CRITERIA FOR SOLVABILITY (CARTAN) AND SEMISIMPLICITY (KILLING FORM)

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Abstract. We illustrate one criterion (Cartan’s criterion) for a Lie algebra to be solvable, and one criterion for a Lie algebra to be semi-simple (using Killing form). We give two consequences of these criteria, one saying every semi-simple Lie algebra can be decomposed into direct sum of simple ideals; one saying for a semi-simple Lie algebra, every derivation of $L$ is inner. The later allows us to define abstract Jordan Chevalley decomposition for a semi-simple Lie algebra.

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The first goal of this exposition is to prove Cartan’s criterion, which is a criterion for a Lie algebra $L$ to be solvable. One of the tools involved will include Jordan Chevalley decompositon and its corollary:

**Proposition 0.1** (Jordan Chevalley Decomposition for Algebraically Closed Fields). Let $F$ be an algebraically close field. Let $V$ be a finite dimensional $F-$vector space. Let $x \in \text{End}(V)$ be an endomorphism.

1. There exists unique endomorphisms $x_s, x_n \in \text{End}(V)$ such that $x = x_s + x_n$, $x_s$ is semi-simple, $x_n$ is nilpotent, and $x_s, x_n$ commute with each other.
2. There exists polynomials $p(X), q(X) \in F[X]$ with one variable without constant term, such that $x_s = p(x), x_n = q(x)$.

So it follows that \( x_s \) and \( x_n \) commutes with any endomorphism that commutes with \( x \).

(3) Suppose \( A \subseteq B \subseteq V \) are \( \mathbb{F} \)-vector subspaces. If \( x \) maps \( B \) into \( A \) then \( x_s, x_n \) also map \( B \) into \( A \).

The decomposition \( x = x_s + x_n \) is called the additive Jordan-Chevalley decomposition of \( x \). \( x_s \) is called the semi-simple part of \( x \), and \( x_n \) is called the nilpotent part of \( x \).

**Corollary 0.2** (Adjoint Representation Preserves Jordan Chevalley Decomposition). Consider the adjoint representation of \( \mathfrak{gl}(V) \):

\[
\text{ad} : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(\mathfrak{gl}(V)).
\]

If \( x \in \mathfrak{gl}(V) \) is a nilpotent endomorphism on \( V \) with Jordan Chevalley decomposition

\[
x = x_s + x_n.
\]

Then \( \text{ad} x \in \text{End}(\text{End}(V)) \) has Jordan Chevalley decomposition

\[
\text{ad} x = \text{ad} x_s + \text{ad} x_n.
\]

Jordan Chevalley decomposition requires the ground field \( \mathbb{F} \) to be algebraically closed, so throughout this exposition, the ground field will be assumed to be algebraically closed. It turns out we also need to assume \( \mathbb{F} \) has characteristic 0 so \( \mathbb{F} \) contains the rationals \( \mathbb{Q} \) as prime field.

The second goal of this exposition is to define Killing form \( \kappa \) of a Lie algebra, then use Killing form to state a criterion for a Lie algebra \( L \) to be semi-simple.

After we achieving these two goals, it will follow we can define Jordan Chevalley decomposition for a semi-simple Lie algebra.

### 1. Cartan's Criterion for Solvability

The following lemma is a trace criterion for an endomorphism to be nilpotent.

**Lemma 1.1** (Trace Criterion). Let \( \mathbb{F} \) be an algebraically closed field with characteristic 0. Let \( V \) be a finite dimensional \( \mathbb{F} \)-vector space. Let \( A \subseteq B \subseteq \mathfrak{gl}(V) \) be two subspaces. Set

\[
M = \{ x \in \mathfrak{gl}(V) : [x, B] \subseteq A \}.
\]

Suppose \( x \in M \) is such that \( \text{Tr}(xy) = 0 \) for all \( y \in M \). Then \( x \) is an nilpotent endomorphism in \( \mathfrak{gl}(V) \).

**Proof.** Let \( x \in M \) be such that \( \text{Tr}(xy) = 0 \) for all \( y \in M \), as in the hypothesis. Since the ground field is assumed to be algebraically closed, we can write \( x = x_s + x_n \) as in 0.1, where \( x_s \) is semi-simple and \( x_n \) is nilpotent.
The goal is clearly to show \( x_s = 0 \). Fix a basis \( B = \{v_1, \cdots, v_m\} \) of \( V \) with respect to which \( x_s \) is diagonal with diagonal entries \( a_1, \cdots, a_m \). \( \mathbb{F} \) contains \( \mathbb{Q} \) as a rational field. Consider

\[
E = \text{Span}_\mathbb{Q} \{a_1, \cdots, a_m\} \subseteq \mathbb{F}.
\]

By construction \( E \) is a finite dimensional \( \mathbb{Q} \)-vector subspace of \( \mathbb{F} \). To show \( x_s = 0 \), it actually suffices to show the \( \mathbb{Q} \)-dual space

\[
E^* = \{ \text{linear functionals } f : E \to \mathbb{Q} \}
\]

is zero.

Let \( f : E \to \mathbb{Q} \) be a linear functional. Let \( y \in \text{gl}(V) \) be the endomorphism whose matrix with respect to \( B \) is diagonal with diagonal entries \( f(a_1), \cdots, f(a_m) \). Let \( e_{ij} \in \text{gl}(V) \) be the endomorphism whose matrix with respect ot \( B \) has a 1 at \((i, j)\)-entry and 0 everywhere else. Let

\[
\text{ad} : \text{gl}(V) \to \text{gl}(\text{gl}(V))
\]

be the adjoint representation. An easy computation shows

\[
\text{ad} x_s(e_{ij}) = (a_i - a_j)e_{ij}, \quad \text{ad} y(e_{ij}) = (f(a_i) - f(a_j))e_{ij}.
\]

Now we use Lagrange interpolation to deal with all data point

\[
(X, Y) = (a_i - a_j, f(a_i) - f(a_j)), \quad \text{all possible pairs of } i, j.
\]

Since \( f \) is linear, these data points allow us to construct \( r(X) \in \mathbb{Q}[X] \) without constant term satisfying

\[
r(a_i - a_j) = f(a_i) - f(a_j).
\]

More precisely,

\[
r(X) = \sum_{a_i - a_j} f(a_i - a_j) \prod_{a_i' - a_j' \neq a_i - a_j} \frac{X - (a_i - a_j)}{(a_i' - a_j') - (a_i - a_j)}.
\]

We claim now

\[
\text{ad} y = r(\text{ad} x_s).
\]

To check this, say \( r(X) = q_n X^n + \cdots + q_1 X \), and we just need to apply both side to \( e_{ij} \):

\[
(\text{ad} x_s)(e_{ij}) = q_n (\text{ad} x_s)(e_{ij}) + \cdots + q_1 \text{ad} x_s(e_{ij}) = \left(q_n (a_i - a_j)^n + \cdots + q_1 (a_i - a_j)\right) e_{ij}
\]

\[
= r(a_i - a_j)e_{ij} = (f(a_i) - f(a_j))e_{ij} = \text{ad} y(e_{ij}) = \text{ad} x_s(e_{ij})
\]

By 0.2, \( \text{ad} x_s \) is the semi-simple part of \( \text{ad} x \). By (2) of 0.1, \( \text{ad} x_s \) can be written in a polynomial in \( \text{ad} x \) without constant term. Combining these two pieces of information and the relation

\[
\text{ad} y = r(\text{ad} x_s)
\]
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together, we see that \( \text{ad} y \) is also a polynomial in \( \text{ad} x \) without constant term.
The hypothesis \( x \in M \) implies \( \text{ad} x \) maps \( B \) into \( A \). It follows that \( \text{ad} y \) also maps \( B \) into \( A \), so \( y \in M \). Now,

\[
0 = \text{Tr}(xy) = \sum_{i=1}^{m} a_i f(a_i) \Rightarrow 0 = f(\sum_{i=1}^{m} a_i f(a_i)) = \sum_{i=1}^{m} f(a_i)^2 f(a_i) = m \sum_{i=1}^{m} f(a_i) \Rightarrow f(a_i) = 0.
\]

We conclude \( f \) is zero functional and this finishes the proof. \( \square \)

An easy but useful identity that will be deployed in the proof of Cartan’s criterion is the following:

**Lemma 1.2.** Let \( V \) be a finite dimensional \( \mathbb{F} \)-vector space. Let \( x,y,z \in \mathfrak{gl}(V) \) then

\[
\text{Tr}([x,y]z) = \text{Tr}(x[y,z]).
\]

**Proof.** We have \([x,y]z = xyz - yxz, x[y,z] = xyz - xzy, \) and \( \text{Tr}(yxz) = \text{Tr}(xzy) \). \( \square \)

**Theorem 1.3** (Cartan’s Criterion). Let \( \mathbb{F} \) be an algebraically closed field with characteristic 0. Let \( V \) be a finite dimensional \( \mathbb{F} \)-vector space. Let \( L \subseteq \mathfrak{gl}(V) \) be a Lie subalgebra. Suppose that \( \text{Tr}(xy) = 0 \) for all \( x \in [L,L], y \in L \). Then \( L \) is a solvable Lie algebra.

**Proof.** It suffices to show that \([L,L] \) is nilpotent. This is because if \([L,L] \) is nilpotent then \([L,L] \) is solvable. Then \( L \) will be solvable. To show \([L,L] \) is nilpotent, it suffices to show every endomorphism \( x \in [L,L] \subseteq \mathfrak{gl}(V) \) is nilpotent, by Engel’s theorem [3, Theorem and Lemma on page 12].

Let \( x \in \mathfrak{gl}(\mathfrak{M}) \) be an endomorphism. Now let’s set

\[
A = [L,L], \quad B = L, \quad M = \{ x' \in \mathfrak{gl}(V) : [x',L] \subseteq [L,L] \}.
\]

Clearly, \( A = [L,L] \subseteq M, B = L \subseteq M \). To apply the trace criterion 1.1 to these \( A,B,M \), we need to show that \( \text{Tr}(xy) = 0 \) for all \( y \in M \).

Since \( x \in [L,L], x \) can be written as a linear combination of elementary generators \([x_1,x_2]\) with \( x_1,x_2 \in L \). Let \( y \in M \) then

\[
\text{Tr}([x_1,x_2]y) \overset{1.2}{=} \text{Tr}(x_1[x_2y]) = \text{Tr}([x_2,y]x_1).
\]

\([x_2,y] \in [L,L] \) since \( y \in M \). Our hypothesis is that \( \text{Tr}(x'y') = 0 \) for all \( x' \in L, y' \in B = L \). Hence setting \( x' = [x_2,y] \) and \( y' = x_1 \) allows us to conclude \( \text{Tr}([x_1,x_2]y) = 0 \). It then follows, by linearity, that \( \text{Tr}(xy) = 0 \) for all \( y \in M \). This finishes the proof. \( \square \)

An immediate corollary of Cartan’s criterion is stated in the following. While Cartan’s criterion is for Lie subalgebras of \( \mathfrak{gl}(V) \), the following corollary applies to any Lie algebra, provided the ground field \( \mathbb{F} \) is algebraically closed with characteristic zero.
Corollary 1.4. Let $L$ be a Lie algebra such that $\text{Tr}(\text{ad} x \text{ad} y) = 0$ for all $x \in [L, L], y \in L$. Then $L$ is solvable.

Proof. Consider the adjoint representation of $L$:

$$\text{ad} : L \to \mathfrak{gl}(L).$$

The Lie subalgebra $\text{ad}(L) \subseteq \mathfrak{gl}(L)$ clearly satisfies the hypothesis of Cartan’s criterion 1.3. So $\text{ad}(L)$ is solvable.

The kernel of the adjoint representation $\ker \text{ad} = Z(L)$ is the center of $L$, which is solvable. This means $L/Z(L) \cong \mathfrak{gl}(L)$ is solvable. By [3, Proposition 3.1 (b)], $L$ is solvable. \qed

2. KILLING FORM AND CRITERION FOR SEMISIMPLICITY

After we have shown the criterion for solvability, we turn our attention into a criterion for semi-simplicity. From 1.4, we have encountered an object of form $\text{Tr}(\text{ad} x \text{ad} y)$. This will turn out be a symmetric bilinear form on a Lie algebra $L$, known as Killing form $\kappa$. Our criterion for semi-simplicity will be based on this Killing form. The proof of the criterion will be dependent on Cartan’s criterion, so it holds for ground field $\mathbb{F}$ being algebraically closed with characteristic zero. However, the killing form can be defined for any field:

Definition 2.1 (Killing Form). Let $\mathbb{F}$ be any field. Let $L$ be a $\mathbb{F}$–Lie algebra. Recall the adjoint representation of $L$ is

$$\text{ad} : \mathbb{F} \to \mathfrak{gl}(\mathbb{F}), \; \text{ad} x(z) = [x, z] \text{ for } x, z \in L.$$ 

If $x, y \in L$ then $\text{ad} x \text{ ad} y$ is a $\mathbb{F}$–linear map with

$$\text{ad} x \text{ ad} y : L \to L, \; z \mapsto [x, [y, z]].$$

The killing form $\kappa$ on $L$ is defined to be the trace:

$$\kappa(x, y) = \text{Tr}(\text{ad} x \text{ ad} y).$$

The following are some easy properties for killing forms:

Lemma 2.2 (Basic Properties of Killing Form). Let $L$ be a $\mathbb{F}$–Lie algebra. Let $\kappa$ be the killing form of $L$.

1. $\kappa$ is a symmetric bilinear form.
2. $\kappa$ is associate in the sense that

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

3. If $I$ is an ideal of $L$ and $\kappa_I$ is the killing form of $I$, then

$$\kappa_I = \kappa|_{I \times I}.$$
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Proof. (1) To show $\kappa$ is symmetric, use property of trace that $\text{Tr}(AB) = \text{Tr}(BA)$:

$$\kappa(x, y) = \text{Tr}(\text{ad} x \text{ ad} y) \overset{\text{Property of trace}}{=} \text{Tr}(\text{ad} y \text{ ad} x) = \kappa(y, x).$$

To show $\kappa$ is bilinear, use the fact that ad is a representation:

$$c \in \mathbb{F} \Rightarrow \kappa(cx, y) = \text{Tr}(\text{ad} cx \text{ ad} y) = c \text{Tr}(\text{ad} x \text{ ad} y) = c \kappa(x, y);$$

$$\kappa(x_1 + x_2, y) = \text{Tr}(\text{ad}(x_1 + x_2) \text{ ad} y) = \text{Tr}((\text{ad} x_1 + \text{ad} x_2) \text{ ad} y)$$

$$= \text{Tr}(\text{ad} x_1 \text{ ad} y) + \text{Tr}(\text{ad} x_2 \text{ ad} y) = \kappa(x_1, y) + \kappa(x_2, y).$$

(2) We need to show

$$\text{Tr}([\text{ad} x, \text{ ad} y] \text{ ad} z) = \text{Tr}(\text{ad} x[\text{ad} y, \text{ ad} z]).$$

This is exactly 1.2.

(3) It suffices to show the following claim:

Claim 2.3. Suppose $W \subseteq V$ is a subspace. Suppose $\varphi \in \text{End}(V)$ maps $V$ into $W$ then

$$\text{Tr}(\varphi : V \to V) = \text{Tr}(\varphi|_W : W \to W).$$

Proof. Extend a basis of $W$ to a basis of $V$. The matrix of $\varphi : V \to$ with respect to this basis has zero on diagonals outside $W$. \qed

Now,

$$\kappa(x, y) = \text{Tr}(\text{ad} x \text{ ad} y) \overset{2.3}{=} \text{Tr}((\text{ad} x \text{ ad} y)|_I) = \text{Tr}(\text{ad}_I x \text{ ad}_I y) = \kappa_I(x, y).$$

\qed

Some important results for bilinear forms are summarized in [1]. The criterion for semi-simplicity we are heading to requires the notion of non-degenerate bilinear form. The following theorem serves as the definition of a non-degenerate bilinear form:

Theorem 2.4. Let $\mathbb{F}$ be any field. Let $V$ be a finite dimensional $\mathbb{F}$-vector space. Let $\beta : V \times V \to V$ be a bilinear form on $V$. The following conditions are equivalent:

1. There exists a basis $\{v_1, \ldots, v_n\}$ such that the $n \times n$ matrix $(\beta(v_i, v_j))$ is invertible.

2. The radical $S$ of $\beta$, which is defined to be $S = \{v \in L : \beta(v, w) = 0 \text{ for all } w \in V\}$, is zero.

3. Every element of the dual $V^\times$ has the form $v \mapsto \beta(w, v)$ for some $w \in V$.

4. Every element of the dual $V^\times$ has the form $v \mapsto \beta(w, v)$ for a unique $w \in V$.

Proof. See [1, Theorem 3.1]. \qed
Definition 2.5 (Non-Degenerate Bilinear Form). A bilinear form is called non-degenerate if it satisfies the equivalent conditions of 2.4.

Another useful fact about non-degenerate symmetric bilinear form is the following:

Theorem 2.6. Let $\mathbb{F}$ be any field. Let $V$ be a finite dimensional $\mathbb{F}$-vector space. Let $\beta : V \times V \to V$ be a symmetric bilinear form on $V$. Let $W$ be a subspace of $V$ and let

$$W^\perp = \{ v \in V : \beta(v, w) = 0 \text{ for all } w \in W \}. $$

The following conditions are equivalent:

1. The restriction $\beta|_{W \times W}$ is non-degenerate.
2. $W \cap W^\perp = 0$.
3. $V = W \oplus W^\perp$.

Proof. [1, Theorem 3.11].

Recall a Lie algebra $L$ is semi-simple if and only if its radical (maximal solvable ideal) $\text{Rad}(L) = 0$. Our criterion for a Lie algebra $L$ being semi-simple is the following:

Theorem 2.7 (Criterion for Semisimplicity). Let $\mathbb{F}$ be any field. Let $L$ be a $\mathbb{F}$-Lie algebra. Let $\kappa$ be the killing form of $L$.

1. If $\kappa$ is non-degenerate, then $L$ is semi-simple.
2. Suppose $\mathbb{F}$ is algebraically closed and $\text{char} \mathbb{F} = 0$. $L$ is semi-simple if and only if $\kappa$ is non-degenerate.

Proof. (1) Suppose the radical $S$ of $\kappa$ is zero. The goal is to show $\text{Rad}(L) = 0$ is zero. Note $\text{Rad}(L)$ is solvable so we have a chain of ideals in $L$:

$$\text{Rad}(L) \supseteq \text{Rad}(L)^{(1)} [\text{Rad}(L), \text{Rad}(L)] \supseteq \cdots \supseteq [\text{Rad}(L)^{(n)}, \text{Rad}(L)^{(n)}] = 0.$$  

The last non-zero ideal $\text{Rad}(L)^{(n)}$ (if there is any) is an abelian ideal of $L$. So to show $\text{Rad}(L) = 0$, it suffices to show every abelian ideal of $L$ is contained in $S = 0$.

Let $I \subseteq L$ be an abelian ideal. Let $x \in I$, $y \in L$. We have a composition of maps

$$L \xrightarrow{\text{ad} y} L \xrightarrow{\text{ad} x} I \xrightarrow{\text{ad} y} I \xrightarrow{\text{ad} x} [I, I] = 0.$$  

This means $(\text{ad} x \text{ad} y)^2 = 0$ and $\text{ad} x \text{ad} y$ is nilpotent. So

$$0 = Tr(\text{ad} x \text{ad} y) = \kappa(x, y).$$

However this means $x \in S = 0$. So $I \subseteq S = 0$.  

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(2) \((\Leftarrow)\) is done by the above argument. Now consider \((\Rightarrow)\) direction:

Suppose \(\text{Rad}(L) = 0\). Let \(S\) be the radical of the Killing form \(\kappa\) of \(L\). Actually \(S\) is an ideal of \(L\). To show \(S\) is zero, it suffices to show \(S\) is solvable so \(S \subseteq \text{Rad}(L) = 0\).

If \(x \in S\), \(y \in L\) then

\[ \text{Tr}(\text{ad} x \text{ad} y) = 0. \]

In particular this holds for \(y \in [S, S]\). We can either use corollary to Cartan’s criterion 1.4 to show \(S\) is solvable; or alternatively, Cartan’s criterion 1.3 implies \(\text{ad}_L(S)\) is solvable then \(S/Z(S) \cong \text{ad}_L(S)\) is solvable so \(S\) is solvable.

\(\blacksquare\)

Remark 2.8. (1) The first part of proof actually shows: let the ground field \(F\) be arbitrary, then every abelian ideal \(I\) of \(L\) is contained in the radical \(S\) of the Killing form \(\kappa\) on \(L\).

(2) The second part of proof actually shows: if the ground field \(F\) is algebraically closed with characteristic zero, then \(S \subseteq \text{Rad}(L)\).

3. Consequences of Killing Form

Throughout this section, we need to assume the ground field \(F\) to be algebraically closed with characteristic zero. The reason is because we will use Cartan’s criterion 1.3 and (2) of 2.7.

The first consequence of Killing form is that every semi-simple Lie algebras is a direct sum of simple ideals. We need a definition:

Definition 3.1 (Direct Sum of Ideals). Let \(L\) be a Lie algebra. \(L\) is said to be direct sum of ideals, denoted

\[ L = I_1 \oplus \cdots \oplus I_n, \]

provided \(L = I_1 \oplus \cdots \oplus I_n\) as vector spaces. This forces \([I_i, I_j] \subseteq I_i \cap I_j = 0\).

The full statement is as follows:

Theorem 3.2 (List of Properties of Semi-simple Lie Algebras). Let \(L\) be a semi-simple Lie algebra. Let \(\kappa\) be the Killing form on \(L\).

(1) There exist simple ideals \(I_1, \cdots , I_n\) of \(L\) such that

\[ L = I_1 \oplus \cdots \oplus I_n. \]

(2) Every simple ideal of \(L\) must be one of \(I_1, \cdots , I_n\).

(3) The Killing form of \(I_i\) is the restriction \(\kappa|_{I_i \times I_i}\).

(4) \(L = [L, L]\).

(5) All ideals and homomorphic image of \(L\) are semi-simple.

(6) Each ideal of \(L\) is a sum of simple ideals of \(L\).
Proof. (1) Let $I$ be an ideal of $L$, define the orthogonal ideal

$$I^\perp = \{ x \in L : \kappa(x, y) = 0 \text{ for all } y \in I \}.$$  

This is also an ideal of $L$.

Apply 1.4 to $I \cap I^\perp$ then we may conclude that $I \cap I^\perp$ is solvable. Being an ideal of semi-simple $L$, $I \cap I^\perp$ must be zero. By 2.6, we then have

$$I \oplus I^\perp = L.$$  

The rest of the proof relies on induction on the dimension of $L$: Let $L$ be semisimple and let $I_1$ be a non-zero proper ideal of $L$ (if no such $I_1$ exists, then $L$ is already simple). By previous paragraph, we have

$$L = I_1 \oplus I_1^\perp.$$  

Apply induction hypothesis on $I_1$ and $I_1^\perp$ respectively. $I_1$ and $I_1^\perp$ are decomposed into simple ideals of $I_1$ and simple ideals of $I_1^\perp$ respectively. These ideals are actually ideals of $L$ because say $I_2$ is an ideal of $I_1$ then

$$[I_2, L] \subseteq [I_2, I_1 \oplus I_1^\perp] \subseteq [I_2, I_1] \subseteq I_2.$$  

Thus we are done.

(2) Now suppose $L$ is decomposed into simple ideals:

$$L = I_1 \oplus \cdots \oplus I_n.$$  

Let $I$ be another simple ideal of $L$, we need to show $I = I_i$ for some $i \in \{1, \cdots, n\}$.

$[I, L]$ is an ideal of $I$. It is non-zero because the center $Z(L)$, being solvable, is zero. So it must be the case that $[I, L] = I$. On the other hand,

$$I = [I, L] = [I, I_1] \oplus \cdots \oplus [I, I_n].$$  

This says exactly one of $[I, I_i]$ is non-zero and it equals $I$. We also have $I = [I, I_i] \subseteq I_i$ but $I_i$ is simple hence $I = I_i$.

(3) This is (3) of 2.2.

(4) This follows from computation

$$L = I_1 \oplus \cdots \oplus I_n \Rightarrow [L, L] = [I_1, I_1] \oplus \cdots \oplus [I_n, I_n] = I_1 \oplus \cdots \oplus I_n = L.$$  

(5) Let $I$ be an ideal of $L$. The killing form on $I$ is non-degenerate by 2.6. So $I$ is semi-simple. Let $L'$ be a homomorphic image of $L$ then $L' \cong L/I \cong I^\perp$ for certain ideal of $L$. $I^\perp$ is semi-simple.

(6) Let $I$ be an ideal of $L = I_1 \oplus \cdots \oplus I_n$ then

$$I = (I \cap I_1) \oplus \cdots \oplus (I \cap I_n).$$  

Each $I_i$ being simple means $I \cap I_i$ is either 0 or $I_i$. □
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The second consequence of Killing form is that for every derivation of a semi-simple Lie algebra is an inner derivation. We record the definition:

**Definition 3.3 (Inner and Outer Derivations).** Let $L$ be a Lie algebra. A derivation $\delta$ of $L$ is a linear map $\delta \in \mathfrak{gl}(L)$ satisfying the product rule:

$$\delta([a,b]) = [a,\delta(b)] + [\delta(a),b].$$

An easy computation shows each $\text{ad} \ x \in \mathfrak{gl}(L)$ is a derivation of $L$. A derivation of this form is called inner. Other derivation is called outer.

**Lemma 3.4.** Let $L$ be a Lie algebra. Let

$$\text{ad} : L \to \mathfrak{gl}(L)$$

be its adjoint representation.

1. $\text{ad}(L)$ is an ideal in $\text{Der}(L)$.
2. $[\delta, \text{ad} \ x] = \text{ad}(\delta x)$ for $x \in L$ and $\delta \in \text{Der}(L)$.

**Proof.**

(1) Omitted.

(2) Let $y \in L$:

$$[\delta, \text{ad} \ x]y = \delta([x,y]) - \text{ad} \ x(\delta y) = [x,\delta(y)] + [\delta(x), y] - [x,\delta(y)] = [\delta(x), y] = \text{ad}(\delta x)y.$$  

□

**Theorem 3.5 (Every Derivation Is Inner).** Let $L$ be a semi-simple Lie algebra. Then

$$\text{ad} \ L = \text{Der} \ L.$$  

**Proof.** Since $L$ is simple, $Z(L) = 0$ so $\text{ad} : L \to \text{ad}(L)$ is an isomorphism.

By 2.7, $M = \text{ad}(L)$ has non-degenerate Killing form $\kappa_M$. Also, $\kappa_M$ is the restriction of the Killing form $\kappa_D$ of $D = \text{Der}(L)$.

Now $M$ is an ideal of $D$, so is $M^\perp$. By 2.6, we have $M^\perp \cap M = 0$ and $M \oplus M^\perp = D$. It follows that

$$[M, M^\perp] \subseteq M \cap M^\perp = 0.$$  

Let $\delta \in M^\perp$. $[M, M^\perp] = 0$ implies that, for all $x \in L$:

$$\text{ad}(\delta x) = [\delta, \text{ad} \ x] = 0.$$  

Since $\text{ad}$ is injective, we have $\delta x = 0$ so $\delta = 0$. We conclude $M^\perp = 0$ and $M = D$. □
4. Abstract Jordan Chevalley Decomposition

In 3.3, we define derivations for a Lie algebra $L$. More generally, derivation can be defined for all $\mathbb{F}$–vector space $\mathfrak{A}$ endowed with a bilinear operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$.

**Definition 4.1** (General Definition of Derivation). Let $\mathfrak{A}$ be $\mathbb{F}$–vector space, endowed with a bilinear operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. This bilinear operation is not assumed to be associative. A derivation of $\mathfrak{A}$ is a linear map $\delta : \mathfrak{A} \to \mathfrak{A}$ satisfying the product rule

$$\delta(a \cdot b) = a \cdot \delta(b) + \delta(a) \cdot b.$$  

Again, the collection of all such derivation is denoted $\text{Der}(\mathfrak{A})$.

Under this definition, 3.3 is just a special case, when we set the bilinear operation to be $[\cdot, \cdot]$ on a Lie algebra.

**Lemma 4.2** (Derivation Contains Semi-Simple and Nilpotent Parts). Let $\mathbb{F}$ be a algebraically closed field, $\mathfrak{A}$ be a finite dimensional $\mathbb{F}$–vector space, endowed with a bilinear operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. If $\delta \in \text{Der}(\mathfrak{A}) \subseteq \text{gl}(\mathfrak{A})$, and say $\delta = \delta_s + \delta_n$ is the Jordan Chevalley decomposition of $\delta$ in the sense of 0.1. Then $\delta_s$ and $\delta_n$ both lie in $\text{Der}(\mathfrak{A})$.

In other words, $\text{Der}(\mathfrak{A})$ contains the semi-simple parts and nilpotent parts of its elements.

**Proof.** Let $\delta \in \text{Der}(\mathfrak{A})$ and let

$$\delta = \delta_s + \delta_n$$

as in the hypothesis. Clearly it suffices to show $\delta_s \in \text{Der}(\mathfrak{A})$.

Set for $a \in \mathbb{F}$:

$$\mathfrak{A}_a = \{ x \in \mathfrak{A} : \exists k \in \mathbb{Z}_{\geq 0} \ (\delta - a \text{Id})^k x = 0 \}.$$ 

From Jordan canonical form, we have

$$\mathfrak{A} = \bigoplus_{\text{Eigenvalues } a \text{ of } \delta} \mathfrak{A}_a.$$  

Also, $\delta_s$ acts on each $\mathfrak{A}_a$ via scalar multiplication by $a$.

**Claim 4.3.** For any $a, b \in \mathbb{F}$, we have

$$\mathfrak{A}_a \cdot \mathfrak{A}_b \subseteq \mathfrak{A}_{a+b}.$$
Proof. Using induction, one can show the following formula holds:

\[
(\delta - (a + b) \text{Id})^n(x \cdot y) = \sum_{i=0}^{n} \binom{n}{i} ((\delta - a \text{Id})^{n-i}x) \cdot ((\delta - b \text{Id})^i y)
\]

for \(x, y \in \mathfrak{A}\).

If \(x \in \mathfrak{A}_a, y \in \mathfrak{A}_b\), then for large enough \(n\),

\[
(\delta - (a + b) \text{Id})^n(x \cdot y) = 0.
\]

Hence \(x \cdot y \in \mathfrak{A}_{a+b}\). \(\Box\)

Again if \(x \in \mathfrak{A}_a, y \in \mathfrak{A}_b\) then

\[
\delta_s(x \cdot y) = (a + b)x \cdot y = \delta_s(x) \cdot y + x \cdot \delta_s(y)
\]

which means \(\delta_s\) is a derivation. \(\Box\)

The version of Jordan Chevalley decomposition we currently have 0.1 applies to endomorphisms \(x \in \text{End}(V)\) where \(V\) is a finite dimensional \(\mathbb{F}-\)vector space. Now, 4.2, 3.5 enable us to define abstract Jordan Chevalley decomposition:

**Definition 4.4** (Abstract Jordan Chevalley Decomposition for Semi-Simple Lie Algebra). Let \(\mathbb{F}\) be an algebraically closed field with characteristic zero. Let \(L\) be a semi-simple \(\mathbb{F}-\)Lie algebra, then the adjoint representation is injective. We have

\[
\text{ad} : L \cong \text{ad}(L) \cong \text{Der}(L).
\]

Each \(x \in L\) unique determine \((\text{ad} x)_s, (\text{ad} x)_n \in \mathfrak{gl}(L)\):

\[
\text{ad} x \overset{0.1}{=} (\text{ad} x)_s + (\text{ad} x)_n.
\]

By 4.2, \((\text{ad} x)_s, (\text{ad} x)_n \in \text{Der}(L)\) so each \(x \in L\) unique determine \(s, n \in L\) such that

\[
x = s + n, \quad [s, n] = 0,
\]

and \(s\) is ad-semisimple, \(n\) is ad-nilpotent. By abuse of language, we call \(s\) semi-simple part of \(x\) and \(n\) nilpotent part of \(x\). This is known as the abstract Jordan Chevalley decomposition.

It will turn out when \(L\) is already a linear Lie algebra, then the abstract Jordan Chevalley decomposition matches up with 0.1.
REFERENCES