Online Supplement For *Asynchronous and Error-prone Longitudinal Data Analysis via Functional Calibration*

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Appendix A: Technical Proofs

A.1 *Preliminaries*

Recall that we use the subscript $*$ to denote the covariates and eigenfunctions evaluated on the same time points as the response, and $\mathbf{X}_{*i} = \{X_i(T_{i1}), \ldots, X_i(T_{im_{y,i}})\}^{\mathrm{T}}$, $\mathbf{\Psi}_i = (\boldsymbol{\psi}_{i1}, \ldots, \boldsymbol{\psi}_{iq})$ and $\mathbf{\Psi}_{*i} = (\boldsymbol{\psi}_{*i1}, \ldots, \boldsymbol{\psi}_{*iq})$, where $\boldsymbol{\psi}_{ik} = \{\psi_k(S_{i1}), \ldots, \psi_k(S_{im_{x,i}})\}^{\mathrm{T}}$ and $\boldsymbol{\psi}_{*ik} = \{\psi_k(T_{i1}), \ldots, \psi_k(T_{im_{y,i}})\}^{\mathrm{T}}$.

Put $\mathbf{X}_* = (\mathbf{X}_{*1}^{\mathrm{T}}, \ldots, \mathbf{X}_{*n}^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{\Lambda} = \mathrm{diag}(\omega_1, \ldots, \omega_K)$, and the FPCA score vectors as $\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{iq})^{\mathrm{T}}$. and the observed covariance matrix is $\mathbf{\Sigma}_i = \mathbf{\Psi}_i \mathbf{\Lambda} \mathbf{\Psi}_i^{\mathrm{T}} + \sigma_u^2 \mathbf{I}$. Specifically, the conditional mean and estimator for $\mathbf{X}_{*i}$ are

$$\widetilde{\mathbf{X}}_{*i} = \mathbf{\Psi}_{*i}\widetilde{\boldsymbol{\xi}}_i = \mathbf{\Psi}_{*i}\mathbf{\Lambda}\mathbf{\Psi}_i^{\mathrm{T}}\left(\mathbf{\Psi}_i\mathbf{\Lambda}\mathbf{\Psi}_i^{\mathrm{T}} + \sigma_u^2\mathbf{I}\right)^{-1}\mathbf{W}_i \quad \text{(A.1)}$$

$$\widehat{\mathbf{X}}_{*i} = \widehat{\mathbf{\Psi}}_{*i}\widehat{\boldsymbol{\xi}}_i = \widehat{\mathbf{\Psi}}_{*i}\widehat{\mathbf{\Lambda}}\widehat{\mathbf{\Psi}}_i^{\mathrm{T}}\left(\widehat{\mathbf{\Psi}}_i\widehat{\mathbf{\Lambda}}\widehat{\mathbf{\Psi}}_i^{\mathrm{T}} + \widehat{\sigma}_u^2\mathbf{I}\right)^{-1}\mathbf{W}_i. \quad \text{(A.2)}$$

Define $\delta_{n1}(h) = \{h^{-1}\log n/n\}^{1/2}$, $\delta_{n2}(h) = \{h^{-2}\log n/n\}^{1/2}$, and $\zeta_n(h) = \sqrt{n}h^2 + h^{1/2} + h^{-1/2}\delta_{n2}(h)$.

LEMMA 1: *Under assumptions described in Section 4.1, for* $t \in \mathcal{T}$,

$$\widehat{\psi}_k(t) - \psi_k(t) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_{x,i}}\sum_{j\neq j'} u_{i,jj'}^{*}\mathcal{G}_k\left(S_{ij}, S_{ij'}, t\right)$$

$$
\begin{aligned}
&+\omega_k^{-1}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_{x,i}}\sum_{j\neq j'}u^*_{i,jj'}\frac{K_{h_R}\left(S_{ij'}-t\right)\psi_k\left(S_{ij}\right)}{f_S(S_{ij})f_S(t)}\\
&+O_p[h_R^2+(nh_R)^{-1/2}\{h_R+\delta_{n2}(h_R)\}],
\end{aligned}
$$

$$
\widehat{\omega}_k-\omega_k = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_{x,i}}\sum_{j'\neq j}u^*_{i,jj'}\frac{\psi_k(S_{ij})\psi_k(S_{ij'})}{f_S(S_{ij})f_S(S_{ij'})}+O_p[h_R^2+n^{-1/2}\{h_R+\delta_{n2}(h_R)\}],
$$

$k=1,\ldots,q,$ *where* $u^*_{i,jj'}=W_{ij}W_{ij'}-R(S_{ij},S_{ij'}), M_{x,i}=m_{x,i}(m_{x,i}-1)$,

$$
\mathcal{G}_k\left(s_1,s_2,s_3\right)=\sum_{\substack{k'=1\\k'\neq k}}^{q}\frac{\omega_{k'}\psi_{k'}\left(s_3\right)}{\left(\omega_k-\omega_{k'}\right)\omega_k}\times\left\{\frac{\psi_k\left(s_1\right)\psi_{k'}\left(s_2\right)}{f_S(s_1)f_S(s_2)}\right\}-\omega_k^{-1}\psi_k\left(s_3\right)\left\{\frac{\psi_k\left(s_1\right)\psi_k\left(s_2\right)}{f_S(s_1)f_S(s_2)}\right\}.
$$

Proof. The asymptotic expansion for $\widehat{\psi}_k(t)-\psi_k(t)$ is a direct result of Lemma S.3.1 in Li et al. (2013) by letting $\mu(t)=0$, and the asymptotic expansion for $\widehat{\omega}_k-\omega_k$ is on page 3349 in Li and Hsing (2010).

A.2 *The Proof for Theorem 1*

Assuming both $X(t)$ and $Y(t)$ have been centered so that $\beta_0=0$, we have

$$
\sqrt{n}(\widehat{\beta}_1-\beta_1)=\left\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i^2(T_{ij})\right\}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i(T_{ij})[\{X_i(T_{ij})-\widehat{X}_i(T_{ij})\}\beta_1+\epsilon_i(T_{ij})]\right).
$$

By (14), the denominator of the expression above is

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i^2(T_{ij}) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\{\widehat{X}_i^2(T_{ij})-\widetilde{X}_i^2(T_{ij})\}+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widetilde{X}_i^2(T_{ij})\\
&= \frac{1}{n}\sum_{i=1}^{n}\widetilde{\mathbf{X}}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}+o_p(1)\\
&\xrightarrow{p} \gamma_x, \qquad \text{(A.3)}
\end{aligned}
$$

where $\gamma_x=\mathrm{E}(\widetilde{\mathbf{X}}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i})$ as defined in (15). Define

$$
\begin{aligned}
\Delta_n &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i(T_{ij})[\{X_i(T_{ij})-\widehat{X}_i(T_{ij})\}\beta_1+\epsilon_i(T_{ij})]\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widetilde{X}_i(T_{ij})[\{X_i(T_{ij})-\widetilde{X}_i(T_{ij})\}\beta_1+\epsilon_i(T_{ij})]\\
&= \frac{1}{\sqrt{n}}\widehat{\mathbf{X}}_*^{\mathrm{T}}\left\{\left(\mathbf{X}_*-\widehat{\mathbf{X}}_*\right)\beta_1+\boldsymbol{\epsilon}\right\}-\frac{1}{\sqrt{n}}\widetilde{\mathbf{X}}_*^{\mathrm{T}}\left\{\left(\mathbf{X}_*-\widetilde{\mathbf{X}}_*\right)\beta_1+\boldsymbol{\epsilon}\right\}\\
&= \mathcal{R}_{1,n}+\mathcal{R}_{2,n}+\mathcal{R}_{3,n}+\mathcal{R}_{4,n}, \qquad \text{(A.4)}
\end{aligned}
$$

where

$$\begin{aligned}
\mathcal{R}_{1,n} &= -\beta_1 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\widehat{\mathbf{X}}_{*i} - \widetilde{\mathbf{X}}_{*i})^{\mathrm{T}} \widetilde{\mathbf{X}}_{*i}, \\
\mathcal{R}_{2,n} &= \beta_1 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\widehat{\mathbf{X}}_{*i} - \widetilde{\mathbf{X}}_{*i})^{\mathrm{T}} (\mathbf{X}_{*i} - \widetilde{\mathbf{X}}_{*i}), \\
\mathcal{R}_{3,n} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\widehat{\mathbf{X}}_{*i} - \widetilde{\mathbf{X}}_{*i})^{\mathrm{T}} \boldsymbol{\epsilon}_i, \\
\mathcal{R}_{4,n} &= -\beta_1 \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\widehat{\mathbf{X}}_{*i} - \widetilde{\mathbf{X}}_{*i})^{\mathrm{T}} (\widehat{\mathbf{X}}_{*i} - \widetilde{\mathbf{X}}_{*i}).
\end{aligned}$$

Lemma 2 shows that $\mathcal{R}_{1,n} = O_p(1)$ and provides its asymptotic expansion. Following similar derivations, $\mathcal{R}_{2,n}$ has a similar decomposition as (A.6) except that $\widetilde{\mathbf{X}}_{*i}$ is replaced by $\mathbf{X}_{*i} - \widetilde{\mathbf{X}}_{*i}$. By lengthy derivations similar to Lemma 2 and using the fact $\mathrm{E}\{(\mathbf{X}_{*i} - \widetilde{\mathbf{X}}_{*i})\mathbf{W}_i^{\mathrm{T}} \mid \mathbf{T}_i, \mathbf{S}_i\} = \mathbf{0}$, we can show $\mathcal{R}_{2,n} = o_p(1)$. By (14) and the fact that $\boldsymbol{\epsilon}_i$ is uncorrelated with $\mathbf{W}_i$, we have $R_{3,n} = o_p(1)$. In addition, $\mathcal{R}_{4,n} = O[\sqrt{n} \times \{h_R^4 + \log n/(nh_R) + h_V^4 + (\log n)^2/(nh_V)^2\}] = o(1)$ a.s. under Assumption (C.4). Combining arguments above,

$$\begin{aligned}
\sqrt{n}(\widehat{\beta}_1 - \beta_1) &= \frac{1}{\gamma_x \sqrt{n}} \sum_{i=1}^{n} \Bigg(\sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij}) [\{X_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}\beta_1 + \epsilon_i(T_{ij})] \\
&\qquad + \frac{\beta_1}{M_{x,i}} \sum_{j \neq j'} u^*_{i,jj'} \mathcal{A}(S_{ij}, S_{i'j'}) \Bigg) + o_p(1) \\
&:= \frac{1}{\gamma_x \sqrt{n}} \sum_{i=1}^{n} \Big(\mathcal{E}_{i1} + \beta_1 \mathcal{E}_{i2} + \beta_1 \mathcal{E}_{i3} \Big) + o_p(1),
\end{aligned}$$

where $\mathcal{E}_{i1} = \sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij})\epsilon_i(T_{ij})$, $\mathcal{E}_{i2} = \sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij})\{X_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}$, and $\mathcal{E}_{i3} = \frac{1}{M_{x,i}} \sum_{j \neq j'} u^*_{i,jj'} \mathcal{A}(S_{ij}, S_{i'j'})$. We can verify $\mathrm{E}\{(\mathbf{X}_{*i} - \widetilde{\mathbf{X}}_{*i})\mathbf{W}_i^{\mathrm{T}} \mid \mathbf{T}_i, \mathbf{S}_i\} = \mathbf{0}$, which means $\mathrm{E}(\mathcal{E}_{i2}) = \mathrm{tr}[\mathrm{E}\{(\mathbf{X}_{*i} - \widetilde{\mathbf{X}}_{*i})\mathbf{W}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\Psi}_i \boldsymbol{\Lambda} \boldsymbol{\Psi}_{*i}^{\mathrm{T}}\}] = 0$. With that, it follows that $(\mathcal{E}_{i1}, \mathcal{E}_{i2}, \mathcal{E}_{i3})$ are zero-mean and independent across i, and that $\mathcal{E}_{i1}$ is uncorrelated with $(\mathcal{E}_{i2}, \mathcal{E}_{i3})$ because ϵ is independent of X and W. We have

$$\begin{aligned}
&\gamma_1 = \mathrm{Var}(\mathcal{E}_{i1}) = \mathrm{E}\Big\{ \mathrm{tr}(\boldsymbol{\Psi}_{*i} \boldsymbol{\Lambda} \boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\Psi}_i \boldsymbol{\Lambda} \boldsymbol{\Psi}_{*i}^{\mathrm{T}} \boldsymbol{\Omega}_i) \Big\}, \\
&\gamma_2 = \mathrm{Var}(\mathcal{E}_{i2} + \mathcal{E}_{i3}) = \mathrm{Var}\Bigg[\sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij}) \{X_i(T_{ij}) - \widetilde{X}_i(T_{ij})\} + \frac{1}{M_{x,i}} \sum_{j \neq j'} u^*_{i,jj'} \mathcal{A}(S_{ij}, S_{i'j'}) \Bigg],
\end{aligned}$$

where $\boldsymbol{\Omega}_i = \mathrm{E}(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^T)$, then by the central limit theory

$$\sqrt{n}(\widehat{\beta}_1 - \beta_1) \xrightarrow{d} \mathrm{Normal}\{0, (\gamma_1 + \beta_1^2\gamma_2)/\gamma_x^2\}.$$

Lemma 2: *Under assumptions described in Section 4.1,*

$$\mathcal{R}_{1,n} = \beta_1 \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{n} \frac{1}{M_{x,i}} \sum_{j \neq j'} u^*_{i,jj'} \mathcal{A}(S_{ij}, S_{ij'}) \right\} \{1 + o_p(1)\}, \tag{A.5}$$

where $\mathcal{A}(s_1, s_2) = -\sum_{k=1}^4 \mathcal{A}_k(s_1, s_2)$*,* $\mathcal{A}_k(s_1, s_2)$*,* $k = 1, \ldots, 4$*, are defined in (A.8), (A.10), (A.12) and (A.14), respectively.*

Proof. We can rewrite $\mathcal{R}_{1,n} = -\beta_1(\mathcal{R}_{11,n} + \mathcal{R}_{12,n} + \mathcal{R}_{13,n} + \mathcal{R}_{14,n})$, where

$$\begin{aligned}
\mathcal{R}_{11,n} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \left(\widehat{\boldsymbol{\Psi}}_{*i} - \boldsymbol{\Psi}_{*i} \right) \boldsymbol{\Lambda} \boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1} \mathbf{W}_i \right\}^{\mathrm{T}} \widetilde{\mathbf{X}}_{*i}, \\
\mathcal{R}_{12,n} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \widehat{\boldsymbol{\Psi}}_{*i} \left(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda} \right) \boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1} \mathbf{W}_i \right\}^{\mathrm{T}} \widetilde{\mathbf{X}}_{*i}, \\
\mathcal{R}_{13,n} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \widehat{\boldsymbol{\Psi}}_{*i} \widehat{\boldsymbol{\Lambda}} \left(\widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}} - \boldsymbol{\Psi}_i^{\mathrm{T}} \right) \boldsymbol{\Sigma}_i^{-1} \mathbf{W}_i \right\}^{\mathrm{T}} \widetilde{\mathbf{X}}_{*i}, \\
\mathcal{R}_{11,n} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \widehat{\boldsymbol{\Psi}}_{*i} \widehat{\boldsymbol{\Lambda}} \widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}} \left(\widehat{\boldsymbol{\Sigma}}_i^{-1} - \boldsymbol{\Sigma}_i^{-1} \right) \mathbf{W}_i \right\}^{\mathrm{T}} \widetilde{\mathbf{X}}_{*i}.
\end{aligned} \tag{A.6}$$

Given a time vector $\mathbf{V} = (V_1, \ldots, V_m)^T$, define

$$\mathbf{A}_1(s_1, s_2, \mathbf{V}) = \mathcal{G}(s_1, s_2, \mathbf{V}) + \left\{ \frac{K_{h_R}(s_2 - V_1)}{f_S(V_1)}, \ldots, \frac{K_{h_R}(s_2 - V_m)}{f_S(V_m)} \right\}^{\mathrm{T}} \mathbf{A}_1^*(s_1) \tag{A.7}$$

where $\mathcal{G}(s_1, s_2, \mathbf{V}) = \{\mathcal{G}_1(s_1, s_2, \mathbf{V}), \ldots, \mathcal{G}_q(s_1, s_2, \mathbf{V})\}$, $\mathcal{G}_k(s_1, s_2, \mathbf{V}) = \{\mathcal{G}_k(s_1, s_2, V_1), \ldots, \mathcal{G}_k(s_1, s_2, V_m)\}^T$, $\mathcal{G}_k(s_1, s_2, V)$ defined in Lemma 1, and

$$\mathbf{A}_1^*(s_1) = \left\{ \frac{\psi_1(s_1)}{f_S(s_1)\omega_1}, \ldots, \frac{\psi_q(s_1)}{f_S(s_1)\omega_q} \right\}.$$

In addition, define

$$\mathcal{A}_1(s_1, s_2) = \mathrm{tr}\left(\mathrm{E}\left(\boldsymbol{\Lambda} \boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\Psi}_i \boldsymbol{\Lambda} \right) \left[\mathrm{E}\left\{ \boldsymbol{\Psi}_{*i}^{\mathrm{T}} \mathcal{G}(s_1, s_2, \mathbf{T}_i) \right\} + \mathrm{E}(m_{y,i}) \boldsymbol{\psi}(s_2) \mathbf{A}_1^*(s_1) \right] \right). \tag{A.8}$$

By Lemma 1,

$$\mathcal{R}_{11,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{W}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\Psi}_i \boldsymbol{\Lambda} \left(\widehat{\boldsymbol{\Psi}}_{*i} - \boldsymbol{\Psi}_{*i} \right)^{\mathrm{T}} \widetilde{\mathbf{X}}_{*i}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\left\{\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{T}_i)^{\mathrm{T}}\right\}\widetilde{\mathbf{X}}_{*i}+O_p\{\zeta_n(h_R)\}\\
&= \frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{T}_i)^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}+O_p\{\zeta_n(h_R)\}\\
&= \left[\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\mathrm{E}\left\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\mathbf{A}_1(s_1,s_2,\mathbf{T}_i)^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\big|_{s_1=S_{i'l},s_2=S_{i'l'}}\right]\\
&\qquad\qquad\times\{1+o_p(1)\}+O_p\{\zeta_n(h_R)\},\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\mathcal{A}_1(S_{i'l},S_{i'l'})\right\}\{1+o_p(1)\}+O_p\{\zeta_n(h_R)\}, \qquad \text{(A.9)}
\end{aligned}
$$

and $\zeta_n(h_R)=o(1)$ by (C.4).

Put

$$
\mathbf{A}_2(s_1,s_2)=\mathrm{diag}\left\{\frac{\psi_1(s_1)\psi_1(s_2)}{f_S(s_1)f_S(s_2)},\ldots,\frac{\psi_q(s_1)\psi_q(s_2)}{f_S(s_1)f_S(s_2)}\right\}, \qquad \text{(A.10)}
$$

$$
\mathcal{A}_2(s_1,s_2)=\mathrm{E}\left\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mathbf{A}_2(s_1,s_2)\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}=\mathrm{tr}\left\{\mathbf{A}_2(s_1,s_2)\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\right)\right\},
$$

then by Lemma 1 and Condition (C.4),

$$
\begin{aligned}
\mathcal{R}_{12,n} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\widehat{\boldsymbol{\Psi}}_{*i}\left(\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\right)\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i,ll'}\mathbf{A}_2(S_{i'l},S_{i'l'})\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\}\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u^*_{i',ll'}\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mathbf{A}_2(S_{i'l},S_{i'l'})\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\}\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u^*_{i',ll'}\mathcal{A}_2(S_{i'l},S_{i'l'})\right\}\{1+o_p(1)\}. \qquad \text{(A.11)}
\end{aligned}
$$

Next, define

$$
\begin{aligned}
\mathcal{A}_3(s_1,s_2) &= \mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Lambda}\mathcal{G}(s_1,s_2,\mathbf{S}_i)^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\right\}\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\right)\right] \qquad \text{(A.12)}\\
&\quad+\mathrm{tr}\left[f_S^{-1}(s_1)\boldsymbol{\psi}(s_1)\mathrm{E}\left\{\sum_{j=1}^{m_{x,i}}\mathrm{E}\left(\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mid S_{ij}=s_2,m_{x,i}\right)\right\}\boldsymbol{\Lambda}\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\right)\right].
\end{aligned}
$$

where $\mathcal{G}(s_1,s_2,\mathbf{T}_i)$ is defined in (A.7), and $\mathbf{e}_j$ is a directional vector of length $m_{x,i}$ with all elements being zero, except that the jth is 1. Following similar derivations as in (A.9), we have

$$\begin{aligned}
\mathcal{R}_{13,n} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left(\widehat{\boldsymbol{\Psi}}_i-\boldsymbol{\Psi}_i\right)\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{\Psi}}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{S}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\}\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{S}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\}\}\\
&= \left[\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\mathrm{E}\left\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{A}_1(s_1,s_2,\mathbf{S}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}|_{s_1=S_{i'l},s_2=S_{i'l'}}\right]\\
&\qquad\times\{1+o_p(1)\}\}\\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u^*_{i',ll'}\mathcal{A}_3(S_{i'l},S_{i'l'})\right\}\{1+o_p(1)\}\}. \qquad (A.13)
\end{aligned}$$

The last equation above follows because

$$\begin{aligned}
&\mathrm{E}\left\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{A}_1(s_1,s_2,\mathbf{S}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\\
=\;&\mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Lambda}\mathbf{A}_1(s_1,s_2,\mathbf{S}_i)^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\right\}\right]\\
=\;&\mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Lambda}\mathcal{G}(s_1,s_2,\mathbf{S}_i)^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\right\}\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\right)\right]\\
&+\mathrm{tr}\left[f_S^{-1}(s_1)\boldsymbol{\psi}(s_1)^{\mathrm{T}}\mathrm{E}\left\{\sum_{j=1}^{m_{x,i}}\mathrm{E}\left(\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mid S_{ij}=s_2,m_{x,i}\right)\right\}\boldsymbol{\Lambda}\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\right)\right]+O(h_R^2).
\end{aligned}$$

For the last term (A.6),

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\widehat{\boldsymbol{\Psi}}_{*i}\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}}\left(\widehat{\boldsymbol{\Sigma}}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{W}_i\right\}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\left(\widehat{\boldsymbol{\Sigma}}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\right)\widehat{\boldsymbol{\Psi}}_i\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{\Psi}}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}.$$

Finally, define

$$\begin{aligned}
\mathcal{A}_4(s_1,s_2)=&-2\mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\mathcal{G}(s_1,s_2,\mathbf{S}_i)^{\mathrm{T}}\right\}\right]\\
&-2\mathrm{E}\left[\sum_{j=1}^{m_{x,i}}\mathrm{E}\left\{\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Lambda}\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\right)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mid S_{ij}=s_2,m_{x,i}\right\}\right]\frac{\boldsymbol{\psi}(s_1)}{f_S(s_1)}\\
&-\mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\boldsymbol{\Psi}_{*i}\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mathbf{A}_2(s_1,s_2)\boldsymbol{\Psi}_i^{\mathrm{T}}\right\}\right]. \qquad (A.14)
\end{aligned}$$

By matrix taylor expansion, $\widehat{\boldsymbol{\Sigma}}_i^{-1}=\boldsymbol{\Sigma}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\left(\widehat{\boldsymbol{\Sigma}}_i-\boldsymbol{\Sigma}_i\right)\boldsymbol{\Sigma}_i^{-1}\{1+o_p(1)\}$. Thus, by Lemma 1 and using similar calculations as for $\mathcal{R}_{11,n}$, we have

$$\mathcal{R}_{14,n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\widehat{\boldsymbol{\Psi}}_{*i}\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}}\left(\widehat{\boldsymbol{\Sigma}}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{W}_i\right\}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}$$

$$
\begin{aligned}
&= \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left(\boldsymbol{\Sigma}_i-\widehat{\boldsymbol{\Sigma}}_i\right)\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\} \\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left(\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}-\widehat{\boldsymbol{\Psi}}_i\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}}\right)\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\} \\
&\quad +(\sigma_u^2-\widehat{\sigma}_u^2)\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-2}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right\}\{1+o_p(1)\} \\
&= -\Bigg[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left\{(\widehat{\boldsymbol{\Psi}}_i-\boldsymbol{\Psi}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i(\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda})\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i\boldsymbol{\Lambda}(\widehat{\boldsymbol{\Psi}}_i-\boldsymbol{\Psi}_i)^{\mathrm{T}}\right\} \\
&\qquad \times\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\Bigg]\{1+o_p(1)\}+o_p(1) \\
&= -\Bigg[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u_{i',ll'}^{*}\Big\{\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{S}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i\mathbf{A}_2(S_{i'l},S_{i'l'})\boldsymbol{\Psi}_i^{\mathrm{T}} \\
&\qquad +\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{S}_i)^{\mathrm{T}}\Big\}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\Bigg]\{1+o_p(1)\}+o_p(1) \\
&= \left\{\frac{1}{\sqrt{n}}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u_{i',ll'}^{*}\mathcal{A}_4(S_{i'l},S_{i'l'})\right\}\{1+o_p(1)\}+o_p(1), \qquad \text{(A.15)}
\end{aligned}
$$

where the last equation is due to that

$$
\begin{aligned}
&-\mathrm{E}\left[\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left\{\mathbf{A}_1(s_1,s_2,\mathbf{S}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i\mathbf{A}_2(s_1,s_2)\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\mathbf{A}_1(s_1,s_2,\mathbf{S}_i)^{\mathrm{T}}\right\}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\widetilde{\mathbf{X}}_{*i}\right] \\
&= \mathcal{A}_4(s_1,s_2)+O(h_R^2).
\end{aligned}
$$

The asymptotic expansion of $\mathcal{R}_{1n}$ provided in the lemma is proven by combining (A.9), (A.11), (A.13) and (A.15).

A.3 *The Proof for Theorem 2*

Under the simplified setting X_i are mean zero random process, by simple algebra we have

$$
\widehat{\beta}_1(t)=\frac{S_2(t)R_0(t)-S_1(t)R_1(t)}{S_2(t)S_0(t)-S_1^2(t)},
$$

where

$$
S_r(t) \;=\; \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i^2(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \quad r=0,1,2,
$$

$$R_r(t) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \widehat{X}_i(T_{ij})Y_{ij}K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \quad r=0,1.$$

Denote $R_r^*(t) = R_r(t) - S_r(t)\beta_1(t) - S_{r+1}(t)h\beta_1'(t)$, $r = 0, 1$, then

$$\widehat{\beta}_1(t) - \beta_1(t) = \frac{S_2(t)R_0^*(t) - S_1(t)R_1^*(t)}{S_2(t)S_0(t) - S_1^2(t)}.$$

By (14) and using the general result from Lemma 2 in Li and Hsing (2010),

$$\begin{aligned} S_0(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \widetilde{X}_i^2(T_{ij})K_h(T_{ij}-t) + O\{h_R^2 + h_V^2 + \sqrt{\log n/(nh_R)} + \log n/(nh_V)\} \\ &= \bar{m}_y f_T(t)\Gamma_x(t) + O_p\{h^2 + \sqrt{\log n/(nh)} + h_R^2 + h_V^2 + \sqrt{\log n/(nh_R)} + \log n/(nh_V)\}, \end{aligned}$$

where $\Gamma_x(t) = \mathrm{Var}\{\widetilde{X}_i(t)\} = \boldsymbol{\psi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\mathrm{E}(\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i)\boldsymbol{\Lambda}\boldsymbol{\psi}(t)$ as defined in the theorem. Similarly, $S_1(t) = \{\Gamma_x'(t)f_T(t) + \Gamma_x(t)f_T'(t)\}\bar{m}_y\sigma_K^2 h + O_p\{h^3 + \sqrt{\log n/(nh)} + h_R^2 + h_V^2 + \sqrt{\log n/(nh_R)} + \log n/(nh_V)\}$ and $S_2(t) = \bar{m}_y\Gamma_x(t)f_T(t)\sigma_K^2 + O_p\{h^2 + \sqrt{\log n/(nh)} + h_R^2 + h_V^2 + \sqrt{\log n/(nh_R)} + \log n/(nh_V)\}$.

Next, we decompose R_r^* as

$$\begin{aligned} R_r^*(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \widehat{X}_i(T_{ij})\Big\{Y_{ij} - \widehat{X}_i(T_{ij})\beta_1(t) - \widehat{X}_i(T_{ij})(T_{ij}-t)\beta_1'(t)\Big\} \\ &\qquad \times K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r \\ &= \sum_{k=1}^{6} R_{r,k}^*(t), \end{aligned} \tag{A.16}$$

where

$$\begin{aligned} R_{r,1}^*(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \widehat{X}_i^2(T_{ij})\{\beta_1(T_{ij}) - \beta_1(t) - (T_{ij}-t)\beta_1'(t)\}K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \\ R_{r,2}^*(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij})\{\beta_1(T_{ij})X_i(T_{ij}) - \beta_1(T_{ij})\widetilde{X}_i(T_{ij}) + \epsilon_i(T_{ij})\}K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \\ R_{r,3}^*(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \{\widetilde{X}_i(T_{ij}) - \widehat{X}_i(T_{ij})\}\widetilde{X}_i(T_{ij})\beta_1(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \\ R_{r,4}^*(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \{\widehat{X}_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}\epsilon_i(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \\ R_{r,5}^*(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}} \{\widehat{X}_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}\{X_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}\beta_1(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r, \end{aligned}$$

$$R^*_{r,6}(t) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\{\widehat{X}_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}\{\widetilde{X}_i(T_{ij}) - \widehat{X}_i(T_{ij})\}\beta_1(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r.$$

By (14) and previous results for $S_r(t)$, using some straightforward calculations and the fact that $K_h(T_{ij}-t) = 0$ if $|T_{ij}-t| > h$,

$$\begin{aligned}
R^*_{0,1}(t) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i^2(T_{ij})\{\beta_1(T_{ij}) - \beta_1(t) - (T_{ij}-t)\beta_1'(t)\}K_h(T_{ij}-t) \\
&= \frac{1}{2}\beta_1^{(2)}(t)h^2\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_{y,i}}\widehat{X}_i^2(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^2\right]\{1+O(h)\} \\
&= \frac{1}{2}\beta_1^{(2)}(t)h^2\bar{m}_y f_T(t)\Gamma_x(t)\sigma_K^2[1 + O_p\{h + \sqrt{\log n/(nh)} + h_R^2 \\
&\quad +h_V^2 + \sqrt{\log n/(nh_R)} + \log n/(nh_V)\}].
\end{aligned}$$

By similar calculations, $R^*_{1,1}(t) = O_p[h^3 + h^2\sqrt{\log n/(nh)} + h^2\{h_R^2 + h_V^2 + \sqrt{\log n/(nh_R)} + \log n/(nh_V)\}]$ is of order $o_p\{(nh)^{-1/2}\}$ by conditions (C.4) and (C.7). In addition, it follows that $R^*_{r,4}(t)$ and $R^*_{r,5}(t)$ are both of order $O_p([h_R^2 + h_V^2 + \{\log n/(nh_R)\}^{1/2} + \log n/(nh_V)] \times \{\log n/(nh)\}^{1/2})$, which is $o_p\{(nh)^{-1/2}\}$, and $R^*_{r,6}(t) = h_R^4 + h_V^4 + \log n/(nh_R) + (\log n)^2/(nh_V)^2] = o_p\{(nh)^{-1/2}\}$ under Condition (C.4), for both $r = 0, 1$.

Define

$$\begin{aligned}
&\mathbf{K}_{r,*i}(t) = \mathrm{diag}\Big[K_h(T_{i1}-t)\{(T_{i1}-t)/h\}^r, \ldots, K_h(T_{im_{y,i}}-t)\{(T_{im_{y,i}}-t)/h\}^r\Big], \\
&\mathbf{K}_i^\dagger(s, \mathbf{T}_i) = \left\{\frac{K_{h_R}(s-T_{i1})}{f_S(T_{i1})}, \ldots, \frac{K_{h_R}(s-T_{im_{y,i}})}{f_S(T_{im_{y,i}})}\right\}^{\mathrm{T}}, \\
&\boldsymbol{\beta}_{*i} = \mathrm{diag}\Big\{\beta_1(T_{i1}), \ldots, \beta_1(T_{im_{y,i}})\Big\},
\end{aligned}$$

then we have

$$R^*_{r,3}(t) = -\sum_{k=1}^{4} R^*_{r,3k}(t) \tag{A.17}$$

where

$$\begin{aligned}
R^*_{r,31}(t) &= \frac{1}{n}\sum_{i=1}^{n}\left\{\left(\widehat{\boldsymbol{\Psi}}_{*i} - \boldsymbol{\Psi}_{*i}\right)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}, \\
R^*_{r,32}(t) &= \frac{1}{n}\sum_{i=1}^{n}\left\{\widehat{\boldsymbol{\Psi}}_{*i}\left(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}\right)\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i},
\end{aligned}$$

$$
\begin{aligned}
R^*_{r,33}(t) &= \frac{1}{n}\sum_{i=1}^{n}\left\{\widehat{\boldsymbol{\Psi}}_{*i}\widehat{\boldsymbol{\Lambda}}\left(\widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}}-\boldsymbol{\Psi}_i^{\mathrm{T}}\right)\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i},\\
R^*_{r,34}(t) &= \frac{1}{n}\sum_{i=1}^{n}\left\{\widehat{\boldsymbol{\Psi}}_{*i}\widehat{\boldsymbol{\Lambda}}\widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}}\left(\widehat{\boldsymbol{\Sigma}}_i^{-1}-\boldsymbol{\Sigma}_i^{-1}\right)\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}.
\end{aligned}
$$

Let $\mathbf{A}_1(s_1,s_2,\mathbf{T}_i)$ be defined as (A.7), using derivations similar to (A.9), by (C.4) we have

$$
\begin{aligned}
R^*_{r,31}(t) &= \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\left\{\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{T}_i)^{\mathrm{T}}\right\}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right]\\
&\qquad\times\{1+o_p(1)\}\\
&= \left[\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{T}_i)^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right\}\right]\\
&\qquad\times\{1+o_p(1)\}\\
&= \left[\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i f_S^{-1}(S_{i'l})\boldsymbol{\psi}(S_{i'l})\mathbf{K}_i^{\dagger}(S_{i'l'},\mathbf{T}_i)^{\mathrm{T}}\right.\right.\\
&\qquad\left.\left.\times\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right\}+O_p(n^{-1/2})\right]\times\{1+o_p(1)\}.
\end{aligned}
$$

Define

$$
\mathcal{Q}(s,t)=\boldsymbol{\psi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\mathrm{E}(\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i)\boldsymbol{\psi}(s)f_T(t)/\{f_S(s)f_S(t)\}, \tag{A.18}
$$

then

$$
\begin{aligned}
&\mathrm{E}\left\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i f_S^{-1}(s_1)\boldsymbol{\psi}(s_1)\mathbf{K}_i^{\dagger}(s_2,\mathbf{T}_i)^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right\}\\
=\;& \mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Psi}_{*i}\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i f_S^{-1}(s_1)\boldsymbol{\psi}(s_1)\mathbf{K}_i^{\dagger}(s_2,\mathbf{T}_i)^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\right\}\right]\\
=\;& \mathrm{E}\left[\sum_{j=1}^{m_{y,i}}\boldsymbol{\psi}^{\mathrm{T}}(T_{ij})\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\psi}(s_1)\frac{K_{h_R}(s_2-T_{ij})}{f_S(s_1)f_S(T_{ij})}\beta_1(T_{ij})K_h(T_{ij}-t)\{(T_{ij}-t)/h\}^r\right]\\
=\;& \bar{m}_y\int_{\mathcal{T}}\mathcal{Q}(s_1,x)K_{h_R}(s_2-x)\beta_1(x)K_h(x-t)\{(x-t)/h\}^r dx,\\
=\;& \bar{m}_y\int_{\mathcal{T}}\mathcal{Q}(s_1,uh_R+s_2)K(u)\beta_1(uh_R+s_2)K_h(uh_R+s_2-t)\{(uh_R+s_2-t)/h\}^r du\\
=\;& \bar{m}_y\int_{\mathcal{T}}\mathcal{Q}(s_1,s_2)\beta_1(s_2)K(u)K_h(uh_R+s_2-t)\{(uh_R+s_2-t)/h\}^r du\{1+o(1)\}\\
=\;& \bar{m}_y\mathcal{Q}(s_1,s_2)\beta_1(s_2)K_h(s_2-t)\left\{\frac{s_2-t}{h}\right\}^r\{1+o(1)\},
\end{aligned}
$$

where the last equation is due to assumption $h_R/h=o(1)$ in Condition (C.7). Define

$$
\mathcal{A}_1^{\dagger}(s_1,s_2)=\bar{m}_y\mathcal{Q}(s_1,s_2)\beta_1(s_2), \tag{A.19}
$$

then

$$\begin{aligned} R^*_{r,31}(t) &= \left\{\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'} u^*_{i',ll'}\mathcal{A}^{\dagger}_1(S_{i'l},S_{i'l'})K_h(S_{i'l'}-t)\left(\frac{S_{i'l'}-t}{h}\right)^r\right\}\{1+o_p(1)\} \\ &= O_p\{(nh)^{-1/2}\} \end{aligned} \quad \text{(A.20)}$$

for both $r=0$ and 1. Next, similar to (A.11),

$$\begin{aligned} R^*_{r,32}(t) &= \left[\frac{1}{n}\sum_{i=1}^{n}\left\{\boldsymbol{\Psi}_{*i}\left(\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\right)\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right]\{1+o_p(1)\} \\ &= \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\left\{\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i,ll'}\mathbf{A}_2(S_{i'l},S_{i'l'})\right\}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right] \\ &\qquad \times\{1+o_p(1)\} \\ &= \left[\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u^*_{i',ll'}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mathbf{A}_2(S_{i'l},S_{i'l'})\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right\}\right] \\ &\qquad \times\{1+o_p(1)\}. \end{aligned}$$

Define

$$\mathcal{A}^{\dagger}_2(s_1,s_2,t)=\bar{m}_y f_T(t)\beta_1(t)\boldsymbol{\psi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\mathrm{E}\left(\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\right)\mathbf{A}_2(s_1,s_2)\boldsymbol{\psi}(t), \quad \text{(A.21)}$$

then it follows that

$$\begin{aligned} &\mathrm{E}\left\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\mathbf{A}_2(s_1,s_2)\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{0,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right\} \\ =\;& \mathrm{tr}\left\{\boldsymbol{\Lambda}\mathrm{E}\left(\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\right)\mathbf{A}_2(s_1,s_2)\mathrm{E}\left(\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{0,*i}(t)\boldsymbol{\beta}_{*i}\boldsymbol{\Psi}_{*i}\right)\right\} \\ =\;& \mathcal{A}^{\dagger}_2(s_1,s_2,t)+O(h^2), \end{aligned}$$

and therefore

$$R^*_{0,32}(t)=\left[\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u^*_{i',ll'}\mathcal{A}^{\dagger}_2(S_{i'l},S_{i'l'},t)\right]\{1+o_p(1)\}=O_p(n^{-1/2}). \quad \text{(A.22)}$$

Using similar derivations we can prove that $R^*_{1,32}(t)=O_p[\{h+(nh)^{-1/2}\}n^{-1/2}]$.

Next, similarly

$$\begin{aligned} R^*_{r,33}(t) &= \left[\frac{1}{n}\sum_{i=1}^{n}\left\{\boldsymbol{\Psi}_{*i}\boldsymbol{\Lambda}\left(\widehat{\boldsymbol{\Psi}}_i-\boldsymbol{\Psi}\right)^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right]\{1+o_p(1)\} \\ &= \left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left\{\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i,ll'}\mathbf{A}_1(S_{i'l},S_{i'l'},\mathbf{S}_i)\right\}\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right] \end{aligned}$$

$$\times\{1+o_p(1)\}$$

$$= \left[\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'}u^*_{i',ll'}\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}f_S^{-1}(S_{i'l})\mathbf{K}_i^{\dagger}(S_{i'l'},\mathbf{S}_i)\boldsymbol{\psi}(S_{i'l})^{\mathrm{T}}\boldsymbol{\Psi}_{*i}^{\mathrm{T}} \times\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right\}+O_p(n^{-1/2})\right]\times\{1+o_p(1)\}.$$

Denote by $\sigma_i^{(j,j')}$ the (j,j')-th entry of $\boldsymbol{\Sigma}_i^{-1}$, and define

$$\mathcal{A}_3^{\dagger}(s_1,s_2,t) = \frac{\bar{m}_y\beta_1(t)f_T(t)}{f_S(s_1)}\boldsymbol{\psi}(s_1)^{\mathrm{T}}\boldsymbol{\psi}(t)\sum_{k=1}^{q}\omega_k\psi_k(t) \times\mathrm{E}\left[\sum_{j=1}^{m_{x,i}}\sum_{j'=1}^{m_{x,i}}\mathrm{E}\left\{\psi_k(S_{ij})\sigma_i^{(j,j')}\mid S_{ij'}=s_2\right\}\right], \qquad (A.23)$$

then

$$\begin{aligned}
&\mathrm{E}\{\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}f_S^{-1}(s_1)\mathbf{K}_i^{\dagger}(s_2,\mathbf{S}_i)\boldsymbol{\psi}(s_1)^{\mathrm{T}}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{0,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\}\\
=\ & f_S^{-1}(s_1)\mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{K}_i^{\dagger}(s_2,\mathbf{S}_i)\boldsymbol{\psi}(s_1)^{\mathrm{T}}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{0,*i}(t)\boldsymbol{\beta}_{*i}\boldsymbol{\Psi}_{*i}\right\}\right]\\
=\ & \bar{m}_y\beta_1(t)f_T(t)f_S^{-1}(s_1)\mathrm{tr}\left[\mathrm{E}\left\{\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{K}_i^{\dagger}(s_2,\mathbf{S}_i)\boldsymbol{\psi}(s_1)^{\mathrm{T}}\boldsymbol{\psi}(t)\boldsymbol{\psi}^{\mathrm{T}}(t)\right\}\right]+O(h^2)\\
=\ & \bar{m}_y\boldsymbol{\psi}(s_1)^{\mathrm{T}}\boldsymbol{\psi}(t)\beta_1(t)f_T(t)f_S^{-1}(s_1)\mathrm{E}\left\{\boldsymbol{\psi}^{\mathrm{T}}(t)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\mathbf{K}_i^{\dagger}(s_2,\mathbf{S}_i)\right\}+O(h^2)\\
=\ & \mathcal{A}_3^{\dagger}(s_1,s_2,t)+O(h_R^2)+O(h^2).
\end{aligned}$$

Therefore

$$R^*_{0,33}(t)=\left[\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l'\neq l}u^*_{i',ll'}\mathcal{A}_3^{\dagger}(S_{i'l},S_{i'l'},t)\right]\{1+o_p(1)\}=O_p(n^{-1/2}), \qquad (A.24)$$

and following the same line of derivation we can show $R^*_{1,33}(t)=O_p[\{h+(nh_R^2)^{-1/2}\}n^{-1/2}]$.

Using similar but lengthier derivations,

$$\begin{aligned}
R^*_{r,34}(t) &= \left[\frac{1}{n}\sum_{i=1}^{n}\left\{\boldsymbol{\Psi}_{*i}\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\Sigma}_i-\widehat{\boldsymbol{\Sigma}}_i)\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i\right\}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right]\{1+o_p(1)\}\\
&= -\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\mathbf{W}_i^{\mathrm{T}}\boldsymbol{\Sigma}_i^{-1}\left\{(\widehat{\boldsymbol{\Psi}}_i-\boldsymbol{\Psi}_i)\boldsymbol{\Lambda}\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i(\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda})\boldsymbol{\Psi}_i^{\mathrm{T}}+\boldsymbol{\Psi}_i\boldsymbol{\Lambda}(\widehat{\boldsymbol{\Psi}}_i-\boldsymbol{\Psi}_i)^{\mathrm{T}}\right\}\right.\right.\\
&\qquad\left.\left.\times\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\Psi}_i\boldsymbol{\Lambda}\boldsymbol{\Psi}_{*i}^{\mathrm{T}}\mathbf{K}_{r,*i}(t)\boldsymbol{\beta}_{*i}\widetilde{\mathbf{X}}_{*i}\right]\right\}\{1+o_p(1)\}\\
&= \begin{cases} O_p(n^{-1/2}), & \text{if } r=0,\\ O_p[\{h+(nh_R^2)^{-1/2}\}n^{-1/2}], & \text{if } r=1.\end{cases} \qquad (A.25)
\end{aligned}$$

Combining (A.20), (A.22), (A.24) and (A.25), we obtain that

$$R_{0,3}^*(t) = -\left\{\frac{1}{n}\sum_{i'=1}^{n}\frac{1}{M_{x,i'}}\sum_{l\neq l'} u_{i',ll'}^* \mathcal{A}_1^\dagger(S_{i'l}, S_{i'l'})K_h(S_{i'l'}-t)\right\}\{1+o_p(1)\}, \tag{A.26}$$

and $R_{1,3}^*(t) = O_p\{(nh)^{-1/2}\}$. Combining the derivations above, we have

$$\begin{aligned}\widehat{\beta}_1(t) - \beta_1(t) &= R_0^*(t)/S_0(t) \times \{1+O_p(h^2)\} + O_p\{h\times (nh)^{-1/2}\} \qquad \text{(A.27)}\\ &= \frac{1}{2}\beta_1^{(2)}(t)\sigma_K^2 h^2 + \frac{1}{n\bar{m}_y f_T(t)\Gamma_x(t)}\sum_{i=1}^{n}(\mathcal{Z}_{i1}+\mathcal{Z}_{i2}+\mathcal{Z}_{i3}) + o_p\{(nh)^{-1/2}\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{Z}_{i1} &= \sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij})\epsilon_i(T_{ij})K_h(T_{ij}-t),\\ \mathcal{Z}_{i2} &= \sum_{j=1}^{m_{y,i}} \widetilde{X}_i(T_{ij})\beta_1(T_{ij})\{X_i(T_{ij}) - \widetilde{X}_i(T_{ij})\}K_h(T_{ij}-t),\\ \mathcal{Z}_{i3} &= -\frac{\bar{m}_y}{M_{x,i}}\sum_{l\neq l'} u_{i,ll'}^*\mathcal{Q}(S_{il}, S_{il'})\beta_1(S_{il'})K_h(S_{il'}-t).\end{aligned}$$

As $\mathcal{Z}_{i1}$, $\mathcal{Z}_{i1}$ and $\mathcal{Z}_{i1}$ are independent, zero-mean variables, straightforward calculations show

$$\begin{aligned}\mathrm{E}(\mathcal{Z}_{i1}^2) &= h^{-1}\bar{m}_y\Gamma_x(t)\Omega(t,t)f_T(t)\nu_0 + o(h^{-1}) := h^{-1}\Gamma_1(t) + o(h^{-1}),\\ \mathrm{E}(\mathcal{Z}_{i2}^2) &= h^{-1}\beta_1^2(t)\bar{m}_y\mathrm{E}[\widetilde{X}^2(t)\{X(t)-\widetilde{X}(t)\}^2]\nu_0 + o(h^{-1}) := h^{-1}\beta_1^2(t)\Gamma_2(t) + o(h^{-1}),\\ \mathrm{E}(\mathcal{Z}_{i3}^2) &= h^{-1}\beta_1^2(t)\bar{m}_y^2 f_S(t)\nu_0\Big[\mathrm{E}(M_{x,i}^{-1})\int \Pi(t,s_2,s_2)\mathcal{Q}^2(s_2,t)f_S(s_2)ds_2\\ &\quad + \mathrm{E}\{M_{x,i}^{-1}(m_{x,i}-2)\}\int \Pi(t,s_2,s_3)\mathcal{Q}(s_2,t)\mathcal{Q}(s_3,t)f_S(s_2)f_S(s_3)ds_2ds_3\Big]\\ &\quad + o(h^{-1}).\\ &:= h^{-1}\beta_1^2(t)\Gamma_3(t) + o(h^{-1}),\end{aligned}$$

where $\Pi(s_1,s_2,s_3) = \mathrm{E}\{X^2(s_1)X(s_2)X(s_3)\} + R(s_2,s_3)\sigma_u^2 - R(s_1,s_2)R(s_1,s_3) + I(s_2 = s_3)\{R(s_1,s_1)\sigma_u^2 + \sigma_u^4\}$. Since $\epsilon_i(\cdot)$ is independent of $X_i(\cdot)$ and $\mathbf{W}_i$, it follows that $\mathrm{E}(\mathcal{Z}_{i1}\mathcal{Z}_{i2}) = 0$, and $\mathrm{E}(\mathcal{Z}_{i1}\mathcal{Z}_{i3}) = 0$. We can also show $\mathrm{E}(\mathcal{Z}_{i2}\mathcal{Z}_{i3}) = O(1) = o_p(h^{-1})$. Therefore, we conclude $\mathcal{Z}_{i1}$, $\mathcal{Z}_{i1}$ and $\mathcal{Z}_{i1}$ are asymptotically independent. The theorem is proven by applying the central limit theorem to (A.27).

Appendix B: An Extension to Multiple Asynchronous Time-Varying Covariates

B.1 *Multiple Asynchronous Covariates*

The functional calibration method can be easily extended to accommodate multiple time-varying covariates, which are asynchronous with the response. Our strategy is to apply the multivariate functional principal component analysis (Chiou et al., 2014) to reconstruct the trajectory for each time-varying covariate, then apply time-varying and time-invariant regression analysis using the imputed covariate values that are synchronized with the response.

Suppose there are p_x asynchronous time-varying covariates $\mathbf{X}_i(t) = (X_{i1}, X_{i2}, \ldots, X_{ip_x})^{\mathrm{T}}(t)$. The time-invariant and time-varying regression models (1) and (2) can be generalized to

$$Y_i(t) = \beta_0 + \boldsymbol{\beta}_z^{\mathrm{T}}\mathbf{Z}_i + \boldsymbol{\beta}_x^{\mathrm{T}}\mathbf{X}_i(t) + \epsilon_i(t), \tag{B.1}$$

$$Y_i(t) = \beta_0(t) + \boldsymbol{\beta}_z^{\mathrm{T}}(t)\mathbf{Z}_i + \boldsymbol{\beta}_x^{\mathrm{T}}(t)\mathbf{X}_i(t) + \epsilon_i(t), \tag{B.2}$$

where $\mathbf{Z}_i$ is a p_z-dim time-invariant covariate. Consider $\mathbf{X}_i(t)$ as independent realizations of a multivariate stochastic process with the mean and (cross-)covariance functions as

$$\mu_v(t) = \mathrm{E}\{X_{iv}(t)\}, \quad R_v(s,t) = \mathrm{Cov}\{X_{iv}(s), X_{iv}(t)\}, \quad s,t, \in \mathcal{T}, \quad v = 1, \ldots, p_x;$$
$$R_{vv'}(s,t) = \mathrm{Cov}\{X_{iv}(s), X_{iv'}(t)\}, \quad v \neq v'.$$

The discrete, error-prone observations on the time-varying covariates are

$$W_{iv,j} = X_{iv}(S_{iv,j}) + U_{iv,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m_{x,iv}, \tag{B.3}$$

where $U_{iv,j}$ are zero mean measurement errors independent of $\mathbf{X}_i(t)$ with variance σ_{uv}^2. On the other hand, the response $Y_i(t)$ are observed on $\mathbf{T}_i = (T_{i1}, \ldots, T_{im_{y,i}})^{\mathrm{T}}$. Under the asynchronous longitudinal design, the observation time points from different variables, $\mathbf{T}_i$ and $\mathbf{S}_{iv} = (S_{iv,1}, \ldots, S_{iv,m_{x,iv}})^{\mathrm{T}}$, $v = 1, \ldots, p_x$, can be different from each other.

B.2 *Multivariate Functional Calibration*

We consider each asynchronous, time-varying covariate as a stochastic process with a Karhunen–Loève expansion,

$$X_{iv}(t) = \mu_v(t) + \textstyle\sum_{k=1}^{q_v} \xi_{iv,k}\psi_{vk}(t), \quad t \in \mathcal{T}, \tag{B.4}$$

for $v = 1, \ldots, p_x$, $i = 1, \ldots, n$, where the $\xi_{iv,k}$ are the principal component scores with mean zero and variance ω_{vk}, $\psi_{vk}(t)$ are orthonormal functions in the sense $\int_{\mathcal{T}} \psi_{vk}(t)\psi_{vk'}(t)dt = 1$ if $k = k'$ and 0 otherwise. Since different components of $\mathbf{X}_i(t)$ can be correlated, these correlations are modeled by cross-covariates between the principal component scores, $\omega_{vv',kk'} = \mathrm{Cov}(\xi_{iv,k}, \xi_{iv',k'}) = \iint R_{vv'}(s,t)\psi_{vk}(s)\psi_{v'k'}(t)dsdt$, $v, v' = 1, \ldots, p_x$, $k \leqslant q_v$ and $k' \leqslant q_{v'}$. As discussed in Happ and Greven (2018), the univariate Karhunen–Loève representation used in (B.4) is equivalent to the multivariate representation in Chiou et al. (2014).

As in Section 3.1, we obtain mean, covariance and eigenfunction estimates for each time-varying variable, namely $\widehat{\mu}_v$, $\widehat{R}_v$, $\widehat{\psi}_{vk}$ for $v = 1, \ldots, p_x$, $k = 1, \ldots, q_v$. We then estimate the cross covariance function $R_{vv'}(s,t)$ by $\widehat{R}_{vv'}(s,t) = \widehat{a}_0$ where $(\widehat{a}_0, \widehat{a}_1, \widehat{a}_2)$ minimizes

$$\frac{1}{n}\sum_{i=1}^{n}\Bigg[\frac{1}{m_{x,iv}m_{x,iv'}}\sum_{j=1}^{m_{x,iv}}\sum_{l=1}^{m_{x,iv'}}\{L_{iv,j}L_{iv',l} - a_0 - a_1(S_{iv,j} - s) - a_2(S_{iv',l} - t)\}^2 \times K_{h_{vv'}}(S_{iv,j} - s)K_{h_{vv'}}(S_{iv',l} - t)\Bigg],$$

where $L_{iv,j} = W_{iv,j} - \widehat{\mu}_v(S_{iv,j})$. We then estimate cross covariance of the FPC scores by

$$\widehat{\omega}_{vv',kk'} = \int\int \widehat{R}_{vv'}(s,t)\widehat{\psi}_{vk}(s)\widehat{\psi}_{v'k'}(t)dsdt,$$

which is implemented by numerical integration. We follow the multivariate PACE method of Chiou et al. (2014) to estimate the FPCA scores. Let $\boldsymbol{\mu}_{iv} = \{\mu_{iv}(S_{iv,1}), \ldots, \mu_{iv}(S_{iv,m_{x,iv}})\}^{\mathrm{T}}$, $\mathbf{W}_{iv} = (W_{iv,1}, \ldots, W_{iv,m_{x,iv}})^{\mathrm{T}}$ for $v = 1, \ldots, p_x$. Let $\boldsymbol{\omega}_{vv'k} = (\omega_{vv',k1}, \cdots, \omega_{vv',kq_{v'}})$, note

$\omega_{vv,kk'} = 0$ if $k \neq k'$. Define Ω_v as the matrix containing all eigenvalues related to v,

$$\Omega_v = \begin{bmatrix} \boldsymbol{\omega}_{v11} & \boldsymbol{\omega}_{v21} & \cdots & \boldsymbol{\omega}_{vp_x1} \\ \boldsymbol{\omega}_{v12} & \boldsymbol{\omega}_{v22} & \cdots & \boldsymbol{\omega}_{vp_x2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\omega}_{v1q_v} & \boldsymbol{\omega}_{v2q_v} & \cdots & \boldsymbol{\omega}_{vp_xq_v} \end{bmatrix}.$$

Put $\boldsymbol{\Psi}_{iv} = (\boldsymbol{\psi}_{iv1}, \dots, \boldsymbol{\psi}_{ivq_v})$, where $\boldsymbol{\psi}_{ivk} = \{\psi_{vk}(S_{iv,1}), \dots, \psi_{vk}(S_{iv,m_{x,iv}})\}^{\mathrm{T}}$ for $k = 1, \dots, q_v$, and define $\boldsymbol{\Psi}_i = \mathrm{diag}(\boldsymbol{\Psi}_{i1}, \dots, \boldsymbol{\Psi}_{ip_x})$ as a diagonal block matrix containing eigenfunctions evaluated at $\mathbf{S}_i$. Similar to the univariate case (5), the BLUP for $\boldsymbol{\xi}_{iv}$ is

$$\widetilde{\boldsymbol{\xi}}_{iv} = (\widetilde{\xi}_{iv1}, \cdots, \widetilde{\xi}_{ivK})^{\mathrm{T}} = \Omega_v \boldsymbol{\Psi}_i^{\mathrm{T}} \boldsymbol{\Sigma}_i^{-1}(\mathbf{W}_i - \boldsymbol{\mu}_i),$$

where $\boldsymbol{\mu}_i = (\boldsymbol{\mu}_{i1}^{\mathrm{T}}, \cdots, \boldsymbol{\mu}_{ip_x}^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{W}_i = (\mathbf{W}_{i1}^{\mathrm{T}}, \dots, \mathbf{W}_{ip_x}^{\mathrm{T}})^{\mathrm{T}}$, and $\boldsymbol{\Sigma}_i = (\boldsymbol{\Sigma}_{i,vv'})_{v,v'=1}^{p_x}$ with $\boldsymbol{\Sigma}_{i,vv'} = \mathrm{Cov}(\mathbf{W}_{iv}, \mathbf{W}_{iv'})$. Substituting all unknown functions and parameters with their FPCA estimators, the empirical FPC score estimators are given by

$$\widehat{\boldsymbol{\xi}}_{iv} = \widehat{\Omega}_v \widehat{\boldsymbol{\Psi}}_i^{\mathrm{T}} \widehat{\boldsymbol{\Sigma}}_i^{-1}(\mathbf{W}_i - \widehat{\boldsymbol{\mu}}_i), \tag{B.5}$$

and we can predict the trajectories of the time-varying covariate processes by

$$\widehat{X}_{iv}(t) = \widehat{\mu}_v(t) + \textstyle\sum_{k=1}^{q_v} \widehat{\xi}_{iv,k} \widehat{\psi}_{vk}(t), \quad t \in \mathcal{T}.$$

We then calibrate the covariate values synchronized with the response, $\widehat{\mathbf{X}}_{*ij} = \{\widehat{X}_{i1}(T_{ij}), \dots, \widehat{X}_{ip_x}(T_{ij})\}^{\mathrm{T}}$, and use them to fit regression models (B.1) and (B.2). Denote $\boldsymbol{\beta} = (\beta_0, \boldsymbol{\beta}_z^{\mathrm{T}}, \boldsymbol{\beta}_x^{\mathrm{T}})^{\mathrm{T}}$, $\mathbb{X}_{ij} = (1, \mathbf{Z}_i^{\mathrm{T}}, \widehat{\mathbf{X}}_{*ij}^{\mathrm{T}})^{\mathrm{T}}$ and $\mathbb{X}_i = (\mathbb{X}_{i1}, \dots, \mathbb{X}_{im_{y,i}})^{\mathrm{T}}$, then Model (B.1) can be fitted by a least square estimator similar as (12). For Model (B.2), denote $\boldsymbol{\beta}(t) = (\beta_0, \boldsymbol{\beta}_z^{\mathrm{T}}, \boldsymbol{\beta}_x^{\mathrm{T}})^{\mathrm{T}}(t)$. For any fixed t, denote by $\mathbf{b}_0 = \boldsymbol{\beta}(t)$ and $\mathbf{b}_1 = \boldsymbol{\beta}'(t)$. Then $\boldsymbol{\beta}(t)$ can be estimated by solving a kernel weighted local least square as (13).

Appendix C: Additional Tables and Graphs

[Table C.1 about here.]

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[Figure C.1 about here.]

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[Figure C.9 about here.]

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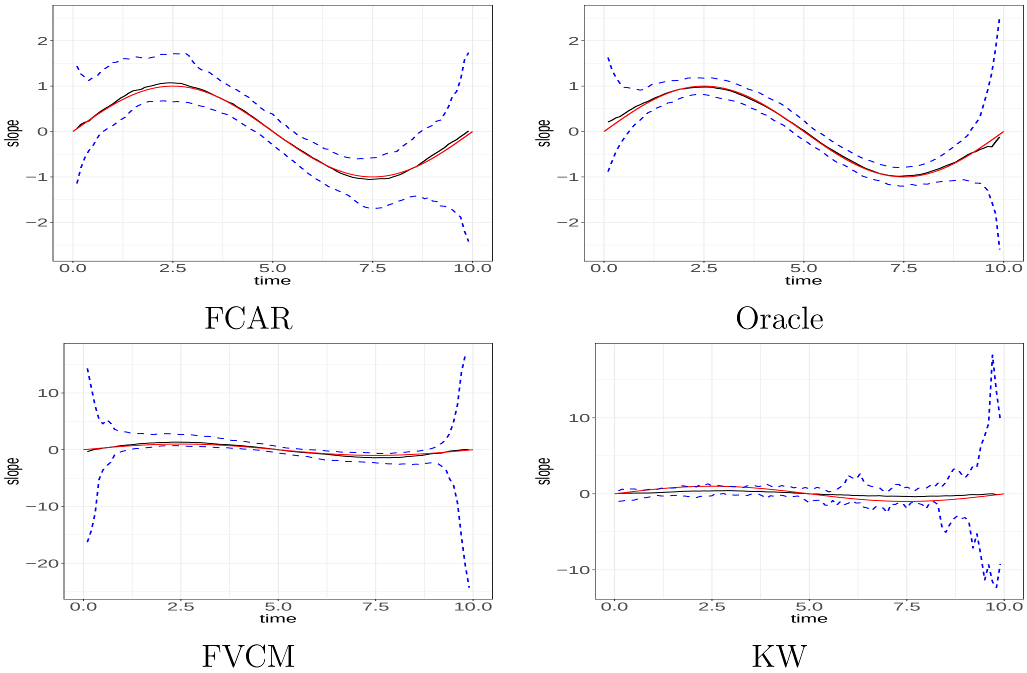

Figure C.1: Summary of $\widehat{\beta}_1(t)$ under Simulation 2, Setting II using various methods. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

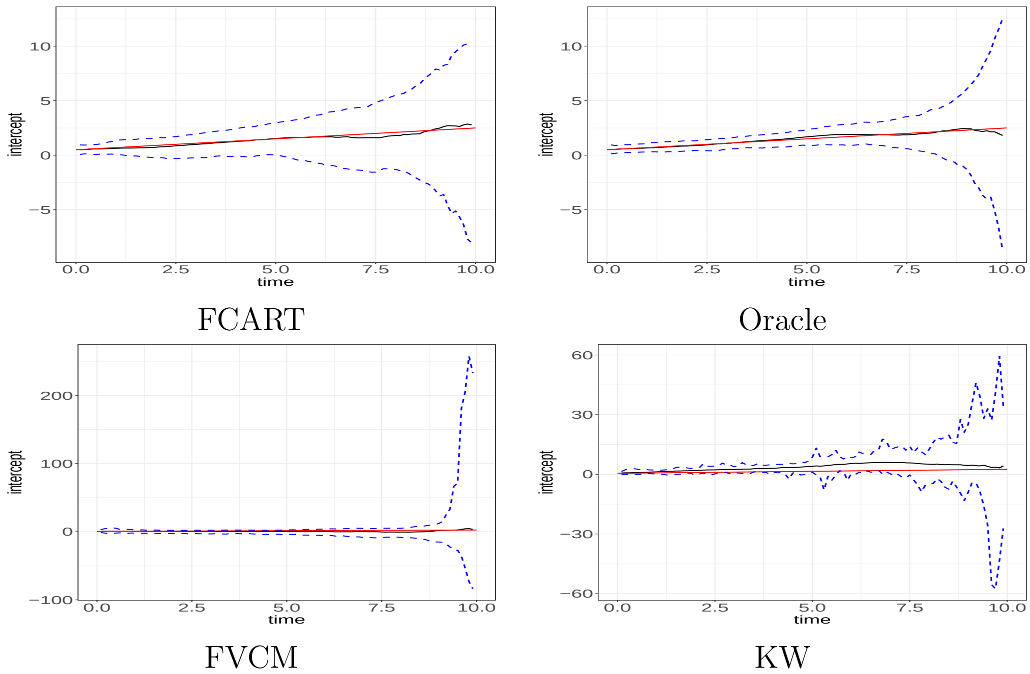

Figure C.2: Summary of $\widehat{\beta}_0(t)$ under Setting I of Simulation 2 using various methods. In each panel, black: median of $\widehat{\beta}_0(t)$; red: true $\beta_0(t)$; dashed blue: 0.975 and 0.025 quantiles.

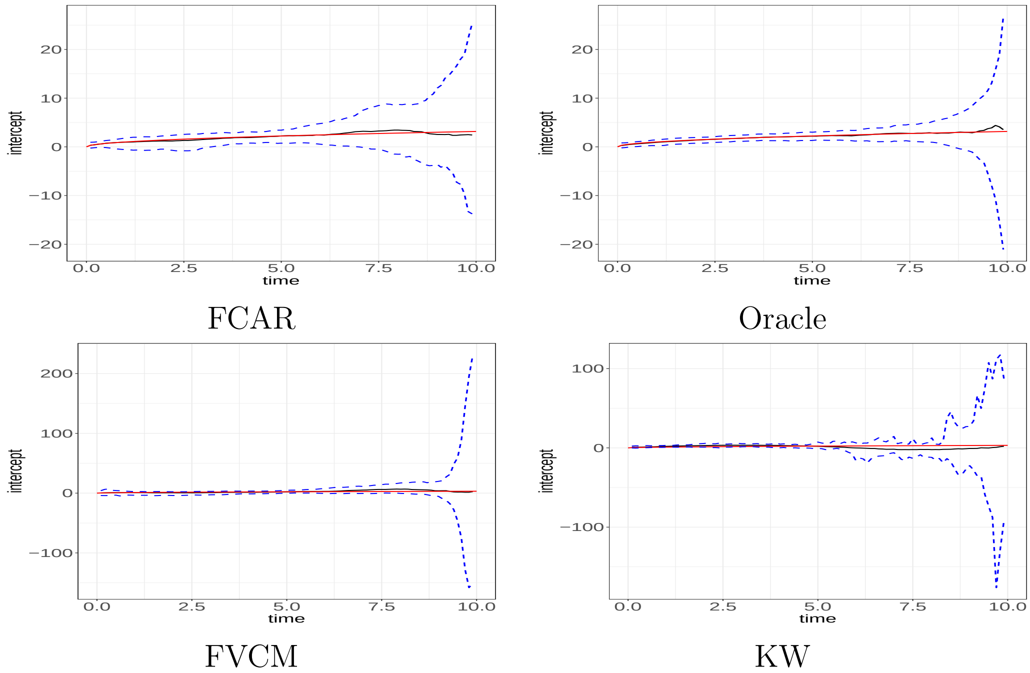

Figure C.3: Summary of $\widehat{\beta}_0(t)$ under Setting II of Simulation 2 using various methods. In each panel, black: median of $\widehat{\beta}_0(t)$; red: true $\beta_0(t)$; dashed blue: 0.975 and 0.025 quantiles.

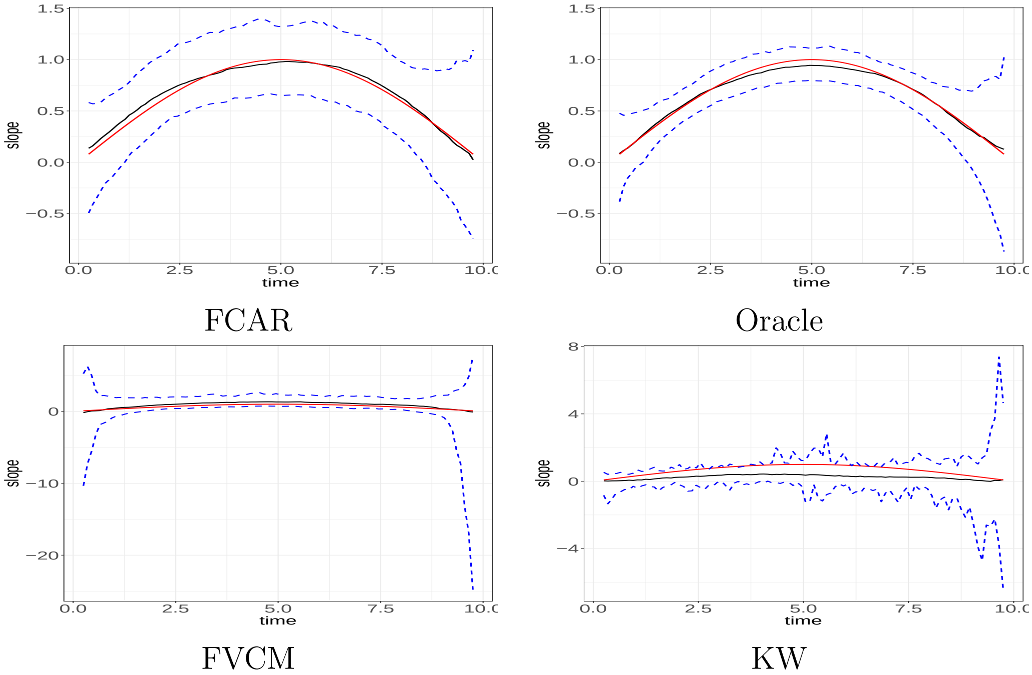

Figure C.4: Summary of $\widehat{\beta}_1(t)$ under Setting I of Simulation 2 with being reported at 95% time domain. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

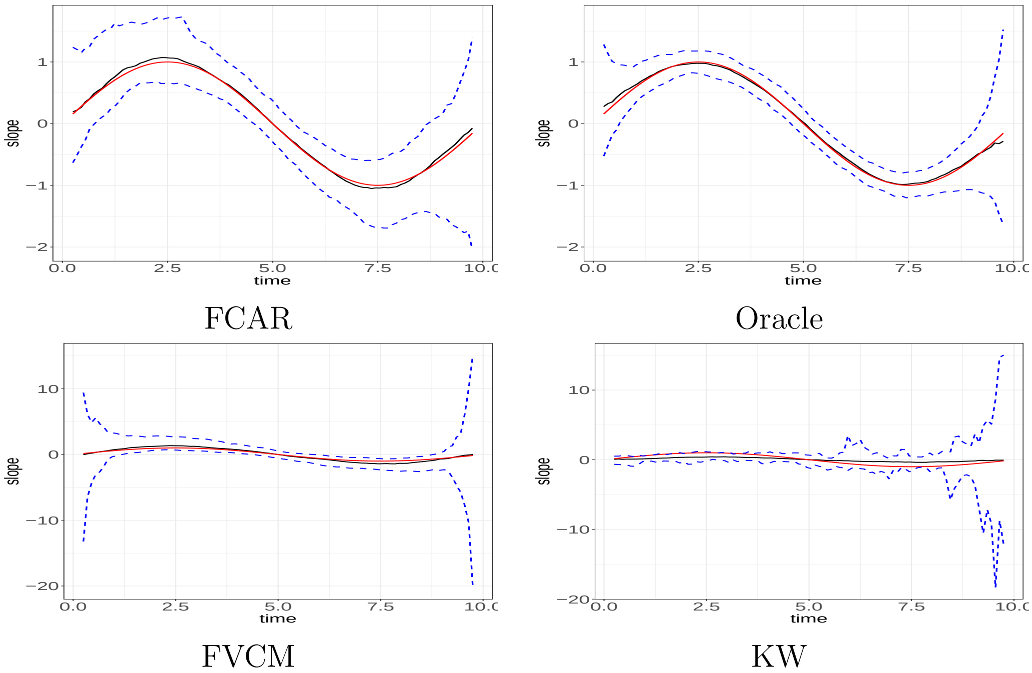

Figure C.5: Summary of $\widehat{\beta}_1(t)$ under Setting II of Simulation 2 with being reported at 95% time domain. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

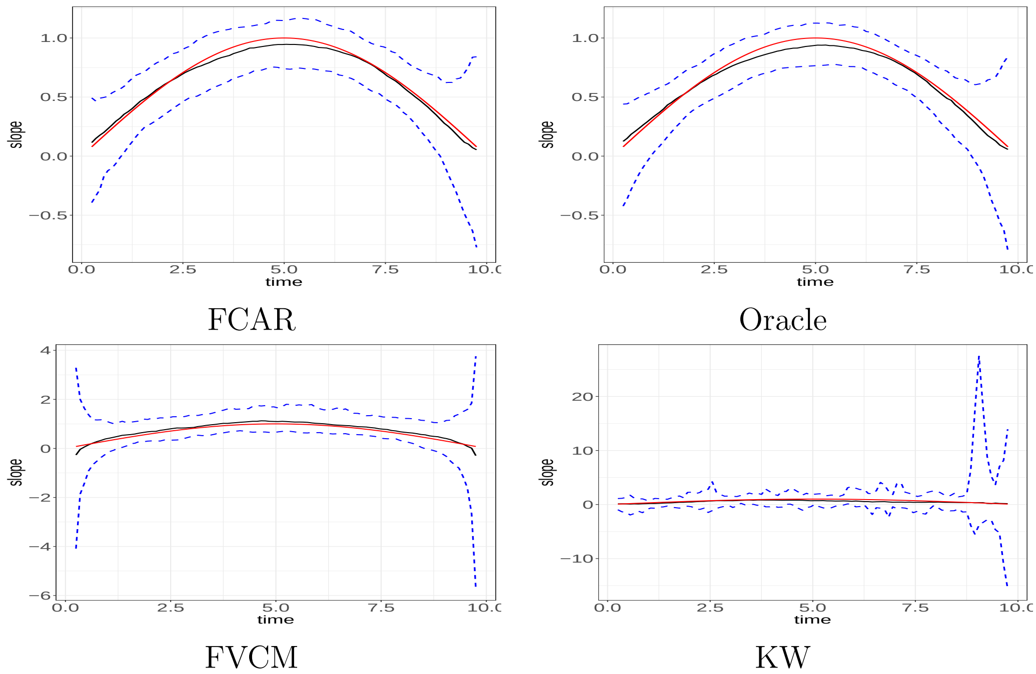

Figure C.6: Summary of $\widehat{\beta}_1(t)$ under Setting I of Simulation 2 with MEF and being reported at 95% time domain. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

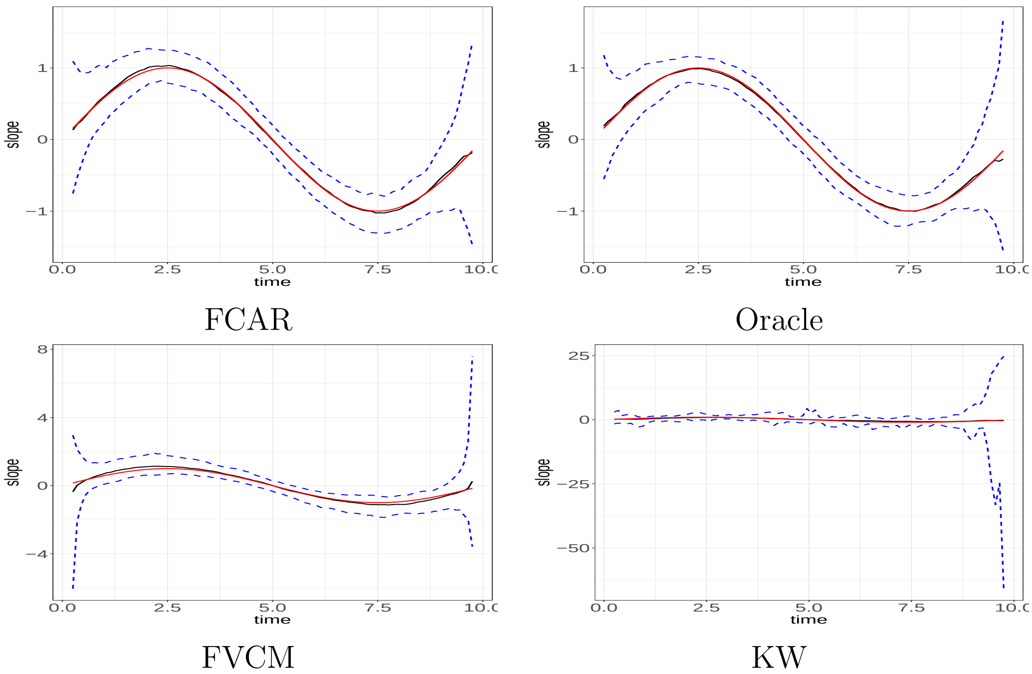

Figure C.7: Summary of $\widehat{\beta}_1(t)$ under Setting II of Simulation 2 with MEF and being reported at 95% time domain. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

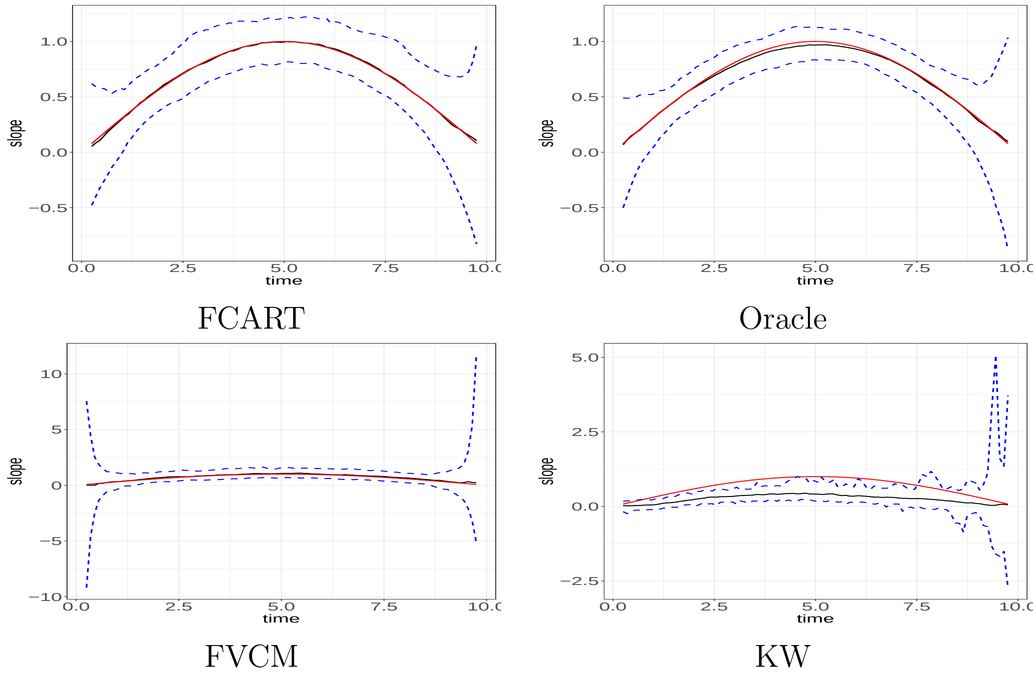

Figure C.8: Summary of $\widehat{\beta}_1(t)$ under Setting I of Simulation 2 with $m_i = 15$ and being reported at 95% time domain. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

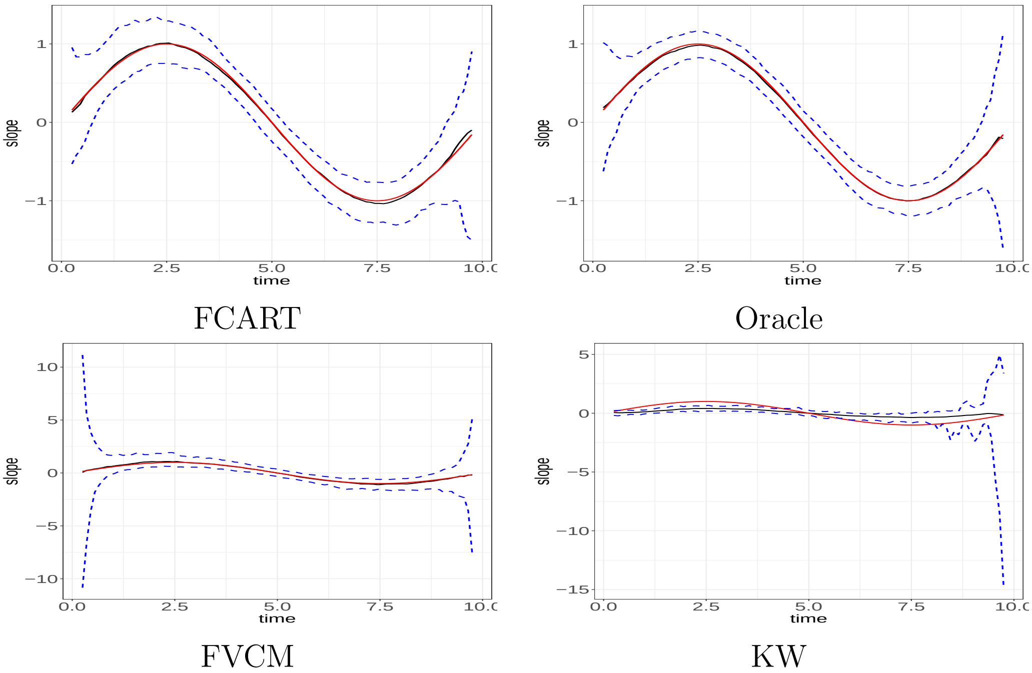

Figure C.9: Summary of $\widehat{\beta}_1(t)$ under Setting II of Simulation 2 with $m_i = 15$ and being reported at 95% time domain. In each panel, black: median of $\widehat{\beta}_1(t)$; red: true $\beta_1(t)$; dashed blue: 0.975 and 0.025 quantiles.

	Setting I			Setting Ⅱ		
Error type	IE	DE	MEF	IE	DE	MEF
Bias	-0.010	0.003	0.015	0.007	0.008	0.002
SD	0.165	0.189	0.114	0.086	0.093	0.067
Naive SE	0.112	0.103	0.071	0.057	0.052	0.035
Naive CP	0.825	0.740	0.765	0.800	0.725	0.720
Bootstrap SE	0.180	0.187	0.118	0.089	0.100	0.068
Bootstrap CP	0.965	0.935	0.955	0.950	0.950	0.940

Table C.1: Simulation 1: the performance of $\widehat{\beta}_0$ obtained by the proposed FCAR method under Settings 1 and 2 with different error structures. SD: standard deviation; Naive SE: mean of the naive standard error; Naive CP: coverage rate of a 95% confidence interval using the naive SE; Bootstrap SE: mean of the bootstrap standard error; Bootstrap CP: coverage rate of a 95% confidence interval using the bootstrap SE. IE: independent errors; DE: dependent errors; MEF: measurement-error free with dependent errors.

Error type	Setting I IE	DE	MEF	*Dense*	Setting II IE	DE	MEF	*Dense*
Bias	0.007	0.004	-0.002	0.002	-0.008	-0.013	0.025	0.003
SD	0.028	0.029	0.017	0.019	0.127	0.122	0.060	0.068
Naive SE	0.019	0.017	0.012	0.008	0.064	0.058	0.034	0.024
Naive CP	0.830	0.770	0.820	0.585	0.670	0.640	0.725	0.485
Bootstrap SE	0.030	0.030	0.019	0.018	0.117	0.119	0.064	0.062
Bootstrap CP	0.955	0.950	0.965	0.955	0.925	0.930	0.940	0.960

Table C.2: Simulation 1: the performance of $\widehat{\beta}_1$ obtained by the proposed FCAR method under Settings 1 and 2 with different measurement error scenarios. SD: standard deviation; Naive SE: mean of the naive standard error; Naive CP: coverage rate of a 95% confidence interval using the naive SE; Bootstrap SE: mean of the bootstrap standard error; Bootstrap CP: coverage rate of a 95% confidence interval using the bootstrap SE. IE: independent errors; DE: dependent errors; MEF: measurement-error free with dependent errors; Dense: 15 observations per subject.

		IE		DE		MEF	
		FCAR	KW	FCAR	KW	FCAR	KW
Setting I	Bias	-0.010	1.163	0.003	1.092	0.015	0.123
	SD	0.165	0.400	0.189	0.418	0.114	0.267
	SE	0.180	0.300	0.187	0.315	0.118	0.195
	CP	0.965	0.115	0.935	0.125	0.955	0.815
Setting II	Bias	0.007	0.166	0.008	0.162	0.002	0.009
	SD	0.086	0.126	0.093	0.117	0.067	0.091
	SE	0.089	0.121	0.100	0.126	0.068	0.084
	CP	0.950	0.705	0.950	0.730	0.940	0.915

Table C.3: Simulation 1: comparisons of $\widehat{\beta}_0$ using the proposed FCAR method with the kernel weighted (KW) method of (Cao et al., 2015) in bias, standard deviation (SD), mean of standard error (SE) and coverage rate of a 95% confidence interval using standard error (CP) under two settings and three error structures (IE: independent errors; DE: dependent errors; MEF: measurement-error free with dependent errors).

		IE		DE		MEF		*Dense*	
		FCAR	KW	FCAR	KW	FCAR	KW	FCAR	KW
Setting I	Bias	0.007	-0.225	0.004	-0.213	-0.002	-0.024	0.002	-0.209
	SD	0.028	0.067	0.029	0.067	0.017	0.045	0.019	0.032
	SE	0.030	0.049	0.030	0.051	0.019	0.032	0.018	0.023
	CP	0.955	0.060	0.950	0.065	0.965	0.830	0.955	0.000
Setting II	Bias	-0.008	-0.978	-0.013	-0.978	0.025	-0.060	0.003	-0.954
	SD	0.127	0.097	0.122	0.092	0.060	0.108	0.068	0.055
	SE	0.117	0.076	0.119	0.077	0.064	0.069	0.062	0.043
	CP	0.925	0.000	0.930	0.000	0.940	0.735	0.960	0.000

Table C.4: Simulation 1: comparisons of $\widehat{\beta}_1$ using the proposed FCAR method with the kernel weighted (KW) method in bias, standard deviation (SD), mean of standard error (SE) and coverage rate of a 95% confidence interval using standard error (CP) under two settings and three error structures (IE: independent errors; DE: dependent errors; MEF: measurement-error free with dependent errors; Dense: 15 observations per subject).

	Method	Criterion	Mean(SD)	Median	25%	75%
Setting I	FCAR	MADE	0.278(0.136)	0.249	0.183	0.342
	FVCM		0.980(1.241)	0.769	0.582	1.040
	KW		1.358(3.510)	0.965	0.815	1.154
	Oracle		0.166(0.073)	0.155	0.108	0.204
	FCAR	WASE	0.249(0.275)	0.166	0.082	0.329
	FVCM		200.092(2589.147)	1.535	0.831	3.253
	KW		1044.667(14234.328)	3.349	1.551	10.016
	Oracle		0.114(0.129)	0.061	0.034	0.145
Setting II	FCAR	MADE	0.220(0.087)	0.205	0.155	0.265
	FVCM		0.540(0.410)	0.441	0.338	0.610
	KW		1.013(1.609)	0.657	0.556	0.931
	Oracle		0.123(0.04)	0.118	0.093	0.147
	FCAR	WASE	0.191(0.199)	0.131	0.066	0.230
	FVCM		10.103(94.566)	0.692	0.303	1.475
	KW		181.166(1533.938)	2.123	1.004	11.570
	Oracle		0.076(0.071)	0.058	0.027	0.097

Table C.5: Simulation 2 reported at 95% time domain: MADE and WASE of various methods. SD: standard deviation; 25%: 25% quantile; 75%: 75% quantile.

	Method	Criterion	Mean(SD)	Median	25%	75%
Setting I	FCAR	MADE	0.186(0.071)	0.176	0.131	0.218
	FVCM		0.416(0.136)	0.382	0.314	0.510
	KW		1.867(3.668)	1.011	0.755	1.596
	Oracle		0.168(0.067)	0.160	0.124	0.198
	FCAR	WASE	0.113(0.103)	0.078	0.042	0.144
	FVCM		0.638(0.547)	0.466	0.267	0.819
	KW		995.093(6506.490)	8.830	2.527	42.406
	Oracle		0.103(0.102)	0.069	0.034	0.129
Setting II	FCAR	MADE	0.135(0.041)	0.133	0.106	0.158
	FVCM		0.252(0.080)	0.237	0.192	0.304
	KW		1.658(3.983)	0.726	0.511	1.240
	Oracle		0.120(0.036)	0.118	0.095	0.142
	FCAR	WASE	0.075(0.060)	0.054	0.034	0.103
	FVCM		0.253(0.245)	0.169	0.109	0.298
	KW		1438.554(11407.122)	5.614	1.491	31.745
	Oracle		0.067(0.063)	0.050	0.027	0.082

Table C.6: Simulation 2 and measurement-error free (MEF) scenario reported at 95% time domain: MADE and WASE of various methods. SD: standard deviation; 25%: 25% quantile; 75%: 75% quantile.

	Method	Criterion	Mean(SD)	Median	25%	75%
Setting I	FCAR	MADE	0.196(0.083)	0.181	0.138	0.239
	FVCM		0.438(0.138)	0.409	0.342	0.514
	KW		0.881(0.602)	0.738	0.686	0.815
	Oracle		0.156(0.062)	0.145	0.112	0.195
	FCAR	WASE	0.139(0.145)	0.091	0.052	0.173
	FVCM		1.176(2.651)	0.569	0.346	1.072
	KW		26.846(148.108)	1.035	0.832	1.687
	Oracle		0.099(0.108)	0.063	0.031	0.123
Setting II	FCAR	MADE	0.131(0.041)	0.123	0.101	0.151
	FVCM		0.289(0.071)	0.276	0.245	0.318
	KW		0.916(2.815)	0.496	0.455	0.564
	Oracle		0.105(0.034)	0.099	0.082	0.123
	FCAR	WASE	0.065(0.058)	0.045	0.030	0.079
	FVCM		0.478(0.865)	0.299	0.197	0.516
	KW		1271.494(15648.487)	0.638	0.460	1.358
	Oracle		0.052(0.057)	0.037	0.020	0.059

Table C.7: Simulation 2 and $m_i = 15$ reported in a 95% time domain: MADE and WASE of various methods. SD: standard deviation; 25%: 25% quantile; 75%: 75% quantile.