SEMISIMPLE GROUPS

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Abstract. We define a connected affine algebraic group of positive dimension to be semisimple provided the only closed connected commutative normal subgroup is the trivial group. Assuming the ground field has characteristic zero, we show that the Lie algebra of a semisimple group is semisimple and vice versa. Based on this, we show that the image of adjoint representation of a semisimple group is the identity component of Lie algebra automorphisms; the center of a semisimple algebraic group is finite; and a semisimple algebraic group is decomposed almost into a product of almost simple subgroups.

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In [3], we have shown two results regarding the Lie algebra of centralizers and centers in affine algebraic groups, which follows from the correspondence between subgroups and Lie subalgebras.

Theorem 0.1 (Correspondence of Subgroups and Lie Subalgebras). Assume char \( k = 0 \). Let \( G \) be a connected algebraic group. The assignment

\[ \{ \text{Closed connected subgroup of } G \} \to \{ \text{Lie subalgebra of } \mathfrak{g} \}, \; H \mapsto \mathcal{L}(H) = \mathfrak{h} \]

is injective. Hence there is an one-to-one, inclusion preserving correspondence \( H \leftrightarrow \mathfrak{h} \) between the collection of closed connected subgroups and the collection of algebraic Lie subalgebras. Under this correspondence, closed connected normal subgroups correspond to ideals of \( \mathfrak{g} \).

Theorem 0.2 (Centralizers and Centers). Assume char \( k = 0 \). Let \( G \) be a connected algebraic group. Let \( x \in G \).

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(1) The Lie algebra of centralizers of $x$ is the centralizers of $\text{Ad } x$:
$$\mathcal{L}(C_G(x)) = c_G(x) = \{ \delta \in g : \text{Ad } x(\delta) = \delta \}.$$

(2) The kernel of the adjoint representation $\text{Ad} : G \to \text{GL}(g)$,
$\text{Ad } x(\delta) = \rho_x \delta \rho_x^{-1}$
is the center of $G$:
$$\ker \text{Ad} = Z(G).$$

(3) The Lie algebra of the center of $G$ is the center of $g$:
$$\ker \text{ad} = Z(g) = \{ \delta \in g : [\delta, g] = 0 \}.$$

In view of (c) of 0.2, we immediately have the following corollary.

**Corollary 0.3.** Assume $\text{char } k = 0$. Let $G$ be a connected algebraic group. $G$ is commutative if and only if its Lie algebra $g$ is commutative.

*Proof.* Suppose $G$ is commutative. $Z(G) = G$ then $Z(g) = g$, meaning $g$ is commutative. Suppose $g$ is commutative. $\mathcal{L}(Z(G)) = Z(g) = g = \mathcal{L}(G)$. Since $G$ is connected, the correspondence theorem 0.1 implies $G = Z(G)$ so $G$ is commutative. □

**Definition 0.4 (Semisimple Affine Algebraic Groups).** Let $G$ be a connected affine algebraic group of positive dimension. $G$ is said to be semisimple provided that the only closed, connected, commutative, normal subgroup of $G$ is the trivial group 1.

This definition is evidently related to the definition of a semisimple Lie algebra, which we now review. From [2], we see that for any Lie algebra $L$, there exists a unique maximal solvable ideal $I \subseteq L$, known as the radical $\text{Rad}(L)$. (Recall that a Lie algebra $I$ is said to be solvable provided that the chain
\[(0.5) \quad I^{(0)} = I \supseteq I^{(1)} = [I, I] \supseteq I^{(2)} = [[I, I], [I, I]] \supseteq \cdots, I^{(n+1)} = [I^{(n)}, I^{(n)}], \]
terminates at $I^{(n)} = 0$ for some $n$.)

**Definition 0.6 (Semisimple Lie Algebra).** A Lie algebra $L$ is said to be semisimple provided that the unique maximal solvable ideal $\text{Rad}(L)$ of $L$ is trivial.

Properties of semi-simple Lie algebra can be found in [4]. They are all immediate consequences of a criterion for semisimplicity. Let $k$ denote the ground field of the Lie algebra $L$ then the killing form $\kappa$ on $L$ is defined by
$$\kappa : L \times L \to k, \quad \kappa(x, y) = \text{Tr}(\text{ad } x \text{ ad } y).$$
The criterion for a Lie algebra $L$ to be semisimple is that $L$ is semisimple if and only if the killing form $\kappa$ on $L$ is non-degenerate. We now briefly record the key results of semisimple Lie algebras here.

**Theorem 0.7 (List of Properties of Semi-simple Lie Algebras).** Let $L$ be a semi-simple Lie algebra. The following hold for $L$. 

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There exist simple ideals $I_1, \cdots, I_n$ of $L$ such that 
\[ L = I_1 \oplus \cdots \oplus I_n. \]

Every simple ideal of $L$ must be one of $I_1, \cdots, I_n$.


All ideals and homomorphic image of $L$ are semi-simple.

Each ideal of $L$ is a sum of simple ideals of $L$.

Every derivation is inner, i.e.,
\[ \text{ad } L = \text{Der } L. \]

(Recall a derivation $\delta$ of $L$ is a linear map $\delta \in \text{gl}(L)$ satisfying the product rule: $\delta([a, b]) = [a, \delta(b)] + [\delta(a), b]$. A derivation is said to be inner if it is $\text{ad } x \in \text{gl}(L)$.)

1. **Semisimple Groups and Semisimple Lie Algebras**

The relation between semisimplicity of algebraic groups and Lie algebras is that the Lie algebra of a semisimple group is semisimple, and vice versa. This is exactly what one expects when one sees the definition of semisimplicity for groups and Lie algebras.

**Theorem 1.1.** Assume $\text{char } k = 0$. Let $G$ be a connected affine algebraic group. $G$ is semisimple if and only if its Lie algebra $\mathfrak{g}$ is semisimple.

**Lemma 1.2.** Assume $\text{char } k = 0$. Let $G$ be a connected algebraic group, $\mathfrak{n}$ a subalgebra of $\mathfrak{g}$. Then the Lie algebra of $C_G(\mathfrak{n})$ is 
\[ \mathcal{L}(C_G(\mathfrak{n})) = \mathfrak{c}_\mathfrak{g}(\mathfrak{n}) = \{ \delta \in \mathfrak{g} : [\delta, \mathfrak{n}] = 0 \}. \]

**Proof.** Let $\{n_1, \cdots, n_m\}$ be a basis of $\mathfrak{n}$. Define $G_{n_i} = \{ x \in G : \text{Ad } x(n_i) = n_i \}$, $\mathfrak{g}_{n_i} = \{ \delta \in \mathfrak{g} : [\delta, n_i] = 0 \}$.

Then
\[ C_G(\mathfrak{n}) = G_{n_1} \cap \cdots \cap G_{n_m}, \quad \mathfrak{c}_\mathfrak{g}(\mathfrak{n}) = \mathfrak{g}_{n_1} \cap \cdots \cap \mathfrak{g}_{n_m}. \]

By [3, Theorem 2.3] and [3, Theorem 1.1]
\[ \mathcal{L}(G_{n_i}) = \mathfrak{g}_{n_i} \Rightarrow \mathcal{L}(C_G(\mathfrak{n})) = \mathfrak{c}_\mathfrak{g}(\mathfrak{n}). \]

\[ \square \]
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Proof of 1.1. \((\Rightarrow)\) Suppose \(G\) is semisimple. In view of the chain 0.5, if a Lie algebra \(I\) is solvable then the last non-zero \(I^{(n)}\), which is an ideal of \(I\), is a commutative Lie algebra. To show \(g\) is semi-simple, it suffices to show all commutative ideal of \(g\) is trivial.

Let \(n\) be a commutative ideal of \(g\). Define
\[
H = C_G(n)^\circ = \{x \in G : \text{Ad} x(\delta) = \delta \text{ for } \delta \in n\}^\circ.
\]
Its Lie algebra is \(h = c_g(n)\) by 1.2. It is not hard to show that \(h\) is actually an ideal so \(H\) is normal by 0.1. Then \(Z(H)^\circ\) is also normal in \(G\). By 0.2,
\[
\mathcal{L}(Z(H)^\circ) = Z(h).
\]
Thus \(\mathcal{L}(Z(H)^\circ) \supseteq n\). \(G\) is semisimple so \(Z(H)^\circ = 0\) which imples that \(Z(h) = 0\) and \(n = 0\), as desired.

\((\Leftarrow)\) Suppose \(g\) is semisimple. Let \(N\) be a closed connected commutative normal subgroup of \(G\). By 0.3 and 0.1, \(n = 0\) and \(N = 1\). \(\Box\)

This theorem tells us the structure of the center of a semi-simple group.

Corollary 1.3. Assume \(\text{char } k = 0\). Let \(G\) be a connected algebraic group. If \(G\) is semisimple, then \(Z(G)\) is a finite group.

Proof. By 1.1, \(g\) is semiple so \(Z(g) = 0\). (c) of 0.2 says
\[
\mathcal{L}(Z(G)) = Z(g).
\]
It follows that \(Z(G)\) is of dimension zero, hence a finite group. \(\Box\)

2. The Adjoint Representations of Semisimple Groups and Semisimple Lie Algebras

Theorem 2.1. Assume \(\text{char } k = 0\). Let \(G\) be a connected algebraic group. If \(G\) is semisimple, then
\[
\begin{align*}
(1) & \quad \text{ad } g = \text{Der } g. \\
(2) & \quad \text{Ad}(G) = (\text{Aut } g)^\circ = \{\text{Lie algebra automorphisms of } g\}^\circ.
\end{align*}
\]
This requires a result from [1] we won’t prove here:

Proposition 2.2. Assume \(\text{char } k = 0\). Let \(g\) be a \(k\)-Lie algebra. \(H = \text{Aut}(g)\), the Lie algebra automorphisms of \(g\), which can be viewed as an affine algebraic group contained in \(\text{GL}(g)\). Then \(h\) equals \(\text{Der}(g)\).

For a proof see [1, Corollary 13.2].

Proof of 2.1. (1) This is (6) of 0.7.
(2) Compare

\[ \text{Ad} : G \to \text{Ad}(G) \subseteq \text{Aut}(g)^\circ, \quad \text{ad} : g \to \text{ad}(g) = \text{Der}(g). \]

We have

\[ \dim \text{Ad}(G) = \dim G - \dim \ker \text{Ad} \overset{\text{1.2}}{=} \dim G - \dim Z(G) \overset{\text{1.3}}{=} \dim G, \]

\[ \dim \text{Aut}(g)^\circ \overset{\text{2.2}}{=} \dim \text{Der}(g) = \dim g - \dim \ker \text{ad} = \dim g. \]

Since \( \dim G = \dim g \), we have \( \dim \text{Ad}(G) = \dim \text{Aut}(g)^\circ \), implying that \( \text{Ad}(G) = \text{Aut}(g)^\circ \).

\[ \square \]

3. Decomposition of Semisimple Groups and Semisimple Lie Algebras

(1) and (2) of 0.7 says that a semi-simple Lie algebra \( L \) can be uniquely decomposed into direct sum of simple ideals

\[ L = I_1 \oplus \cdots \oplus I_n. \]

Using this one can show that a semisimple algebraic group \( G \) can be almost viewed as a product of almost simple groups \( G_1, \cdots, G_n \), meaning that \( G_i \) is noncommutative and contains no proper closed connected normal subgroups.

**Theorem 3.1.** Assume \( \text{char} k = 0 \). Let \( G \) be a connected affine algebraic group. If \( G \) is semisimple, then

\[ G = G_1 \cdot G_2 \cdots G_{n-1} \cdot G_n \]

where each \( G_i \) is noncommutative connected closed normal subgroup of \( G \) containing no proper closed connected normal subgroups. Furthermore, \( G_i \) centralizes \( G_j \) for \( i \neq j \). And

\[ G_i \cap G_1 \cdots G_{i-1} \cdot G_{i+1} \cdots G_n \]

is finite.

**Proof.** Let \( g \) decompose into simple ideals

\[ g = g_1 \oplus \cdots \oplus g_n. \]

It is easy to see that

\[ c_g (g_1 \oplus \cdots \oplus g_{i-1} \oplus g_{i+1} \cdots \oplus g_n) = g_i \]

and also

\[ c_g (g_1 \oplus \cdots \oplus g_{i-1} \oplus g_{i+1} \cdots \oplus g_n) \overset{\text{1.2}}{=} \mathcal{L}(C_G(g_1 \oplus \cdots \oplus g_{i-1} \oplus g_{i+1} \cdots \oplus g_n)). \]

Let \( G_i \) be the identity component of

\[ \mathcal{L}(C_G(g_1 \oplus \cdots \oplus g_{i-1} \oplus g_{i+1} \cdots \oplus g_n)). \]
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It is connected for sure, not commutative by 0.3, and is normal in $G$ by 0.1. $G_i$ will not have proper closed connected normal subgroup otherwise $g_i$ will have ideal, contradicting with simplicity.

The Lie algebra of the product $G_1 \cdots G_n$ contains $g$, hence $G = G_1 \cdots G_n$. If $i \neq j$ then $[g_i, g_j] = 0$. This implies that $G_i$ centralizes $G_j$. Then $G_i \cap G_1 \cdots G_{i-1} \cdot G_{i+1} \cdots G_n \subseteq Z(G_i)$ is finite due to 1.3.

□

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