{10}\}, \{\bar{7}\}, \{\bar{8}\}\}$.



A medial graph can also be defined for a given cactus network under the assumption that every edge of the medial graph is contained in one circle of the hollow cactus. Sometimes it is more convenient to draw the medial graph of a cactus network in a disk instead of in a cactus. Similarly, we say a cactus network is **critical** if its medial graph is lensless.



Medial graph in cactus network



Medial graph in a disk

Note that for a cactus network Γ if we put the same voltage on the identified vertices, we can still measure the electrical current flowing into or out of the boundary vertices. Thus, the response matrix $\Lambda(\Gamma)$ can also be defined. We have the following propositions.

Proposition 3.2 ([Lam])**.**

(1) *Every cactus network is electrically-equivalent to a critical cactus network.*
(2) *If two critical cactus networks have the same response matrix, then they are related by doing a sequence star-triangle transformations.*
(3) *Any medial pairing can be obtained as by the medial graph of some cactus network.*

An easy enumeration shows that there are $\dfrac{(2n)!}{2^n n!}$ medial pairings for cactus networks.

3.2. **Grove Measurements as Projective Coordinates.** The definition of grove $L_\sigma(\Gamma)$ for a circular planar electrical network Γ and a non-crossing planar partition σ in Subsection 2.2 can be naturally extended to cactus networks. Let $\mathbb{P}^{\mathcal{NC}_n}$ be the projective space with homogeneous coordinates indexed by non-crossing partitions. The map

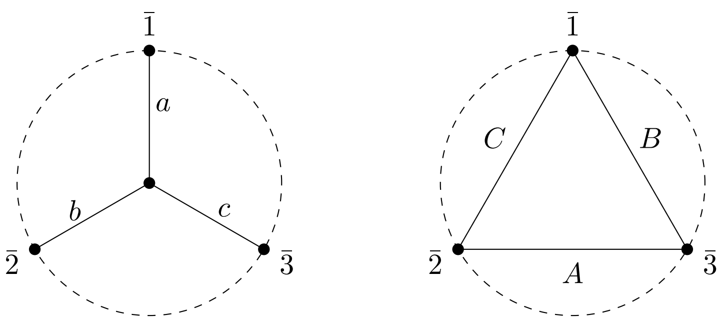
$$\Gamma \longmapsto (L_\sigma(\Gamma))_\sigma$$

sends a cactus network Γ to a point $\mathcal{L}(\Gamma) \in \mathbb{P}^{\mathcal{NC}_n}$.

Proposition 3.3 ([Lam]). *If Γ and Γ' are electrically-equivalent cactus networks, then $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.*

Remark 3.4. From Prosition 3.3, the map $\Gamma \longmapsto \mathcal{L}(\Gamma)$ can be lifted to a map from the electrically-equivalent classes of cactus networks to $\mathbb{P}^{\mathcal{NC}_n}$.

Example 3.5. The following are two electrically-equivalent circular planar electrical networks (a special case of cactus networks) Γ and Γ'. We would like to see that their images $\mathcal{L}(\Gamma)$ and $\mathcal{L}(\Gamma')$ are equal.



Γ Γ'

We see that

$L_{\{\bar1\},\{\bar2\},\{\bar3\}}(\Gamma) = a+b+c, \quad L_{\{\bar1,\bar2\},\{\bar3\}}(\Gamma) = ab, \quad L_{\{\bar1\},\{\bar2,\bar3\}}(\Gamma) = bc, \quad L_{\{\bar1,\bar3\},\{\bar2\}}(\Gamma) = ac,$
$L_{\{\bar1,\bar2,\bar3\}}(\Gamma) = abc.$
$L_{\{\bar1\},\{\bar2\},\{\bar3\}}(\Gamma') = 1, \quad L_{\{\bar1,\bar2\},\{\bar3\}}(\Gamma') = C, \quad L_{\{\bar1\},\{\bar2,\bar3\}}(\Gamma') = A, \quad L_{\{\bar1,\bar3\},\{\bar2\}}(\Gamma') = B,$
$L_{\{\bar1,\bar2,\bar3\}}(\Gamma') = AC + AB + BC$

By Theorem 2.6, we have $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$ in $\mathbb{P}^{\mathcal{NC}_n}$.

3.3. **Compactificaiton and Main Results for Cactus Networks.** By Theorem 2.4, and the fact that $L_{\sigma_{singleton}}(\Gamma) \neq 0$ for every circular planar electrical network Γ, we know that there is a one-to-one correspondence between the grove measurements $\mathcal{L}(\Gamma)$ of Γ and the response matrix of $\Lambda(\Gamma)$ of Γ. Thus, we can also define the space of circular planar electrical networks E'_n with n boundary vertices to be the space of grove measurements of such networks.

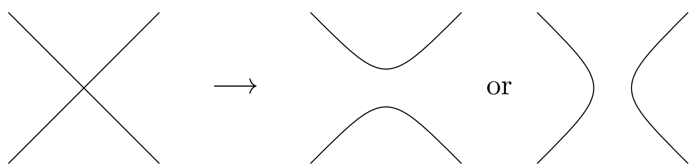
Define the closure in the Hausdorff topology $E_n = \overline{E'_n} \subset \mathbb{P}^{\mathcal{NC}_n}$ to be the compactification of the space of circular planar electrical networks. Let P_n be the set of medial pairings (or matchings) of $[2n]$. Note that two electrically-equivalent cactus networks will have the same medial pairing.

Theorem 3.6 ([Lam])**.**

(1) *The space E_n is exactly the set of grove measurements of cactus networks. A cactus network is determined by its grove measurement uniquely up to electrical equivalences.*
(2) *Let $E_\tau = \{\mathcal{L}(\Gamma) | \tau(\Gamma) = \tau\} \subset E_n$. Each stratum E_τ is parameterized by choosing a cactus network Γ such that $\tau(\Gamma) = \tau$ with edge weights being the parameters. So we have $E_\tau = \mathbb{R}_{>0}^{c(\tau)}$. Moreover,*

$$E_n = \bigsqcup_{\tau \in P_n} E_\tau.$$

3.4. **Matching Partial Order on P_n and Bruhat Order.** A partial order on P_n can be defined as follows: Let τ be a medial pairing and $\mathcal{G}$ be any lensless medial graph representing τ, denoted as $\tau(\mathcal{G}) = \tau$. Next uncrossing one crossing of $\mathcal{G}$ in either of the following two ways:



Suppose the resulting graph $\mathcal{G}'$ is also lensless. Let $\tau' = \tau(\mathcal{G}')$. Then we say $\tau' \lessdot \tau$ is a covering relation on P_n. The transitive closure of $\lessdot$ defines a partial order on P_n.

Remark 3.7. We can define another partial order on P_n, by uncross the crossings of $\mathcal{G}$, and use lens and loop removal in Figure 2.4 to reduce the resulting graph to a lensless graph $\mathcal{G}'$. Let $\tau' = \tau(\mathcal{G}')$. Then define $\tau' < \tau$ to be the partial order on P_n. Lam [Lam] showed that these two partial orders on P_n are the same.

Recall that $c(\tau)$ be the number of crossings in a lensless representative of τ. We have the following two theorems.

Theorem 3.8 (**??**)**.** *Let $\hat{P}_n$ be obtained from P_n by adding a unique minimal element $\hat{0}$ with rank -1. Then the poset $\hat{P}_n$ is Eulerian.*

Theorem 3.9 ([Lam])**.**

(1) *The poset $(P_n, \leq)$ is graded by the crossing number $c(\tau)$.*
(2) *This partial order on P_n is exactly the closure partial order for the stratafication $E_n = \bigsqcup_{\tau \in P_n} E_\tau$. In another word, $\overline{E_\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'}$.*

A **bounded affine permutation** of type (k, n) is a bijection $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying

(1) $i \leq f(i) \leq i + n$,
(2) $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$,
(3) $\sum_{i=1}^{n} (f(i) - i) = kn$.

We can associate $\tau \in P_n$ with an affine permutation g_τ as the following,

$$g_\tau(i) = \begin{cases} \tau(i) & \text{if } i < \tau(i), \\ \tau(i) + 2n & \text{if } i > \tau(i). \end{cases}$$

It is straightforward to check that g_τ is a bounded affine permutation of type $(n, 2n)$.

Theorem 3.10 ([Lam]). *We have $l(g_\tau) = 2(\binom{n}{2} - c(\tau))$. Then $\tau \longmapsto g_\tau$ gives an isomorphism between $(P_n, \leq)$ and an induced subposet of the dual Bruhat order of bounded affine permutations of type A. In other words, $g_\tau \leq g_{\tau'}$ in the affine Bruhat order if and only if $\tau' \leq \tau$ in $(P_n, \leq)$.*

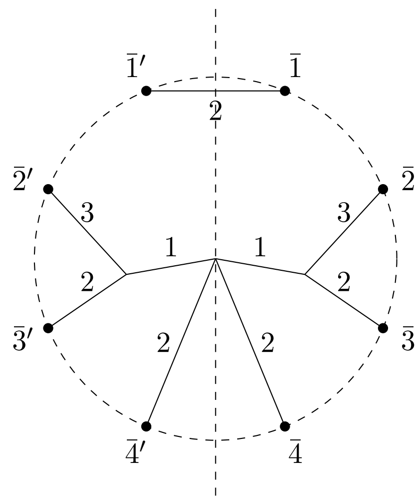
4. Mirror Symmetric Circular Planar Electrical Networks and Type B Electrical Lie Theory

4.1. Mirror Symmetric Circular Planar Electrical Networks.

Definition 4.1. *A **mirror symmetric circular planar electrical network** of rank n is a circular planar electrical network with $2n$ boundary vertices, which is also mirror symmetric to itself with respect to a mirror line that does not contain any boundary vertex.*

The boundary vertices are labeled as $\{\bar{1}, \bar{2}, \ldots, \bar{n}, \bar{1}', \bar{2}', \ldots, \bar{n}'\}$.

Example 4.2. The following is a mirror symmetric circular planar electrical network with boundary vertices $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{1}', \bar{2}', \bar{3}', \bar{4}'\}$.



The definitions of response matrices and grove measurements of mirror symmetric electrical networks are the same as the ones for circular planar electrical networks in Subsection 2.1 and 2.2. Let ME'_n be the sapce of response matrices of mirror symmetric circular planar electrical networks.

4.2. Electrically-Equivalent Transformations of Mirror Symmetric Networks.

In this section, we discuss under what reductions and transformations, the response matrix or the grove measurements will be unchanged. Note that after each transformation, the resulting electrical network should still be mirror symmetric. Clearly, by Theorem 2.5 we have the following proposition.

Proposition 4.3. *Mirror symmetrically performing the actions in Proposition 2.5, that is, removing interior vertices of degree 1, removing loops, and series and parallel transformations, plus double series and parallel transformation (See Figure 4.1) will do not change the response matrix of a mirror symmetric electrical network.*

We call these operations **symmetric reductions** of a mirror symmetric network. As for non-reduction transformations, we have:

Theorem 4.4.

(1) *Mirror symmetrically performing star-triangle transformations will not change the response matrix or the grove measurement of the mirror symmetric network.*

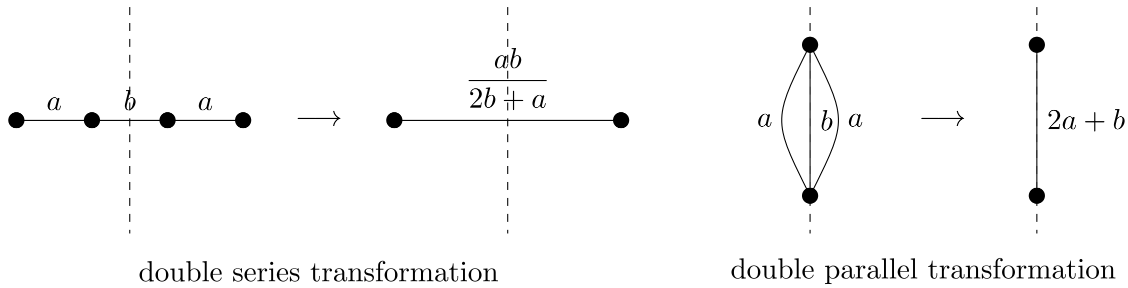
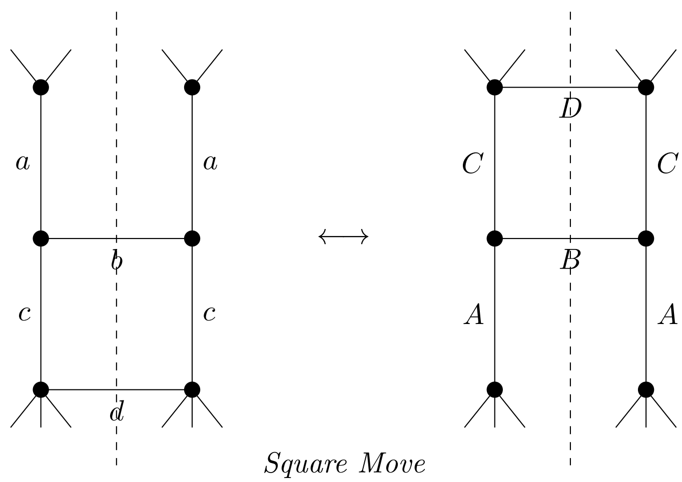


Figure 4.1. Double Series and Parallel Transformation

(2) *(**Square transformation** or **square move**) locally changing between the following two configurations will not change the response matrix or the grove measurement of the mirror symmetric network.*



Square Move

where the weights are given by the rational transformation $\phi : (a, b, c, d) \longrightarrow (A, B, C, D)$.

$$A = \frac{ad + bc + cd + 2bd}{b}$$

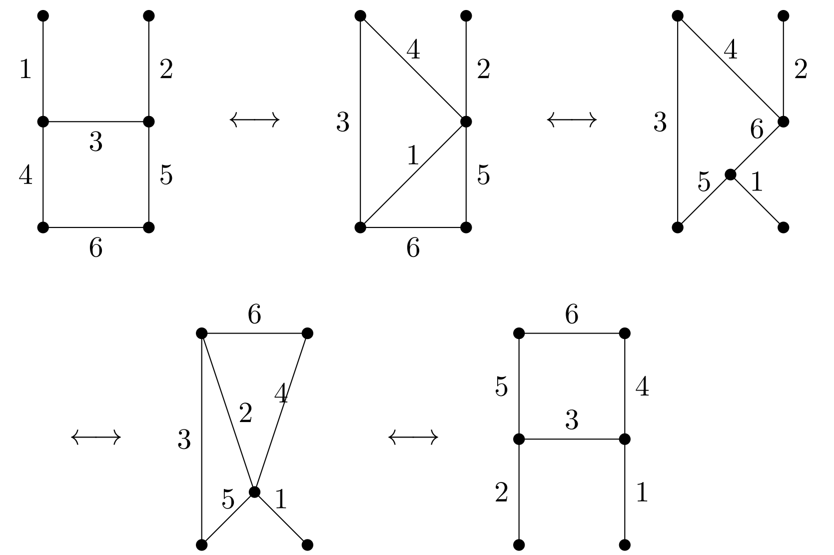
$$B = \frac{(ad + bc + cd + 2bd)^2}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}$$

$$C = \frac{ac(ad + bc + cd + 2bd)}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}$$

$$D = \frac{a^2 bd}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}$$

We also have $\phi(A, B, C, D) = (a, b, c, d)$, so ϕ is an involution.

Proof. By Theorem 2.6, (1) is clear. As for (2), we would like to decompose this transformation into star-triangle transformations, where the numbers on the edge keep track of the corresponding edges changes in the star triangle transformations.



The transformation ϕ is obtained by the composition of star-triangle transformations. ϕ is an involution can be directly verified. □

4.3. **Generators of Mirror Symmetric Networks and Electrical Lie Theory of Type B.** We introduce the following operations on mirror symmetric circular planar electrical networks with $2n$ boundary vertices (See Figure 4.2)

Adjoining two boundary spikes mirror symmetrically: For all $i \in [n]$, define $v_{2i}(t) \cdot \Gamma$ to be the action of adding boundary spikes with weights $1/t$ on both vertices $\bar{i}$ and $\bar{i}'$, and treating the newly added vertices as new boundary vertices $\bar{i}$ and $\bar{i}'$, and old boundary vertices $\bar{i}$ and $\bar{i}'$ as interior vertices.

Adjoining two boundary edges mirror symmetrically: For $i \in [n]\backslash\{1\}$, define $v_{2i-1}(t) \cdot \Gamma$ to be the action of adding boundary edges between vertices $\overline{i-1}$ and $\bar{i}$, and between $\overline{i-1}'$ and $\bar{i}'$, both with weight t.

Adjoining a boundary edge across the mirror line: Define $v_1(t)$ be the action of adding an boundary edge between vertices $\bar{1}$ and $\bar{1}'$ with weight $t/2$.

These operations can be seen as the generators of mirror symmetric circular planar electrical networks. Now we introduce electrical Lie algebra of type B of even rank, denoted as $e_{B_{2n}}$, defined by Lam-Pylyavskyy [LP]. $e_{B_{2n}}$ is generated by $\{e_1, e_2, \ldots, e_{2n}\}$ under the relations:

$$
\begin{aligned}
&[e_i, e_j] = 0 && \text{if } |i-j| \geq 2 \\
&[e_i, [e_i, e_j]] = -2e_i && \text{if } |i-j| = 1, i \neq 2 \text{ and } j \neq 1 \\
&[e_2, [e_2, [e_2, e_1]]] = 0
\end{aligned}
$$

Theorem 4.5 ([Su]). *The Lie algebra $e_{B_{2n}}$ is isomorphic to $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2n-1}$, where the odd symplectic Lie algebra $\mathfrak{sp}_{2n-1}$ was a Lie subalgebra of $\mathfrak{sp}_{2n}$ studied by Gelfand-Zelevinsky* [GZ] *and Proctor* [RP].

Let $E_{B_{2n}}$ be the direct product of Sp_{2n} and Sp_{2n-1} in Theorem 4.5. Its Lie algebra is $e_{B_{2n}}$. Let $u_i(t) = \exp(te_i)$ for all i. Define the nonnegative part $(E_{B_{2n}})_{\geq 0}$ to be the Lie subsemigroup generated by all $u_i(t)$ for all $t \geq 0$.

Theorem 4.6 ([LP]). *If $t > 0$, then $u_i(t)$'s satisfy the following relation:*

(1) $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i-j| \geq 2$
(2) $u_i(a)u_i(b) = u_i(a+b)$

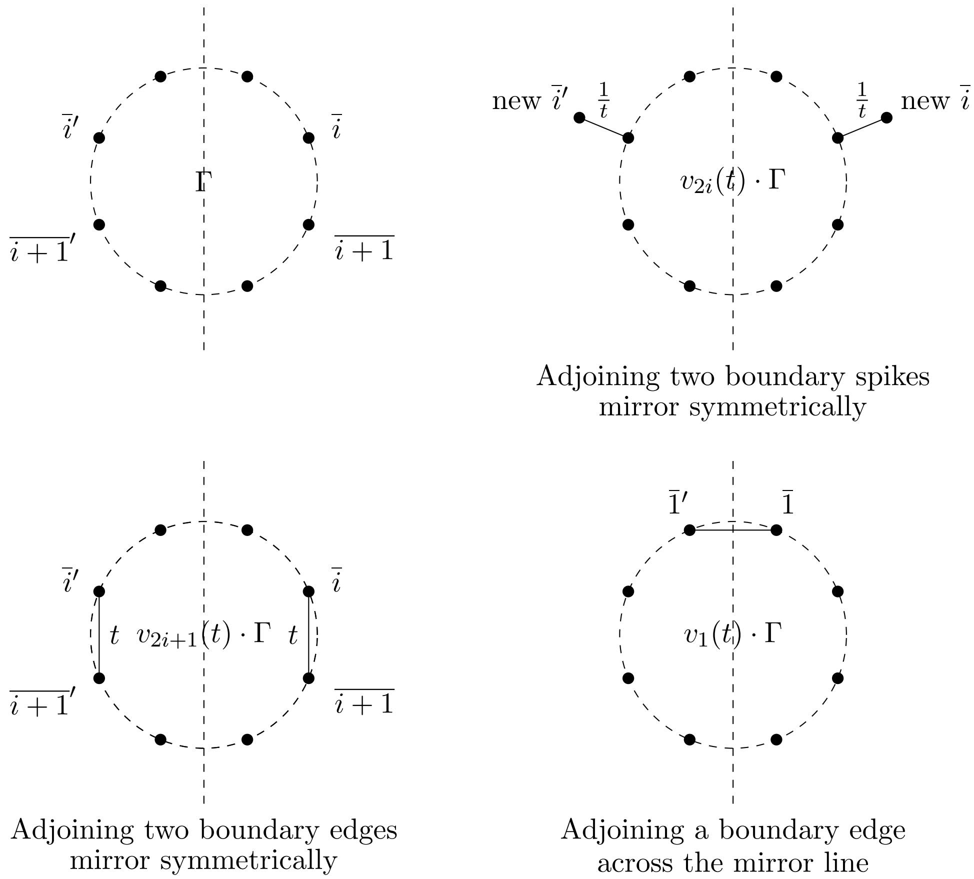


FIGURE 4.2. Generators of Mirror Symmetric Circular Planar Electrical Networks

(3) $u_i(a)u_j(b)u_i(c) = u_j\left(\frac{bc}{a+c+abc}\right)u_i(a+c+abc)u_j\left(\frac{ab}{a+c+abc}\right)$ *if* $|i-j|=1, i,j \neq 1$

(4) $u_2(t_1)u_1(t_2)u_2(t_3)u_1(t_4) = u_1(p_1)u_2(p_2)u_1(p_3)u_2(p_4)$, *with*

$$p_1 = \frac{t_2t_3^2t_4}{\pi_2}, p_2 = \frac{\pi_2}{\pi_1}, p_3 = \frac{\pi_1^2}{\pi_2}, p_4 = \frac{t_1t_2t_3}{\pi_1},$$

where

$$\pi_1 = t_1t_2 + (t_1+t_3)t_4 + t_1t_2t_3t_4, \pi_2 = t_1^2t_2 + (t_1+t_3)^2t_4 + t_1t_2t_3t_4(t_1+t_3).$$

This Lie subsemigroup $(E_{B_{2n}})_{\geq 0}$ is related to the operation $v_i(t)$'s in the following way.

Theorem 4.7. *The generators $v_i(t)$'s also satisfy the relation of $u_i(t)$'s in Theorem 4.6 for $t > 0$. Therefore, $e_{B_{2n}}$ has an infinitesimal action on the space of mirror symmetric circular planar electrical networks.*

Proof. $v_i(t)$'s satisfying the first three relations is a consequence of Theorem 2.8. It suffice to show relation (4) in Theorem 4.6. Note that the action $v_2(t_1)v_1(t_2)v_2(t_3)v_1(t_4)$ and $u_1(p_1)u_2(p_2)u_1(p_3)u_2(p_4)$ will give the following two configurations.

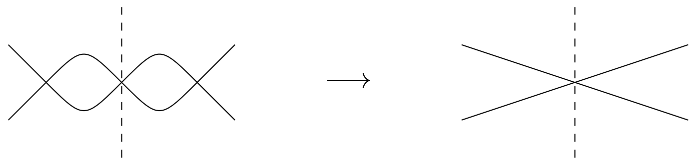
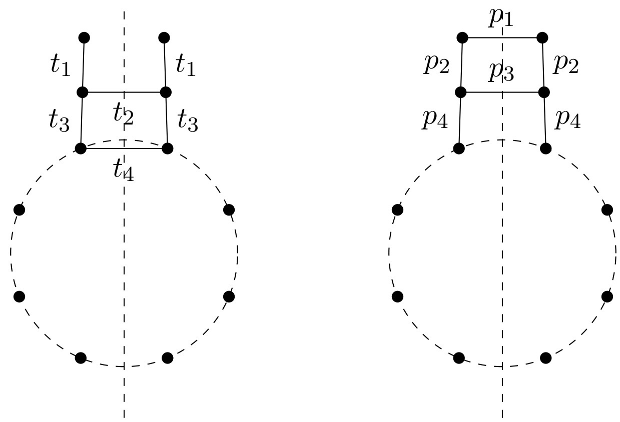


FIGURE 4.3. Double Lenses Removal



We note that this is exactly the square transformation in Theorem 4.4 with $\phi(t_1, t_2, t_3, t_4) = (p_4, p_3, p_2, p_1)$.□

4.4. **Medial Graph and Some Results for Mirror Symmetric Electrical Networks.** Each mirror symmetric circular planar electrical network Γ is associated with a **(Symmetric) medial graph** $\mathcal{G}(\Gamma)$. The notion of **lensless** medial graph and its medial pairing are the same as the ones for ordinary electrical networks. A **mirror symmetric medial pairing** is a matching of $\{\bar{1}, \bar{2}, \ldots, \overline{2n}, \bar{1}', \bar{2}', \ldots, \overline{2n}'\}$ such that if $\{\bar{i}, \bar{j}\}$, $\{\bar{i}', \bar{j}'\}$ or $\{\bar{i}, \bar{j}'\}$ are in the matching, so are $\{\bar{i}', \bar{j}'\}$, $\{\bar{i}, \bar{j}\}$ or $\{\bar{i}', \bar{j}\}$, respectively. Similar to ordinary medial pairings, the number of pairs of mirror symmetric crossings (if the crossing is on the mirror line, it counts as one pair of symmetric crossing) of a mirror symmetric medial pairing τ is independent of the choice of medial graph. Define $mc(\tau)$ to be the number of pairs of symmetric crossings.

Proposition 4.8. *Let Γ and Γ' be two mirror symmetric networks.*

(1) *If Γ and Γ' are related by symmetric leaf and loop removals, series and parallel transformations, and double series and parallel transformations, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by symmetric lens and loop removals and double lenses removals (See Figure 4.3).*
(2) *If Γ and Γ' are related by symmetric star-triangle transformations, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by two mirror symmetric Yang-Baxter transformations.*
(3) *If Γ and Γ' are related by the square move in Theorem 4.4, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by the* ***crossing interchanging transformation*** *(See Figure 4.4).*

Proof. The first two claims are consequences of Theorem 2.11. The third claim is proved via the following picture.□

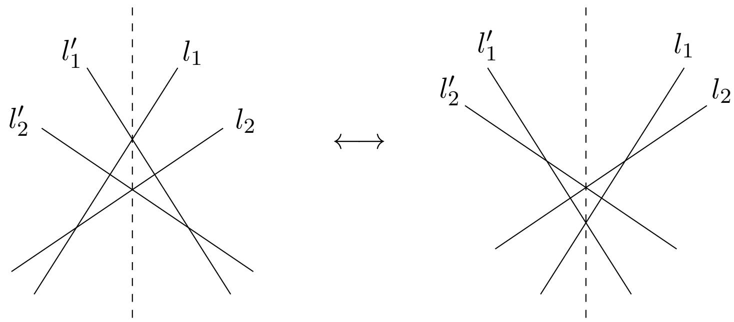
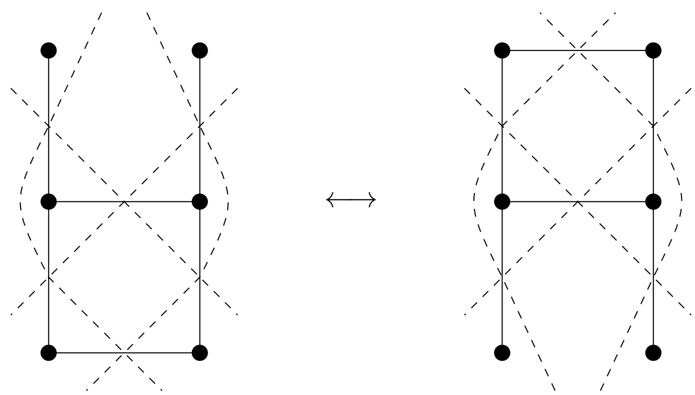


FIGURE 4.4. Crossing Interchanging Transformation



Theorem 4.9. *Every mirror symmetric circular planar electrical network can be transformed into a critical mirror symmetric network through the symmetric operations in Proposition 4.3 and Theorem 4.4.*

Proof. Let Γ be a mirror symmetric electrical network. Consider its symmetric medial graph $\mathcal{G}$. Claim that we can use symmetric lens and loop removal, double lenses removal, symmetric Yang-Baxter transformations, and crossing interchanging transformations to remove all lenses in $\mathcal{G}$. Thus on the level of mirror symmetric electrical network, the resulting network Γ' will be a critical network, and it is obtained from Γ by doing symmetric leaf and loop removals, series-parallel transformations, double series-parallel transformations, symmetric Yang-Baxter transformations, and square transformations. Thus, we have the theorem.

It suffices to show the above claim. First pick a lens in $\mathcal{G}$. Two medial strands of this lens are denoted as l_1 and l_2, which intersect at a and b. Let al_1b be the arc of this lens which lies on l_1. Define al_2b likewise. Let the interior of the lens be the region enclosed by al_1b and al_2b. We can assume this lens does not contain any other lens in its interior. Otherwise, pick a smaller lens in its interior. We have two cases: (1) the mirror line does not pass through both of a and b; (2) the mirror line pass through both a and b, which means l_1 is symmetric to l_2. Our goal is to remove this lens via the operations in Proposition 4.8.

Case 1: There is another lens enclosed by l_1' and l_2' which are the mirror symmetric counterparts of l_1 and l_2 respectively. In the following discussion, for every transformation we perform mirror symmetric operations simultenously on both lenses.

Let $H = \{h_1, h_2, \ldots, h_k\}$ be the set of medial strands that intersect both of al_1b and al_2b. The h_i can possibly intersect with other h_j's in the interior of the lens. We claim that we can use symmetric Yang-Baxter transformations to make h_i's have no intersections among themselves in the interior of the lens, and the same is true for the mirror images of h_i's.

We proceed by induction. The base case is trivial. Now assume there is at least one intersection. Among those intersections, let r_i be the intersection point on h_i that is closest to the arc al_1b. Two medial strands which intersect at r_i and the arc al_1b form a closed region D_i. We pick r_k such that the number of regions in D_k is minimized. We do the same construction for the mirror symmetric lens (these two lenses are possibly the same). Note the number of regions in D_k has to be one. Otherwise, there must be another strand l_s intersecting D_k at r_s, and the region enclosed by two strands which intersect at r_s and the arc al_1b will have smaller number of regions in it, contradiction. Hence, we can use symmetric Yang-Baxter transformation to remove the region D_k and its mirror image in order to reduce the number of intersections by one in this lens and in its mirror image. Thus the claim is true.

Note that at least one of a and b is not on the mirror line. Without loss of generality, say the point a does not lie on the mirror line. Among the strands in H, pick the strand h_t that is the "closest" to a. Then l_1, l_2, and h_t enclose a region which does not have subregions inside. So we can use symmetric Yang-Baxter transformation to move h_t such that it does not intersect the arcs al_1b and al_2b, and likewise for the mirror image. Repeat this until no strand intersects this lens and its mirror image.

Lastly, use the symmetric lens removal or double lenses removal to remove these two lenses symmetrically.

Case 2: Again let $H = \{h_1, h_2, \ldots, h_k\}$ be the set of medial strands that intersect both of al_1b and al_2b. We can use the same argument as in Case 1 to reduce the number of intersections among h_i's which are in the lens, but not on the mirror line. Thus, we can assume all of the intersections stay on the mirror line. Pick an intersection point r_k that is closest to the point a. Say two mirror symmetric strands h_1 and h_1' intersect at a. Then we can use crossing interchanging transformation to move h_1 and h_1' so that they intersect neither al_1b nor al_2b. Repeat this until no strand intersects this lens. Then we use symmetric lens removal to decrease the number of lenses.

By iterating the above lens removal procedures, we can change $\mathcal{G}$ into a lensless medial graph.□

Next we prove a lemma.

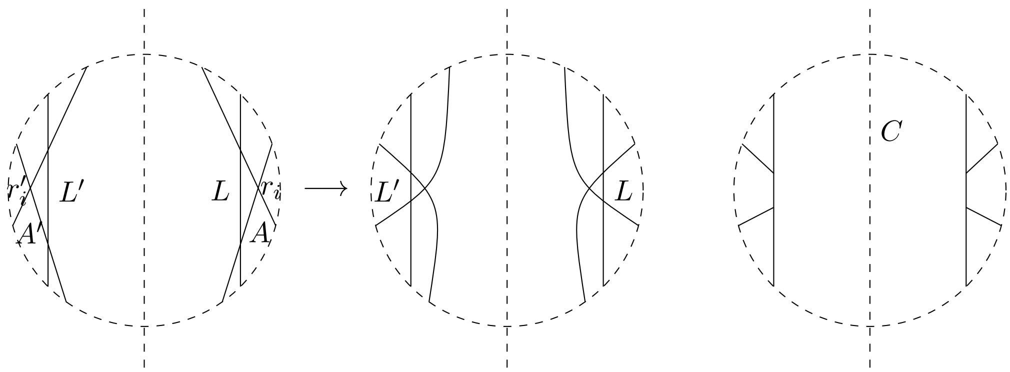
Lemma 4.10. *Let $\mathcal{G}$ and $\mathcal{H}$ be two lensless mirror symmetric medial graphs. If $\mathcal{G}$ and $\mathcal{H}$ have the same medial pairing, then they can be obtained from each other by symmetric Yang-Baxter transformations and crossing interchanging transformations in Proposition 4.8 (2) and (3).*

Proof. We will proceed by induction on the number of medial pairings. The base case is the empty network, which is trivially true. Pick a medial strand L such that L divides the circle into two parts, say A and B, and there is no other chord completely contained within A. Consequently, the mirror image L' of L divides the circle into two parts A' and B', mirror image of A and B. There are three cases:

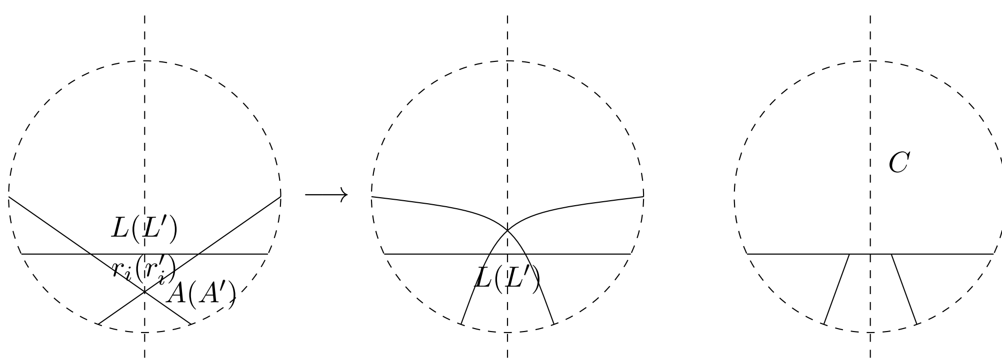
1 L does not intersect L'.
2 L coinsides with L'.
3 L intersects L' at one point.

Let $P_1, P_2, \ldots, P_k$ be the medial strands intersecting L. WE claim that one can use symmetric Yang-Baxter transformations and crossing interchanging transformations to make $P_1, P_2, \ldots, P_k$ have no intersection point among themselves in the region A. Suppose otherwise. Let r_j be the intersection point on the medial strand P_j that is closest to L. For each r_j, we know that L and two medial strand where r_j lies on form a closed region S_j. Pick r_i such that the number of subregions in S_i is mininized. Similarly we can define the mirror image $L', P'_1, P'_2, \ldots, P'_k, r'_j$ and S'_i. Next consider three cases seperately.

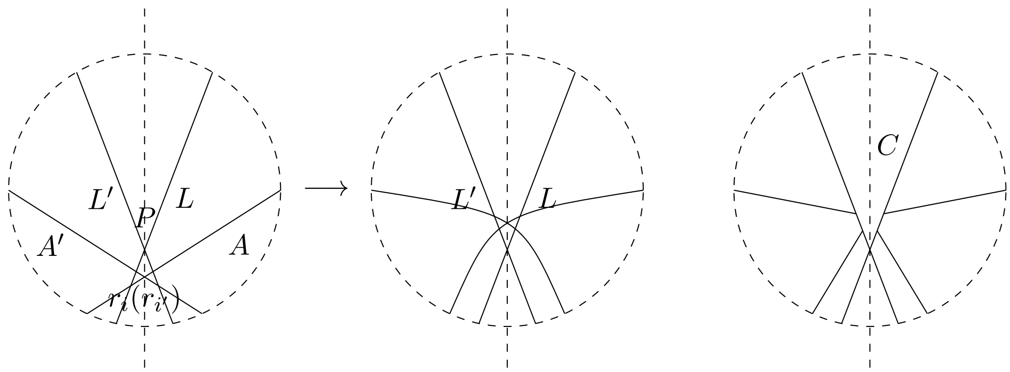
Case 1: First note that the number of regions in S_i must be one, otherwise, one can find i' on one of the medial strand where r_i lies on, such that $S_{i'}$ has fewer regions, contradiction. Then by mirror symmetry, the number of regions in S'_i is also 1. By symmetric Yang-Baxter transformations, we can remove the closed regions S_i and S'_i so that the the number of intersections in A and A' decreases respectively, while the medial parings remain the same. So without loss of generality, we can assume the number of intersections in A and A' is 0. Hence we can use symmetric Yang-Baxter transformations to change both $\mathcal{G}$ and $\mathcal{H}$ as in the following picture. The regions C enclosed by L, L' and the rest of the boundary circle has fewer number of medial pairings. Therefore, by induction, we are done (See the following pictures).



Case 2: A and A' coincide with each other. The argument is almost the same as above except that the region S_i and S'_i can coincide with each other, which means the region S_i is symmetric to itself and r_i is on the mirror line. In this situation, we can still apply symmetric Yang-Baxter move to decrease the number of intersections in region A, and apply the induction hypothesis (See the following pictures).



Case 3: L and L' intersect at point P. One can assume that all the medial strands cross the mirror line, otherwise, we can perform Case 1 first. Then by similar argument to Case 1, regions S_i and $S_{i'}$ both have one region in it. Moreover, r_i coincides with r'_i. And S_i is mirror symmetric to S'_i and they only intersect at r_i. The region enclosed by L, L' and the two medial strands whose intersection is r_i only has one region in it. In this case we can perform crossing exchanging transformations to swap P and r_i, so that the number of intersections in A and A' decrease. By repeating these operations, we can also assume the number of intersections in A and A' is 0. Hence we can use symmetric Yang-Baxter transformations to change both $\mathcal{G}$ and $\mathcal{H}$ as in the following picture, and apply induction hypothesis. (See the following pictures).□



The following theorem is analogus to Theorem 2.12 (2).

Theorem 4.11. *If two mirror symmetric planar electrical networks Γ and Γ' have the same repsonse matrix, then they can be connected by symmetric leaf and loop removal, symmetric series-parallel transformations (in Proposition 4.3), symmetric star-triangle transformations, and square transformations (in Theorem 4.4). Furthermore, if both Γ and Γ' are critical, only the symmetric star-triangle transformations and square transformations are required.*

Proof. By Theorem 4.9, we can assume that Γ and Γ' are both critical.

As Γ and Γ' have the same response matrix, by Theorem 2.4 and 3.6, we know that the medial pairing of these two networks $\tau = \tau(\mathcal{G}(\Gamma))$ and $\tau' = \tau(\mathcal{G}(\Gamma'))$ are the same. Then by Lemma 4.10, τ and τ' can be obtained from each other by symmetric Yang-Baxter transformations and crossing interchanging transformations. Therefore, the underlying graph of Γ and Γ' are related by mirror symmetric star-triangle transformations and square transformations. Now let $T(\Gamma)$ be the network by doing such transformations on the underlying graph of Γ as well as its weights on the edges. It can be seen that the underlying graph of $T(\Gamma)$ is the same as underlying graph of Γ'. On the other hand, $T(\Gamma)$ and Γ' have the same response matrix. Thus by Theorem 2.12 (3), they must have the same weight on each edge, which means $T(\Gamma) = \Gamma'$. This proves the theorem. □

We also have the following theorem:

Theorem 4.12. *The space ME_n' of response matrices of mirror symmetric circular planar networks has a stratification by cells $ME_n' = \bigsqcup D_i$ where each $D_i \cong \mathbb{R}_{>0}^{d_i'}$ can be obtained as the set of response matrices for a fixed critical mirror symmetric circular planar network with varying weights on the pairs of symmetric edges.*

Proof. Recall that E_n' is the space of response matrices of circular planar electrical networks. By Theorem 2.12 (5), $E_n' = \bigsqcup C_i$ where $C_i \cong \mathbb{R}_{>0}^{d_i}$ can be obtained as a set of response matrices for a fixed critical network with varying edge weights. Now for each C_i, if possible, we pick the representative critical network such that the underlying graph is mirror symmetric. Let D_i be the subspace of each such C_j with mirror symmetric edge weights to be the same. It is clear that every mirror symmetric circular planar electrical network is obtained in this way up to electrical equivalences. Thus, $ME_n' = \bigsqcup D_i$. □

5. Compactification of the Space of Mirror Symmetric Circular Planar Electrical Networks

Again, not every mirror symmetric medial pairing can be obtained as a medial pairing of some mirror symmetric circular planar electrical network. We will use a definition similar to cactus networks in Section 3.1 to resolve this.

5.1. **Mirror Symmetric Cactus Network.** A **mirror symmetric cactus network** is a cactus network which is symmetric with respect to the mirror line. The medial graph of mirror symmetric medial graph again is typically drawn in a circle. Similar to usual cactus networks, we have the following proposition.

Proposition 5.1.

(1) *Every mirror symmetric cactus network is electrically-equivalent to a critical cactus network through symmetric reductions.*
(2) *If two mirror symmetric cactus network have the same response matrix, then they are related by doing a sequence of symmetric star-triangle and square transformations.*
(3) *Any symmetric medial pairing can be obtained as the medial pairing of some mirror symmetric cactus network.*

Proof. The proof (1) and (2) are simiar to the proof for mirror symmetric circular planar networks. (3) is proved as following: Let τ be a symmetric medial pairing of $\{-2n+1,\ldots,0,\ldots,2n-1,2n\}$ and $\mathcal{G}$ be any medial graph with $\tau(\mathcal{G})=\tau$. Then the medial strands divide the disk into different regions. If some vertices in $\{\bar{1},\bar{2},\ldots,\bar{n},\bar{1}',\bar{2}',\ldots,\bar{n}'\}$ are in one region, then identify them as one vertex in the mirror symmetric cactus network. This gives a mirror symmetric hollow cactus. Now within each pair of symmetric disks we have symmetric medial pairings. We can reconstruct a pair of circular planar networks which is mirror symmetric to each other within each pair of such disks. Thus we are done. □

5.2. **Grove Measurements as Projective Coordinates of Mirror Symmetric Networks.** Similar to ordinary electrical networks, we can also define grove measurements for mirror symmetric cactus networks. Two partitions σ and σ' are called **mirror images of each other** or **mirrors** if interchanging $\bar{i}$ and $\bar{i}'$ for all $i\in[n]$ in σ and σ' will swap these two partitions. Clearly, if Γ is a mirror symmetric cactus network, $L_\sigma(\Gamma)=L_{\sigma'}(\Gamma)$, where σ and σ' are mirrors. Let $\mathbb{P}^{\mathcal{SNC}_n}$ be the subspace of $\mathbb{P}^{\mathcal{NC}_{2n}}$ in which the grove measurements indexed by a pair of symmetric non-crossing partitions are the same. Thus the map

$$\Gamma \longmapsto (L_\sigma(\Gamma))_\sigma$$

sends a mirror symmetric cactus network to a point $\mathcal{L}(\Gamma)\in\mathbb{P}^{\mathcal{SNC}^n}$.

Proposition 5.2. *If Γ and Γ' are electrically equivalent mirror symmetric cactus networks, then $\mathcal{L}(\Gamma)=\mathcal{L}(\Gamma')$.*

Proof. Since each symmetric reduction, symmetric star-triangle transformation and square transformation can all be decomposed into a sequence of ordinary reductions and star-triangle transformations. Thus by Theorem 3.3, we have $\mathcal{L}(\Gamma)=\mathcal{L}(\Gamma')$.□

5.3. **Compactification and Some Result for Cactus networks.** Recall ME_n' is the space of response matrices of mirror symmetric circular planar electrical networks. Since the grove measurements of mirror symmetric circular planar electrical network is in one-to-one correspondence with the response matrix of $\Lambda(\Gamma)$ by a similar argument in Subsection 3.3, we can regard ME_n' as the space of grove measurements of electrically equivalent mirror symmetric circular planar electrical networks. Theorem 4.11 implies that $ME_n'\longrightarrow\mathbb{P}^{\mathcal{SNC}_n}$ is an injection.

Define the closure in the Hausdorff topology $ME_n=\overline{ME_n'}\subset\mathbb{P}^{\mathcal{SNC}_n}$ to be the **compactification** of the space of mirror symmetric circular planar electrical networks. Let MP_n be the set of mirror symmetric medial pairings of $\{-2n+1,\ldots,0,\ldots,2n-1,2n\}$. Note that two electrically equivalent cactus networks have the same medial pairing.

Theorem 5.3.

(1) *The space ME_n is exactly the set of grove measurements of mirror symmetric cactus networks. A mirror symmetric cactus network is determined uniquely by its grove measurement up to symmetric electrical equivalences.*
(2) *Let $ME_\tau = \{\mathcal{L}(\Gamma)|\tau(\Gamma) = \tau\} \subset ME_n$. Each stratum ME_τ is parametrized by choosing a mirror symmetric cactus network Γ such that $\tau(\Gamma) = \tau$ with varying edge weights. So we have $ME_\tau = \mathbb{R}_{>0}^{mc(\tau)}$. Moreover,*

$$ME_n = \bigsqcup_{\tau \in MP_n} ME_\tau$$

Proof. First we prove (1). By definition, any point $\mathcal{L}$ in ME_n is a limit point of points in ME'_n. On the other hand, the top cell of ME'_n is dense in ME'_n. Thus we can assume $\mathcal{L} = \lim_{i\to\infty} \mathcal{L}(\Gamma_i)$, where Γ_i's are mirror symmetric circular planar electrical networks whose underlying graphs are the same, say G. Note that the response matrices depend continuously on the edge weights. Thus $\mathcal{L}$ is obtained by send some of mirror symmectric edge weights of G to ∞. By doing this we identify some of the boundary vertices, and obtain a mirror symmetric cactus network.

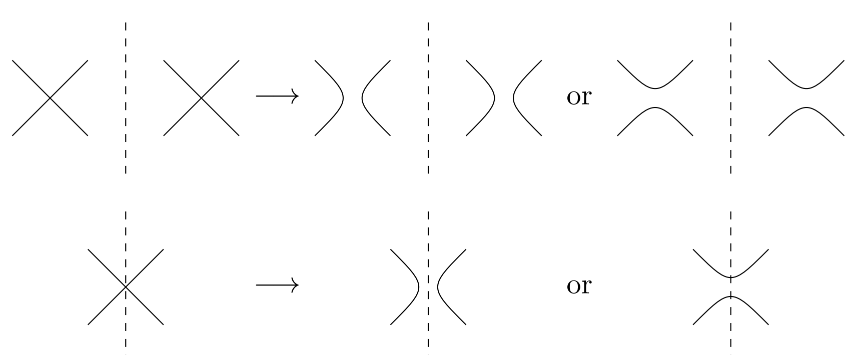
To see how $\mathcal{L}(\Gamma)$ determines Γ, first notice that Γ is a union of circular planar networks Γ_r. We observe that the shape σ of Γ is determined by $\mathcal{L}(\Gamma)$. To reconstruct Γ, it suffices to recover each circular planar networks Γ_r in a symmetric way. Let σ_{ij} be obtained from σ by combining the parts containing i and j. Then we can determine the response matrix $\Lambda(\Gamma_r)$ of Γ_r by the following identity:

$$\Lambda(\Gamma_r)_{ij} = \frac{L_{\sigma_{ij}}(\tau)}{L_\sigma(\tau)}.$$

Each grove of Γ is a union of groves in Γ_r, so in the above ratio the contribution of groves from Γ_s where $s \neq r$ gets cancelled. Thus by Theorem 2.4, the above identity is true. Hence we can recover response matrices $\Lambda(\Gamma_r)$ for each Γ_r. By Theorem 2.12 (3) we can reconstruct Γ_r and its mirror image Γ'_r simultaneously in a mirror symmetric way uniquely up to electrical equivalences.

As for (2), note that each mirror symmetric cactus network is a union of pairs of mirror symmetric circular networks, and then we apply Theorem 2.12 to obtain the first statement. $ME_n = \bigsqcup_{\tau\in MP_n} ME_\tau$ is a consequence of (1) and Proposition 5.1 (3). □

5.4. **Symmetric Matching Partial Order on MP_n and Bruhat Order.** A partial order on MP_n can be defined as follows: Let τ be a medial pairing and $\mathcal{G}$ be a medial graph such that $\tau(\mathcal{G}) = \tau$. Uncross the two different kinds of crossings as in the following:



Suppose the resulting medial graph $\mathcal{G}'$ is also lenseless. Let $\tau' = \tau(\mathcal{G}')$. Then we say $\tau' \lessdot \tau$ is a covering relation in $\mathcal{MP}_n$. The partial order on $\mathcal{MP}_n$ is the transitive closure of these relations. This poset is an induced subposet of $\mathcal{P}_n$ studied in [Lam] .

Lemma 5.4. *Let $\mathcal{G}$ be a medial graph with $\tau(\mathcal{G}) = \tau$, where the labels of vertices of $\mathcal{G}$ on the boundary are $\{-2n+1, \dots, -1, 0, 1, \dots, 2n\}$, and i is the mirror image of $-i+1$. Suppose that (a, b, c, d) are in clockwise order (Some of them possibly coincide), and τ has strands from a to c, and b to d. Correspondingly $(-a+1, -b+1, -c+1, -d+1)$ are in counterclockwise order, and there τ has strands from $-a+1$ to $-c+1$, and $-b+1$ to $-d+1$. Let $\mathcal{G}'$ be obtained from uncrossing the intersections and joining a to d, b to c, $-a+1$ to $-d+1$, and $-b+1$ to $-c+1$. Then G' is lenseless if and only if no other medial strand goes from the arc (a, b) to (c, d) and from the arc $(-a+1, -b+1)$ to $(-c+1, c-d+1)$ and they are one of the configurations in Figure 5.1.*

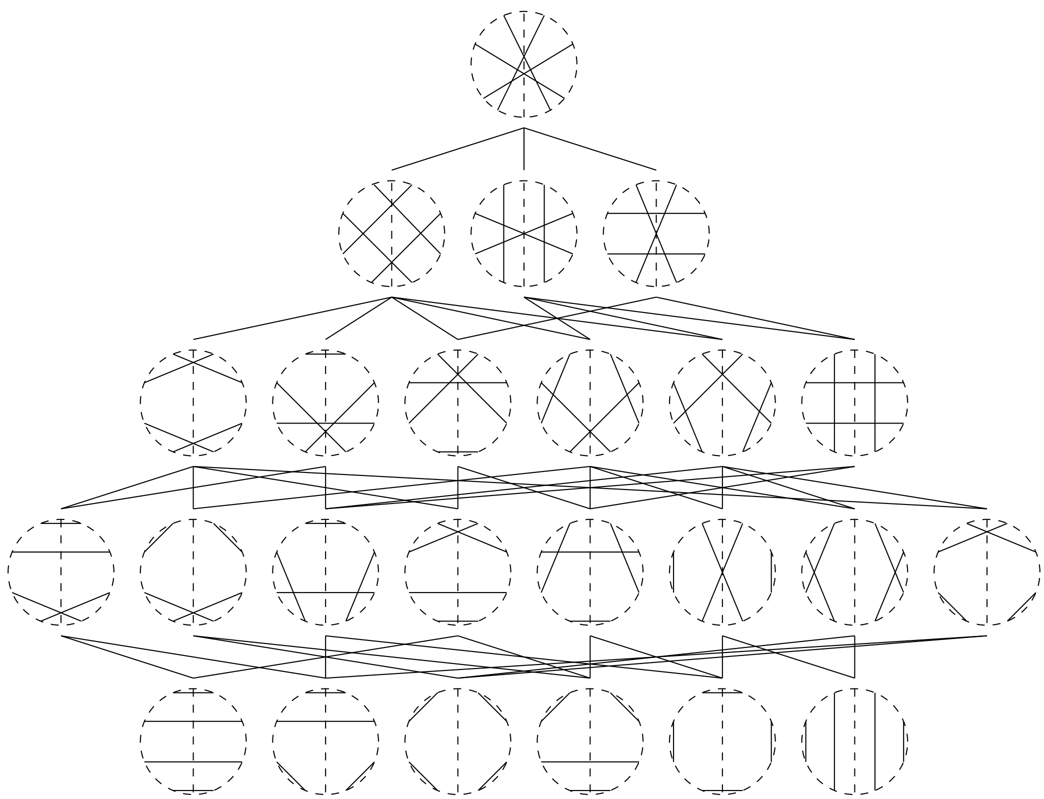
Proof. Straightforward case analysis. □

Recall that $mc(\tau)$ is the number of pairs of symmetric crossings.

Lemma 5.5. *$\mathcal{MP}_n$ is a graded poset with grading given by $mc(\tau)$.*

By straightforward enumeration the highest rank is n^2, and the number of elements on rank 0 is $\binom{2n}{n}$ which is the number of mirror symmetric medial pairings.

Example 5.6. The following is the poset $\mathcal{MP}_2$.



Poset on $\mathcal{MP}_2$

Now consider another partial order on $\mathcal{MP}_n$. Let $\tau \in \mathcal{MP}_n$. Pick a medial graph $\mathcal{G}$ with $\tau(\mathcal{G}) = \tau$. We break a crossing of $\mathcal{G}$ to obtain a graph $\mathcal{H}$. Note that $\mathcal{H}$ may not be lensless any more. Let $\mathcal{H}'$ be the lensless graph obtained from removing all lenses in $\mathcal{H}$. Let $\tau' = \tau(\mathcal{H}') = \tau(\mathcal{H})$. In this case, we say $\tau' \tilde{<} \tau$. We claim these two partial orders are the same.

Theorem 5.7. *Let $\tau, \nu \in (\mathcal{MP}_n, \tilde{<})$. Then τ covers ν if and only if there is a lensless medial graph $\mathcal{G}$ with $\tau(\mathcal{G}) = \tau$, such that a lensless medial graph $\mathcal{H}$ with $\tau(\mathcal{H}) = \nu$ can be obtained by uncrossing*

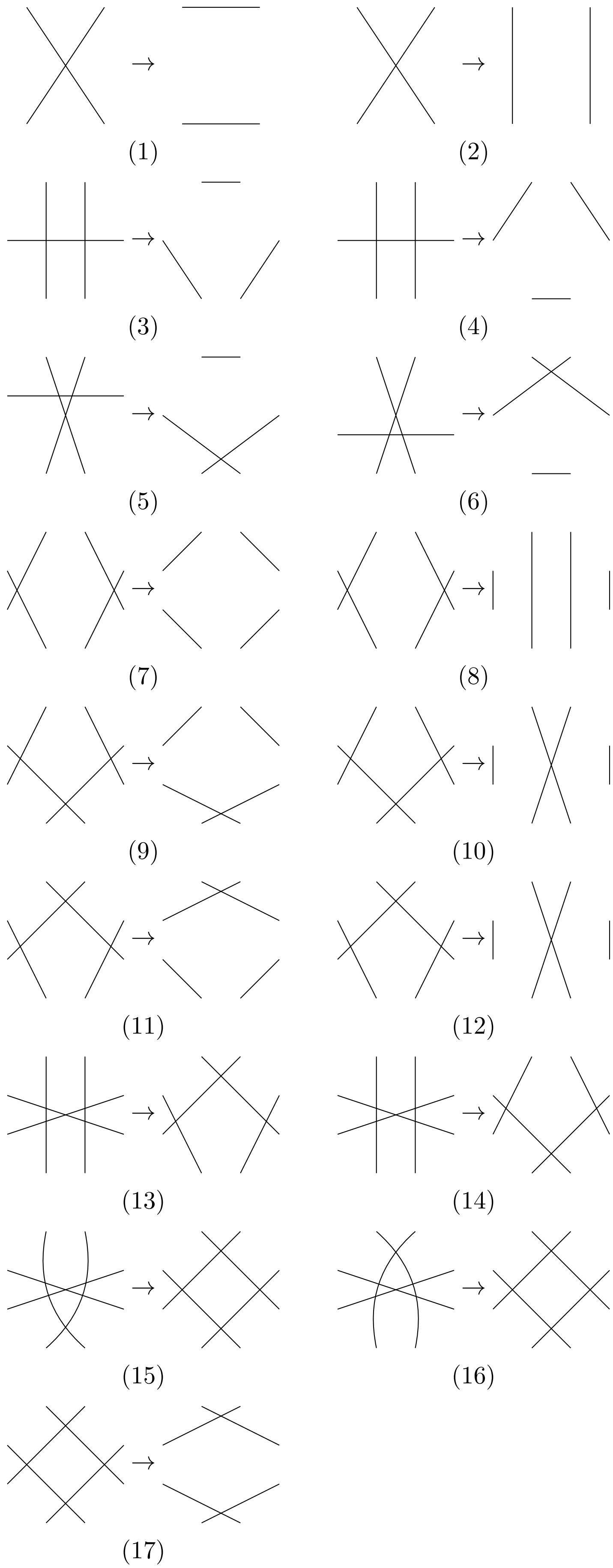


FIGURE 5.1. Covering Relation for $\mathcal{MP}_n$

one pair of symmetric intersection points (one intersection point if the point stays on the mirror line) from $\mathcal{G}$.

Proof. First assume that a lensless medial graph $\mathcal{H}$ is obtained from $\mathcal{G}$ by uncrossing one pair of intersection points, and we want to prove τ covers ν. Suppose otherwise. Then there must be a medial graph $\mathcal{G}'$ such that $\tau = \tau(\mathcal{G}) \tilde{>} \tau(\mathcal{G}') \tilde{>} \tau(\mathcal{H}) = \nu$. But the number of pairs of symmetric crossings in $\mathcal{G}$ is at least two more than that in $\mathcal{H}$, contradicting the hypothesis. Therefore, $\tau(\mathcal{G})$ covers $\tau(\mathcal{H})$.

Now we show the contrapositive of the other direction. Suppose $\mathcal{H}$ is obtained from $\mathcal{G}$ by uncrossing one pair of symmetric intersection points and $\tau(\mathcal{H}) = \nu$, but $\mathcal{H}$ is not lensless. Then we would like to show that there is a lensless medial graph $\mathcal{G}'$ such that $\tau(\mathcal{G}) \tilde{>} \tau(\mathcal{G}') \tilde{>} \tau(\mathcal{H})$.

Suppose points a, b, c and d lie on a circle in a clockwise direction, and strands a, c and b, d intersect at q, and their mirror image strands a', c' and b', d' intersect at q'. Suppose we uncross mirror-symmetrically the intersection q and q' so that the new strands are a, d and b, c, as well as a', d' and b', c'. Since H is not lensless, then there must be a strand connecting a boundary point between a and b to a boundary point between c and d, and likewise in the mirror-symmetric side. Without loss of generality assume that one such strand intersects the sector aqd. Let $L = \{l_1, l_2, \ldots, l_k\}$ be the set of strands intersecting both aq and dq. We want to prove the following claim:

Claim. *We can use mirror symmetric Yang-Baxter transformations to change $\mathcal{G}$ into a medial graph such that the intersection point among strands in L are all outside of the sector aqd.*

Let x_i be the intersection of l_i and aq. Let D_i be the closest intersection point to x_i on l_i among all l_j's that have intersection with l_i within the sector aqd. Let D be the set of D_i's. If D is empty, the statement is already true. Otherwise, we pick D_i such that the number r of regions enclosed by l_i, l_j and aq is smallest, where l_j is the strand that intersects with l_i at D_i. We claim $r = 1$. Otherwise, there is another medial strand l_k intersecting the region enclosed by l_i, l_j and aq. For example, if l_k intersects l_i and aq, then the region enclosed by l_i, l_k, aq has strictly fewer number of subregions, contradicting the minimality of r. Hence, $r = 1$.

The same is true for the mirror image. Thus we can use mirror symmetric Yang-Baxter transformation to move the intersection of l_i, l_j as well as its mirror image out of the sector aqd and its mirror image. $|D|$ gets decreased by 1. By induction, we can reduce D to be empty set. We are done with this claim.

Now pick $l \in L$ such that l intersects with aq at the point x closest to q. And say l intersects dq at y. Then given the above claim, we also want to prove the following claim:

Claim. *We can use mirror symmetric Yang-Baxter transformations to change $\mathcal{G}$ into a medial graph such that no other medial strand intersects the region xqy.*

Let X be the set of medial strands that intersect aq and l. By the similar argument above, we can show that using mirror symmetric Yang-Baxter move, any intersection between two medial strands in X can be moved outside of sector aqd. Then we can use mirror symmetric Yang-Baxter move at v and its mirror image to move all medial strands in X out of sector aqd. Now let Y be the set of medial strands that intersect dq and l. With similar argument, we can show that all medial strand in Y can be moved out of sector aqd and its mirror image by mirror symmetric Yang-Baxter move. We prove this claim.

With the two claims above, we can assume that $\mathcal{G}$ is lensless, and ac, bd, and l form a triangle where no other medial strand enters, and the same configuration on the mirror symmetric image. We are ready to prove the rest of the theorem.

Say $l = ef$, where e, f are two end points of l, e is between arc ab, and f is between arc cd. If a is not mirror symmetric to d or b is not mirror symmetric to c, then we uncross the intersection of strands ac, ef to get a mirror symmetric medial graph $\mathcal{G}'$ with new strands af, ec. After that uncross the intersection of af, bd to get new strands ad, bf. Lastly uncross the intersection of ec, bf to get new strands ef, bc. And perform the same operation for the mirror image simultenously. If a is mirror symmetric to d and b is mirror symmetric to c, we uncross the intersections of bd with ef and ac with ef respectively to obtain strands ad, fb, and ce (See Figure 5.1 (5)). Again call this resulting mirror symmetric medial graph $\mathcal{G}'$. Then uncross the intersection between fb and ce to obtain strands bc and fe. At each step, we always get a lensless medial graph. In particular, $\mathcal{G}'$ is lensless. The whole procedure above is equivalent of uncrossing the intersection of strands ac and bd as well as its mirror image to get $\mathcal{H}$, and remove lenses from $\mathcal{H}$. Thus $\tau(\mathcal{G})\tilde{>}\tau(\mathcal{G}')\tilde{>}\tau(\mathcal{H})$.□

The above theorem implies that two posets are the same. In the following, we would like to explore the relations between symmetric matching partial order and Bruhat order.

For each $\tau \in \mathcal{MP}_n$, we associate an affine bounded permutation g_τ to τ as follows:

$$g_\tau(i) = \begin{cases} \tau(i) & \text{if } \tau(i) > i \\ \tau(i) + 4n & \text{if } \tau(i) < i \end{cases}$$

where τ is thought of as a fixed point free involution on the set $\{-(2n-1), \ldots, -1, 0, 1, \ldots, 2n\}$. Note g_τ is a bounded affine permtation of type $(2n, 4n)$. And $g_0 := g_{\tau_{\text{top}}}$ is defined as $g_0(i) = i + 2n$ with length 0. Then g_0 plays the role of the identity permutation. Let $t_{a,b}$ denote the transposition swapping a and b. Note we have $t_{i,i+1}g_0 = g_0 t_{i+2n,i+1+2n}$.

Let $s_0 = \prod_{k\in\mathbb{Z}} t_{4nk,4nk+1}$, $s_{2n} = \prod_{k\in\mathbb{Z}} t_{2n+4nk,2n+1+4nk}$, and $s_i = \prod_{k\in\mathbb{Z}} t_{i+4nk,i+1+4nk} t_{-i+4nk,-i+1+4nk}$ for $1 \le i \le 2n-1$. Affine permutation of type C, denoted as $\tilde{S}^C_{2n}$, is defined to be the group generated by $\{s_i\}_{i=0}^{2n}$. More precisely $\tilde{S}^C_{2n}$ is the set of injective maps $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ with the condition $\sum_{i=-2n+1}^{2n}(f(i) - i) = 0, f(i + 4n, j + 4n) = f(i, j)$, and $f(i, j) = f(-i + 1, -j + 1)$. Note that $\tilde{S}^C_{2n}$ is a subgroup of $\tilde{S}^0_{4n}$, and the affine Bruhat order on $\tilde{S}^C_{2n}$ is induced from a subposet of the affine Bruhat order on $\tilde{S}^0_{4n}$. Let l_C be the length function of $\tilde{S}^C_{2n}$. It is clear that each g_τ can be viewed as an element in affine permutations of type C with identity shifted to g_0.

Lemma 5.8. *Let $\tau \in \mathcal{MP}_n$. Then there is $w \in \tilde{S}^C_{2n}$ such that*

$$g_\tau = w g_0 w^{-1}$$

where $l_C(w) = n^2 - mc(\tau)$, and $l_C(g_\tau) = 2l_C(w)$

Proof. The claim is trival if $\tau = \tau_{top}$, where $g_\tau = g_0$. Suppose that τ is not the top element. Then there exists some i such that $g_\tau(i) > g_\tau(i+1)$. By symmetry, $g_\tau(-i) > g_\tau(-i+1)$. Then we can swap $i, i+1$ and $g_\tau(i), g_\tau(i+1)$, $-i, -i+1$ and $g_\tau(-i), g_\tau(-i+1)$ (all taken modulo $4n$) to obtain τ'. Thus, we have $\tau \lessdot \tau'$, with $g_\tau = s_i g_{\tau'} s_i$, and $l_C(g_\tau) = l_C(g_{\tau'}) + 2$. By induction on $mc(\tau)$, the claim is true. □

The factorization in Lemma 5.8 is not unique. In fact, later we will see that the number of such w is equal to the number of different paths from τ to τ_{top} in $\mathcal{MP}_n$.

Define the (infinite by infinite) affine rank matrix of an affine permutation f by

$$r_f(i, j) = |\{a \in \mathbb{Z} | a \le i, f(a) \ge j\}|.$$

Note that $r(i, j)$ satisfies $r(i, j) = r(i + 4n, j + 4n)$.

Theorem 5.9 ([BB], Theorem 8.4.8)**.** *Let $u, v \in \tilde{S}^C_{2n}$. Then $u \le v$ if and only if $r_u(i, j) \le r_v(i, j)$ for all $i, j \in \mathbb{Z}$.*

The following theorem identifies the partial order on $\mathcal{MP}_n$ with a subposet of the affine Bruhat order.

Theorem 5.10. *We know $l_C(g_\tau) = 2(n^2 - mc(\tau))$. Thus the map $\tau \to g_\tau$ identifies $\mathcal{MP}_n$ with an induced subposet of the dual Bruhat order of affine permutation of type C. In other words, $\tau' \leq \tau$ if and only if $g_\tau \leq g_{\tau'}$.*

Proof. First assume $\tau' \lessdot \tau$. According to Lemma 5.4, τ' is obtained from τ by uncrossing the intersection points of strands $a \leftrightarrow c$ and $b \leftrightarrow d$, $-a+1 \leftrightarrow -c+1$ and $-b+1 \leftrightarrow -d+1$, and join a to d, b to c, $-a+1$ to $-d+1$, $-b+1$ to $-c+1$ (if the intersection point is on the mirror line, then there will only be one strand) as in Figure 5.1. Then $g_{\tau'} = t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,1b+1}$ (if the intersection point of $a \leftrightarrow c$ and $b \leftrightarrow d$ not on the mirrow line), or $g_{\tau'} = t_{a,b}g_\tau t_{a,b}$ (if the intersection point lies on the mirror line). In both cases, we have that $g_{\tau'} > g_\tau$.

Next assume $g_\tau < g_{\tau.}$. Denote the ordinary Bruhat order for $\tilde{S}^0_{4n}$ as $\leq_A$. Then there exist $a < b$ such that: (i) $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau \gtrdot_A t_{a,b}g_\tau \gtrdot_A g_\tau$ (if the strands $a \leftrightarrow c$ and $b \leftrightarrow d$ are not mirror symmetric), or (ii) $g_{\tau'} >_A t_{a,b}g_\tau \rhd_A g_\tau$ (if the strands $a \leftrightarrow c$ and $b \leftrightarrow d$ are mirror symmetric). The Case (ii) is corresponding to configuration (1) and (2) in Figure 5.1 and was proved in [Lam], Theorem 4.15. We only focus on Case (i).

For Case (i), let $c := g_\tau(a), d := g_\tau(b)$. Now claim $g_{\tau'} >_A g_\tau t_{a,b}t_{-a+1,-b+1}$. Define the group isomorphism $\phi : \tilde{S}^0_{4n} \to \tilde{S}^0_{4n}$ by $t_{i,i+1} \mapsto t_{i+2n,i+2n+1}$. By Lemma 5.8 , $g_\tau = ug_0u^{-1}$, $g_{\tau'} = vg_0v^{-1}$, where $u, v \in \tilde{S}^C_{2n} \subset \tilde{S}^0_{4n}$. Therefore $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau$ is equivalent to $ug_0u^{-1} >_A t_{-a+1,-b+1}t_{a,b}vg_0v^{-1}$, which implies $u\phi(u^{-1})g_0 >_A t_{-a+1,-b+1}t_{a,b}v\phi(v^{-1})g_0$. Hence $u\phi(u^{-1}) >_A t_{-a+1,-b+1}t_{a,b}v\phi(v^{-1})$. Taking the inverse of both side and multiply g_0 on the left side of both terms, we get $g_0\phi(u)u^{-1} >_A g_0\phi(v)v^{-1}t_{a,b}t_{-a+1,-b+1} \iff ug_0u^{-1} >_A vg_0v^{-1}t_{a,b}t_{-a+1,-b+1}$, which is $g_{\tau'} >_A g_\tau t_{a,b}t_{-a+1,-b+1}$.

Let $N = 4n$. We claim $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1}$. Note that modulo symmetry, the order of (a, b, c, d) has to be one of the configurations in Figure 5.1. For (a, b, c, d) as in configurations (3), (4), (5), (6), (8), (10), (12), (13), (14), (15), and (16), the proofs will be similar. Let's first assume it is configuration (8) of Figure 5.1. Then we have

$$-d+1 < -c+1 < -b+1 < -a+1 < a < b < c < d < -d+N$$

Let R_1 be the rectangle with corners $(-d+1+N, -b+2+N), (-d+1+N, -a+1+N), (-c+N, -b+2+N), (-c+N, -a+1+N)$, R_2 be the one with corners $(-b+1, -d+2+N), (-b+1, -c+1+N), (-a, -d+2+N), (-a, -c+1+N)$, R_3 be the one with corners $(a, c+1), (a, d), (b-1, c+1), (b-1, d)$, and R_4 be the one with corners $(c, a+1+N), (c, b+N), (d-1, a+1+N), (d-1, b+N)$. Then the rank matrix of $t_{-a+1,-b+1}t_{a,b}g_\tau$ will only increase by 1 in the rectangles R_3 and R_4 as well as its periodic shifts. The rank matrix of $g_\tau t_{a,b}t_{-a+1,-b+1}$ will only increase by 1 in the rectangles R_1 and R_2 as well as its periodic shifts. Note that R_1, R_2, R_3, R_4 and their periodic shifts will never intersect, which implies:

$$r_{g_{\tau'}}(i,j) \geq \max(r_{t_{-a+1,-b+1}t_{a,b}g_\tau}(i,j), r_{g_\tau t_{a,b}t_{-a+1,-b+1}}(i,j)) \geq r_{g_\tau}(i,j) \quad \forall i,j \in \mathbb{Z}$$

Hence, $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1}$. On the other hand $t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1} = g_{\tau''}$, where $\tau'' < \tau$ is in $\mathcal{MP}_n$ obtained by uncrossing the intersection of strands $a \leftrightarrow c$ and $b \leftrightarrow d$, as well as strands $(-a+1) \leftrightarrow (-c+1)$ and $(-b+1) \leftrightarrow (-d+1)$. Consequently $g_{\tau'} \geq g_{\tau''} > g_\tau$. By induction on $l_C(g_{\tau'}) - l_C(g_\tau)$, we have $\tau' \leq \tau''$. Thus, $\tau' < \tau$.

For (a, b, c, d) as in the rest of the configuration (7), (9), (11), (17), we will use a different different argument.

We will use configuration (7) to illustrate the argument, in which case we have:

$$-d+1 < -c+1 < -b+1 < -a+1 < a < b < c < d < -d+1+N$$

Let R_1 be the rectangle with corners $(-c+1, -a+2), (-c+1, -d+1+N), (-b, -a+2), (-b, -d+1+N)$, R_2 be the one with corners $(d-N, b+1), (d-N, c), (a-1, b+1), (a-1, c)$, R_3 be the one with corners $(b, d+1), (b, a+N), (c-1, d+1), (c-1, a+N)$, and R_4 be the one with corners $(-a+1, -c+2+N), (-a+1, -b+1+N), (-d+N, -c+2+N), (-d+N, -b+1+N)$. Then the rank matrix of $t_{-a+1,-b+1}t_{a,b}g_\tau$ will only increase by 1 in the rectangles R_1 and R_4 as well as its periodic shifts. The rank matrix of $g_\tau t_{a,b}t_{-a+1,-b+1}$ will only increase by 1 in the rectangles R_2 and R_3 as well as its periodic shifts. However, R_1 and R_2 intersect at rectangle R_5 with corners $(-c+1, b+1), (-c+1, c), (-b, b+1), (-b, c)$ as well as its periodic shifts, and R_3 and R_4 intersect at rectangle R_6 with corners $(b, -c+2+N), (b, -b+1+N), (c-1, -c+2+N), (c-1, -b+1+N)$ as well as its periodic shifts. Consider $t_{-b+1,-c+1}g_{\tau'}t_{-b+1,-c+1}$, the entries of the rank matrix of which decrease at regions R_5 and R_6 by 1. Note that $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau >_A g_\tau$ and $g_{\tau'} >_A g_\tau t_{a,b}t_{-a+1,-b+1} >_A g_\tau$. Therefore

$$r_{t_{-b+1,-c+1}g_{\tau'}t_{-b+1,-c+1}}(i,j) \geq r_{g_\tau}(i,j) \forall i,j \in \mathbb{Z}.$$

On the other hand $t_{-b+1,-c+1}g_{\tau'}t_{-b+1,-c+1} = g_{\tau''}$ where τ' is obtained from τ'' by uncrossing the intersection of strands $b(-c+1)$ and $(-b+1)c$. Thus, $\tau'' > \tau' \in \mathcal{MP}_n$. By induction on $l_C(g_{\tau'}) - l_C(g_\tau)$, we have $\tau'' \leq \tau$. Thus, $\tau' < \tau$. □

Note that the poset $\mathcal{MP}_n$ has a unique maximum element, and $\binom{2n}{n}$ minimum elements, which is the Catalan number of type B (see [CA]). Let $\widehat{\mathcal{MP}}_n$ denote $\mathcal{MP}_n$ with a minimum $\hat{0}$ adjoined, where we let $mc(\hat{0}) = -1$. Recall that a graded poset P with a unique maximum and a unique minimum, is **Eulerian** if for every interval $[x, y] \in P$ where $x < y$, the number of elements with odd rank in $[x, y]$ is equal to the number of elements with even rank in $[x, y]$. We have the following theorem.

Theorem 5.11. *$\widehat{\mathcal{MP}}_n$ is an Eulerian poset.*

We need some terminology and a few lemmas before proving the above theorem. For a subset $S \subset \widehat{\mathcal{MP}}_n$, we write $\chi(S) = \sum_{\tau \in S}(-1)^{mc(\tau)}$. We need to show that $\chi([\tau, \eta]) = 0$ for all $\tau < \eta$. Recall that $\tilde{S}_{2n}^C$ denote the poset of affine permutation of type C. By Theorem 5.10, we know that there is an injection $\rho : \mathcal{MP} \hookrightarrow \tilde{S}_{2n}^C, \tau \mapsto g_\tau$ such that $\mathcal{MP}_n$ is dual to an induced subposet of $\tilde{(S)}_{2n}^C$. For $f \subset \tilde{S}_{2n}^C$, let

$$D_L f := \{i \in \mathbb{Z}/2n\mathbb{Z} | s_i f < f\}, \qquad D_R f := \{i \in \mathbb{Z}/2n\mathbb{Z} | f s_i < f\},$$

be the left and right descent set of f. We have the following lemma.

Lemma 5.12 ([BB], Proposition 2.2.7)**.** *Suppose $f \leq g$ in $\tilde{S}_{2n}^C$.*

- *If $i \in D_L(g) \backslash D_L(f)$, then $f \leq s_i g$ and $s_i f \leq g$,*
- *If $i \in D_R(g) \backslash D_R(f)$, then $f \leq g s_i$ and $f s_i \leq g$.*

For $\tau \in \mathcal{MP}_n$, We label the medial strands of a representative of τ by $\{-2n+1,\dots,0,\dots,2n-1,2n\}$ and $i \in \{0,1,2,\dots,2n\}$, let

$$i \in \begin{cases} A(\tau), & \text{if the strands } i \text{ and } i+1 \text{ do not cross, and } -i+1 \text{ and } -i \text{ do not cross} \\ B(\tau), & \text{if the strands } i \text{ and } i+1 \text{ cross, and } -i+1 \text{ and } -i \text{ cross} \\ C(\tau), & \text{if } i \text{ is adjoined with } i+1, \text{ and } -i+1 \text{ is joined with } -i \end{cases}$$

Note that when $i=0$, and $i=2n$, the two mirror symmetric pairs of medial strands $\{i,i+1\}$ and $\{-i+1,-i\}$ will be the same.

Thus, we have $\{0,1,2,\dots,2n\} = A(\tau) \cup B(\tau) \cup C(\tau)$, where $\{0,1,2,\dots,2n\} \in \mathbb{Z}\backslash 4n\mathbb{Z}$. For $s_i \in \tilde{S}^C_{2n}$, $\tau \in \mathcal{MP}_n$, define $s_i \cdot \tau$ such that $g_{s_i\cdot\tau} = s_i g_\tau s_i$. Then by Thorem 5.10 the above conditions translate into the following

$$i \in \begin{cases} A(\tau), & \text{if } s_i g_\tau s_i < g_\tau, \text{ or equivalently } s_i \cdot \tau \gtrdot \tau, \\ B(\tau), & \text{if } s_i g_\tau s_i > g_\tau, \text{ or equivalently } s_i \cdot \tau \lessdot \tau, \\ C(\tau), & \text{if } s_i g_\tau s_i = g_\tau, \text{ or equivalently } s_i \cdot \tau = \tau, \end{cases}$$

For instance, if $i \in B(\tau)$, then $s_i \cdot \tau$ is obtained from τ by uncrossing pairs of medial strands i and $i+1$, and $-i+1$ and $-i$.

Lemma 5.13. *If $\tau < \sigma$, and $i \in A(\tau) \cap B(\sigma)$, then we have $s_i \cdot \tau \le \sigma$ and $\tau \le s_i \cdot \sigma$.*

Proof. Since $i \in A(\tau)$, we have $i \in D_R(g_\tau) \cap D_L(g_\tau)$. Similarly, since $i \notin B(\sigma)$, we have $i \in D_R(\sigma) \cap D_L(\sigma)$. Then we know $i \in D_R(g_\tau)\backslash D_R(g_\sigma)$. Thus, by Lemma 5.12, we have $g_\sigma s_i < g_\tau$. We also know $s_i \notin D_L(g_\sigma s_i)$ because $s_i g_\sigma s_i > g_\tau s_i$. Again, since $i \in D_L(g_\tau)\backslash D_L(g_\sigma s_i)$, by Lemma 5.12, we have $s_i g_\sigma s_i < g_\tau$, or equivalently $\tau < s_i \cdot \sigma$. Similarly, we can show $s_i \cdot \tau < \sigma$. □

The following lemma is trivially true by the definition of sets $A(\tau)$ and $C(\tau)$.

Lemma 5.14. *If $\tau \le \sigma$, and $i \in A(\tau)$, then $i \notin C(\sigma)$.*

Proof of Theorem 5.11. We first prove this theorem for the intever $[\tau,\sigma]$ of $\mathcal{MP}_n$, that is, $\tau \neq \hat{0}$. We will prove by descending induction on $mc(\tau) + mc(\sigma)$.

The base case is σ is the maximum element, and $mc(\tau) = n^2-1$. It is trivial. If $mc(\sigma) - mc(\tau) = 1$, it is also clear. Thus, we may assume $mc(\sigma) - mc(\tau) \ge 2$. Since τ is not maximal, $D_L(g_\tau)$ and $D_R(g_\tau)$ are not empty. So we can pick $i \in A(\tau)$. We have the following three cases:

Case 1: If $i \in A(\sigma)$, then we have

$$[\tau,\sigma] = [\tau, s_i\cdot\sigma]\backslash\{\delta | \tau < \delta < s_i\cdot\sigma, \delta \not\le \sigma\}.$$

We claim that

$$\{\delta|\tau < \delta < s_i\cdot\sigma, \delta \not\le \sigma\} = \{\delta | s_i\cdot\tau < \delta < s_i\cdot\sigma, \delta\not\le\sigma\}.$$

Suppose that $\tau \le \delta \le s_i\sigma$, and $\delta \not\le \sigma$. If $i \in A(\delta)$, then because $i \in A(s_i\cdot\sigma)$, we can apply Lemma 5.13 to $\delta < s_i\cdot\sigma$ and get $\sigma < s_i\cdot(s_i\cdot\sigma) = \sigma$, contradiction. On the other hand, $i \notin C(\delta)$ because of $\tau \le \delta$, $i \in A(\tau)$ and Lemma 5.14. Therefore, we have $i \in B(\delta)$, or equivalently $s_i\cdot\delta \lessdot \delta$. Since $i \in A(\tau)\cap B(\delta)$, applying Lemma 5.13, we get $\tau \le s_i\cdot\delta$. Thus, we proved the claim.

Furthermore, we have

$$\{\delta|s_i\cdot\tau < \delta < s_i\cdot\sigma, \delta\not\le\sigma\} = [s_i\cdot\tau, s_i\cdot\sigma]\backslash[s_i\cdot\tau,\sigma].$$

By induction, we have $\chi([\tau,\sigma]) = \chi([\tau, s_i\cdot\sigma]) - (\chi([s_i\cdot\tau, s_i\cdot\sigma]) - \chi([s_i\cdot\tau,\cdot\sigma])) = 0$. Note that the assumption $mc(\sigma) - mc(\tau) \ge 2$ implies that none of these intervals will contain only one element.

Case (2): $i \in B(\sigma)$. Let $\delta \in [\tau,\sigma]$. Since $i \in A(\tau)$, apply Lemma 5.14 on $\tau \le \delta$, we have $i \notin C(\delta)$. If $i \in A(\sigma)$, apply Lemma 5.13 on $\delta \le \sigma$, and if $i \in B(\sigma)$, apply Lemma 5.13 on $\tau < \delta$. We then get $s_i \cdot \delta \in [\tau, \sigma]$. Hence, we construct an involution of elements in $[\tau, \sigma]$, where the parity of the rank of the elements gets swapped. Thus,

$$\chi([\tau, \sigma]) = 0.$$

Case (3): $i \in C(\sigma)$. Since $i \in A(\tau)$, apply Lemma 5.14 on $\tau \leq \sigma$, we know $i \notin C(\sigma)$. So this case is vacuous.

Therefore, we have proved that $\chi([\tau, \sigma]) = 0$ for intervals $\tau < \sigma$ where $\tau \neq \hat{0}$.

Next assume that $\tau = \hat{0}$, and $\tau \leq \sigma$. We can further assume $mc(\sigma) \geq 1$. Thus, $B(\sigma)$ is nonempty. Pick $i \in B(\tau)$. By applying Lemma 5.13 on $\tau \leq \sigma$, we construct an involution on $\{\delta \in [\hat{0}, \sigma] | \ i \notin C(\delta)\}$. This involution will swap the parity of $mc(\delta)$. Now let $S = \{\delta \in [\hat{0}, \sigma) | \ i \in C(\delta) \text{ or } \delta = \hat{0}\}$. We claim that S is an interval with a unique maximal element. Then we may delete the strands connecting i and $i+1$ and use induction.

We prove the claim by constructing explicitly the maximal element ν. Let G be a representative medial graph of $\sigma \in \mathcal{MP}_n$. The strands l_i and l_{i+1} starting at i and $i+1$ respectively cross each other at some point p since $i \in B(\sigma)$. Similarly, the mirror image strands l_{-i+1} and l_{-i} starting at $-i+1$ and $-i$ respectively will cross at point p'. Assume these strands cross before intersecting with other strands. Let G' be obtained from G by uncrossing p and p' such that i is matched with $i+1$ and $-i+1$ is matched with $-i$. Let ν be the medial pairing of G'.

We need to show that for $\delta \in S$, we have $\delta \leq \nu$. Let H be a lensless medial graph representing δ. Then by Theorem 5.7, H can be obtained from G by uncrossing some subset $\mathcal{A}$ of the mirror symmetric pairs of crossings of G. Because $i \in C(\delta)$, we must have $\{p, p'\} \in \mathcal{A}$. If $\{p, p'\}$ are resolved in H in the same way as in G', then H can be obtained from G' by uncrossing a number of mirror symmetric pairs of crossings, which implies $\delta \leq \nu$. If $\{p, p'\}$ are resolved in H in the different direction to the one in G', then let H' be obtained from H by uncrossing the pair $\{p, p'\}$ of mirror symmetric crossing in another direction. We observe that H' and H both represent δ. H' may contain a closed loop in the interior which can be removed. Thus, we complete the proof of $S = [\hat{0}, \sigma]$, and consequently the theorem. □

Similar to the case of cactus network, we conjecture that the partial order on $\mathcal{MP}_n$ is also the closure partial order of the decomposition $ME_n = \bigsqcup_{\tau \in \mathcal{MP}_n} ME_\tau$:

Conjecture 5.15. *We have*

$$ME_\tau = \bigsqcup_{\tau' \leq \tau} ME_\tau.$$

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