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YORK COLLEGE
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Solving Mathematical Problems:

Solution to $\sum_{n\geq 0}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^{n+j}}{(n+j)!}\right)$ where j ϵ N, j $\geq$ 1

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Abstract

1

The Riemann hypothesis is one of the major problems that are of intense interest to mathematicians and physicist alike. It is to prove or disprove this hypothesis. During the course of this project the Riemann Zeta function is being investigated. The goal is to further understand the zeta function, and hopefully contribute meaningful ideas. Furthermore, another objective of this research is solving mathematical problems from various journals of interest. They include problems involving exponential functions, hyperbolic functions etc. Presented, is one of the problems that was solved.

Introduction

2

$$\sum_{n\geq 0}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^{n+j}}{(n+j)!}\right)$$ where j ϵ N, j $\geq$ 1

At first sight, one can easily conclude that the solution to this problem is zero (knowing that the series of e^x subtracted from e^x is zero). However, it is the terms of the series that go to zero as n tends towards infinity. We therefore have to find a general expression that shows the solution for all n. This problem was proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI in the College Mathematics Journal.

Method

3

In solving the above problem, a mathematica program was first used to find the sum where the limit of n =10, and j = 0. The answer will be in the form of F(x). After that, F(1), F(2), F(3) ...etc was calculated until a pattern was identified and a formula deduced. It was verified algebraically. In addition, an expression was found algebraically when j = 1, 2, 3 …y. Finally, a general expression was worked out for any j.

PROOF

4

Showing the results numerically

First, the sum as n is varied from 1 to10 is calculated using mathematica.

$$\sum_{n=0}^{n=10}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^{n+j}}{(n+j)!}\right), \quad j = 0$$

The solution is represented below:

$$-11 + 11e^x - 10x - \frac{9x^2}{2} - \frac{4x^3}{3} - \frac{7x^4}{24} - \frac{x^6}{144} - \frac{x^7}{1260} - \frac{x^8}{13440} - \frac{x^9}{181440} - \frac{x^{10}}{362880}$$

As seen, the above expression is represented as a F(x). F(1), F(2), F(3), ….F(n) until a pattern is found.

F(1) = 2.71828 e^1

F(2) = 14.778 $2e^2$

F(3) = 60.2548 $3e^3$

F(4) = 218.332 $4e^4$

F(5) = 740.805 $5e^5$

We can see that the expression corresponds to xe^x when j =0. We can move forward to verify this result algebraically and find a general expression for all j.

Proving the results algebraically

Let

$$S_j = \sum_{n\geq 0}\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots - \frac{x^{n+j}}{(n+j)!}\right)$$

Sum as n goes to y, when j = 0

$$S_0 = (e^x - 1) + (e^x - 1 - x) + \left(e^x - 1 - x - \frac{x^2}{2!}\right) + \left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}\right) + \ldots.. + \left(e^x - 1 - x - \frac{x^2}{2!} - \ldots \frac{x^y}{y!}\right)$$

The series of e^x could be substituted into the above equation. The result will give:

$$S_0 = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots\right) + \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots\right) +$$
$$\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots\right) + \left(\frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots\right) + \ldots..$$

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$$= x + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} + \frac{5x^5}{5!} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= xe^x$$

$$S_1 = (e^x - 1 - x) + \left(e^x - 1 - x - \frac{x^2}{2!}\right) + \left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}\right) +$$
$$\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}\right) + \ldots\ldots\ldots\ldots$$
$$= xe^x - e^x + 1$$
$$= (x-1)e^x + 1$$

$$S_2 = \left(e^x - 1 - x - \frac{x^2}{2!}\right) + \left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}\right) + \left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}\right) +$$
$$\left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{x^5}{5!}\right) + \ldots\ldots\ldots$$
$$= (x-2)e^x + 2 + x$$

It would be observed that $S_{j+1} = S_j -$ (the first term of the S_j series)

Therefore,

$$S_3 = ((x-2)e^x + 2 + x) - (e^x - 1 - x)$$
$$= (x-3)e^x + 3 + 2x + \frac{x^2}{2!}$$
$$S_4 = (x-3)e^x + 3 + 2x + \frac{x^2}{2!} - \left(e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!}\right)$$
$$= (x-4)e^x + 4 + 3x + x^2 + \frac{x^3}{3!}$$

Looking at the above expressions we can derive a general expression:

$$(x-j)e^x + j + (j-1)x + (j-2)\frac{x^2}{2!} + (j-3)\frac{x^3}{3!} \ldots\ldots. \frac{2x^{j-2}}{(j-2)!} + \frac{x^{j-1}}{(j-1)!}$$