1 Introduction

This topic discusses rigidity phenomena of lattices in semisimple Lie groups. By combining both the algebraic and geometric structures, it can be seen much of the structure of $G$ is determined by any of its lattices. We discuss Mostow (or strong) rigidity: we give Gromov’s proof for hyperbolic manifolds, and present the higher rank result as a corollary of Margulis superrigidity. Then we use superrigidity to prove Margulis’ results on the arithmeticity and characterization of normal subgroups of higher rank lattices. In the last section, we state Zimmer’s generalization to superrigidity of cocycles, and some implications of these results on the Zimmer program, a collection of conjectures on the possible actions of higher rank lattices on compact manifolds.

2 Lattices in Semisimple Lie Groups

The basic object of study will be a semisimple Lie group $G$, i.e. a Lie group that is isogenous to a product of simple Lie groups. For simplicity, we assume $G$ is connected, linear, has finite center, and no nontrivial compact factors. For example, $G = SL(n, \mathbb{R})$ or $G = SO(p, q)$ (for most $p, q \geq 1$). It is straightforward to see semisimple Lie groups are unimodular, i.e. they admit a bi-invariant Haar measure. Therefore, if $H$ is a subgroup of $G$, we get a $G$-invariant measure on $G/H$.

Definition. Let $G$ be a semisimple Lie group. A subgroup $\Gamma \subseteq G$ is called a lattice if it is discrete and $G/\Gamma$ has finite Haar measure. A lattice is cocompact if $G/\Gamma$ is compact.

If $\Gamma$ is a torsion-free lattice, then $G/\Gamma$ is in fact a finite-volume manifold. A lattice is called irreducible if whenever $N \subseteq G$ is a closed, noncompact, normal subgroup, $\Gamma N$ is dense in $G$. Roughly this means that $\Gamma$ cannot be written as a product of lattices in factors of $G$ (indeed, if $G$ is actually equal to a product of simple Lie groups, this is equivalent).

The following Borel Density theorem shows that algebraic structures on $G$ are determined by their behavior on $\Gamma$:

Theorem 1 (Borel Density Theorem). Any lattice $\Gamma$ in $G$ is Zariski-dense.

Sketch of proof (see [19]). The first step is to prove that given any continuous finite-dimensional representation of $G$, a $\Gamma$-invariant subspace is actually $G$-invariant. This can be seen by considering the induced action on a certain projective space, and using the Poincaré recurrence theorem. Let $V$ be the connected component of the identity of the Zariski-closure of $\Gamma$. $V$ contains $\Gamma^0 := \Gamma \cap V$ as a lattice. $\Gamma^0$ normalizes $V$, so the Lie algebra of $V$ is a $\Gamma^0$-invariant under the adjoint representation. Therefore, it is $G$-invariant, i.e. $V$ is normal in $G$. Since $G$ is semisimple, it follows that $V$ is a factor. Since $V$ contains a lattice, it must be cocompact. Hence $G = V$.

The theorem raises the question how much information about $G$ can be extracted from knowledge about $\Gamma$. This ‘rigidity’ is the subject of the next two sections.
3 Mostow Rigidity

The first rigidity result we will prove is that if $n \geq 3$, a compact hyperbolic $n$-manifold is determined up to isometry by its fundamental group:

**Theorem 2** ((Hyperbolic) Mostow Rigidity). Let $M, N$ be compact hyperbolic $n$-manifolds with $n \geq 3$. If $\varphi : \pi_1(M) \cong \pi_1(N)$ is an isomorphism, then there exists a unique an isometry $f : M \to N$ such that $f_* = \varphi$.

*Sketch of proof, due to Gromov [17].* Assume without loss of generality $M, N$ are oriented. Since complete hyperbolic manifolds are aspherical, $\varphi$ induces a homotopy equivalence $M \to N$. This induces a quasi-isometry $f : \mathbb{H}^n \to \mathbb{H}^n$, equivariant with respect to $\varphi$. By the Morse-Mostow lemma, $f$ induces a boundary map $\partial f : S^{n-1} \to S^{n-1}$. Since isometries of $\mathbb{H}^n$ are in one-to-one correspondence with conformal boundary maps, it suffices to show $\partial f$ is conformal.

The idea is to show $\partial f$ possesses a lot of symmetry; more precisely, an $n$-simplex in $\mathbb{H}^n$ is called regular if it has maximal hyperbolic volume. We claim $\partial f$ preserves such simplices. We define the Gromov norm of $M$: Construct a chain complex with basis certain measures on $C_k(M)$, and equip it with the $\ell^1$-norm (with respect to this basis). This induces a seminorm $|| \cdot ||$ on its homology. The usual singular homology naturally includes into this homology. The Gromov norm of $M$, denoted $||M||$, is then defined to be the number $||[M]||$, where $[M]$ is the fundamental class.

The advantage of defining this new chain complex is that one can perform a certain operation, called ‘smearing’. Using smearing, we see $||M|| = vol M / v_n$, where $v_n$ is the maximum volume of a simplex in $\mathbb{H}^n$, and moreover, we see an arbitrary regular simplex induces a chain representing $[M]$ that realizes the equality. A computation then shows $\partial f$ preserves regular simplices.

By the Haagerup-Munkholm theorem [8], when one vertex of a regular simplex is at infinity in the upper half space model, the remaining $n$ vertices form a regular polyhedron in $\mathbb{R}^{n-1}$. Then an ingenious argument by reflecting in the sides and using the barycenter of such a polyhedron, shows that $\partial f$ is conformal.

This theorem was originally proved by Mostow [14]. Prasad [15] relaxed the compactness assumption to merely finite volume. The analogous theorem is true for other locally symmetric spaces, proved by Mostow [14] in the cocompact case and by Margulis [12] in general; most cases will be handled in the next section. Also note that this theorem fails for hyperbolic surfaces of genus $g \geq 2$.

4 Superrigidity

The (hyperbolic) Mostow rigidity theorem can be restated in algebraic terms. It then says that the set of embeddings of a cocompact lattice $G$ is exactly conjugations by $G$ of the original embedding. A generalization of the Mostow Rigidity theorem above does not merely consider embeddings of a lattice, but a more general class of homomorphisms. On the other hand, we have a slight restriction on the ambient group. This restriction is in terms of its real rank:

**Definition.** The real rank of $G$, denoted $\text{rk}_\mathbb{R}(G)$, is the maximal dimension of a torus that is diagonalizable over $\mathbb{R}$.

A semisimple group has real rank zero if and only if it is compact. The difference between rigidity properties of lattices in groups of real rank 1 and $> 1$ is also enormous, but more
of characteristic zero, and $H$ map
Sketch of proof, due to Zimmer [20].
First let $k$ theorem. The proof if
and facts about algebraic groups. The uniqueness statement follows from the Borel density
a.e. equal to a rational map. This follows from the structure theory of semisimple Lie groups
$H$ of a point in $\text{Prob}(\mathbb{R})$.

(i) If $k = \mathbb{R}$, $H(\mathbb{R})$ has trivial center, and is not compact, then $\varphi$ extends uniquely to an
$\mathbb{R}$-rational morphism $G \to H$.

(ii) If $k = \mathbb{C}$, $H$ has trivial center, then either $\varphi$ has precompact image or $\varphi$ extends uniquely
to a rational morphism $G \to H$.

(iii) If $k = \mathbb{Q}_p$, then $\varphi$ has precompact image.

Sketch of proof, due to Zimmer [20]. First let $k = \mathbb{R}$. Prove it suffices to find a rational $\Gamma$-
equivariant map between homogeneous spaces of $G$ and $H$ by showing that the Zariski-closure
of the graph of $\varphi$ is the graph of a morphism and an application of the Borel density theorem.
Let $P$ be a minimal parabolic subgroup, and set $P_0 := P \cap G(\mathbb{R})^0$. Hence it suffices to show
there exists a proper algebraic $\mathbb{R}$-subgroup $L$ of $H$, and a rational $\Gamma$-equivariant map $\xi : G(\mathbb{R})^0/
P_0 \to H(\mathbb{R})/L(\mathbb{R})$. Let $Q$ be a proper parabolic subgroup of $H$, so that $\Gamma$ acts on the compact
metric space $H(\mathbb{R})/Q(\mathbb{R})$. Since $P(\mathbb{R})$ is amenable, there exists a measurable $\Gamma$-equivariant map

$$\xi : G(\mathbb{R})^0/P_0 \to \text{Prob}(H(\mathbb{R})/Q(\mathbb{R})).$$

By Moore ergodicity, the image is essentially contained in a single $H(\mathbb{R})$-orbit. The stabilizer
of a point in $\text{Prob}(H(\mathbb{R})/Q(\mathbb{R}))$ is of the form $L(\mathbb{R})$, where $L$ is a proper algebraic subgroup
of $H$. So we can view $\xi$ as a map $G(\mathbb{R})^0/P_0 \to H(\mathbb{R})/L(\mathbb{R})$. The final step is to prove $\xi$ is a.e.
equal to a rational map. This follows from the structure theory of semisimple Lie groups and facts about algebraic groups. The uniqueness statement follows from the Borel density theorem. The proof if $k \neq \mathbb{R}$ is similar.

As a special case, consider an $n$-dimensional real or complex representation $\Gamma \to GL(n,k)$. Superrigidity then implies that any such representation actually extends to a representation of $G$.

Superrigidity for $k = \mathbb{R}$ or $\mathbb{C}$ fails for isometries of real and complex hyperbolic space; it is true for other rank one spaces, as proven by Corlette [1].

Margulis superrigidity has amazing consequences for the lattices in such higher rank groups; we will discuss some of them now. The first is Mostow rigidity for higher rank groups:

**Corollary 1 ((Higher rank) Mostow Rigidity).** Let $G_1, G_2$ be semisimple Lie groups with
trivial center (and the usual restrictions), $G_1$ of higher rank, $\Gamma_1 \subseteq G_1$ an irreducible lattice,
and $\Gamma_2 \subseteq G_2$ some lattice. Suppose $\varphi : \Gamma_1 \to \Gamma_2$ is an isomorphism. Then $\varphi$ extends to an
isomorphism $G_1 \cong G_2$.

**Definition.** A lattice $\Gamma \subseteq G$ is called arithmetic if there exists a finite normal subgroup $K$,
a real linear group $G'$ defined over $\mathbb{Q}$, $K' \subseteq G'$ compact normal subgroup, and an isogeny
$\varphi : G/K \to G'/K'$ with $\varphi(\Gamma/K) = G'(\mathbb{Z})/K'$.
Example. Consider \( \Gamma := SL(n, \mathbb{Z}[i]) \subseteq SL(n, \mathbb{C}) \). Let \( G' := SL(2n, \mathbb{R}) \) and \( \varphi : G \hookrightarrow G' \) the standard embedding. Then \( \varphi(\Gamma) = SL(2n, \mathbb{Z}) \), hence \( \Gamma \) is arithmetic.

More complicated examples are \( SL(2, \mathbb{Z}[\sqrt{2}]) \hookrightarrow SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) (by the Galois conjugates) and \( SO(x^2 + y^2 - \sqrt{2}z^2; \mathbb{Z}) \hookrightarrow SO(2, 1) \times SO(3) \).

Amazingly, in a higher real rank group, every irreducible lattice is obtained in this way \([11]\); we will show this with an additional assumption on \( G \):

**Theorem 4** (Margulis Arithmeticity). If \( G \) has no rank one factors, then any irreducible lattice in \( G \) is arithmetic.

*Sketch of proof.* Consider the adjoint representation of \( \Gamma \) on the Lie algebra \( g \) of \( G \). An application of superrigidity with \( k = \mathbb{C} \) shows that the trace of \( \text{ad}(\gamma) \) is an algebraic number for \( \gamma \in \Gamma \). A computation then shows that in some basis, all the matrix entries of \( \text{ad}(\gamma) \) are algebraic. Since \( G \) has no rank one factors, \( \Gamma \) has property (T), so \( \Gamma \) is finitely generated. Therefore, all entries of elements of \( \Gamma \) lie in some algebraic number field. By using restriction of scalars, we can then assume all entries of all elements of \( \Gamma \) are rational. It remains to show that \( \Gamma \) is commensurable with \( G(\mathbb{Z}) \), i.e. that the denominators of entries of elements of \( \Gamma \) are bounded. Again using \( \Gamma \) is finitely generated, we see only finitely many primes appear as factors of these denominators. For all such primes \( p \), apply superrigidity with \( k = \mathbb{Q}_p \).

Actually this theorem is also true for many rank one cases; this follows from the corresponding superrigidity theorem in these cases. However, in \( SO(2, 1) \), the existence of non-arithmetic lattices follows immediately from Teichmüller theory. Non-arithmetic lattices have been constructed in \( SO(n, 1) \) for all \( n \geq 2 \) by Gromov and Piatetski-Shapiro \([7]\) by gluing the manifolds corresponding to certain arithmetic lattices along suitable totally geodesic codimension one submanifolds. In \( SU(n, 1) \), non-arithmetic lattices have been constructed for \( n = 2, 3 \) by Deligne and Mostow \([2]\); existence of non-arithmetic lattices is an open problem for \( n \geq 4 \).

Another major application of superrigidity is the Margulis Normal Subgroups theorem: It says that the normal subgroups of such a lattice are all finite or finite index. This in turn will have enormous consequences for the spaces such lattices can act on nontrivially (see next section). The main part of the proof is the following result:

**Theorem 5** (Margulis Projective Factors). Let \( G \) have higher rank, and let \( P \) be a minimal parabolic subgroup of \( G \). Let \( Y \) is a compact metric space with a continuous \( \Gamma \)-action and suppose there exists an essentially \( \Gamma \)-equivariant measurable map \( \varphi : G/P \rightarrow Y \). Equip \( Y \) with the push-forward measure. Then there is a subgroup \( Q \) of \( G \) containing \( P \), such that \( Y \) is measurably and \( \Gamma \)-equivariantly isomorphic to \( G/Q \).

*Sketch of proof.* The map \( \varphi \) induces a \( \Gamma \)-equivariant inclusion

\[
\varphi^* : L^\infty(Y) \rightarrow L^\infty(G/P).
\]

Restrict this map to the set of characteristic functions on \( Y \); this is a Boolean algebra \( B(Y) \), and \( \varphi^* \) embeds \( B(Y) \) as a weak*-closed \( \Gamma \)-invariant Boolean subalgebra of \( B(G/P) \). We then use structure theory of semisimple Lie groups to show the existence of suitable contracting automorphisms of \( G \); a modified version of the Lebesgue density theorem proves the subalgebra \( L^\infty(Y) \) is actually \( G \)-invariant. It is then a fact from measure theory that \( \varphi^* \) is induced by a \( G \)-equivariant map. Then the \( G \)-action is essentially transitive, hence \( Y \) is measurably and \( \Gamma \)-equivariantly isomorphic to \( G/Q \) for some \( Q \), as desired. \( \square \)
Theorem 6 (Margulis Normal Subgroups). Let $\Gamma$ be an irreducible lattice in a higher real rank semisimple Lie group $G$ without rank one factors. Then every normal subgroup of $\Gamma$ is finite or has finite index.

Sketch of proof. Let $N \triangleleft \Gamma$ be an infinite normal subgroup. Then $\Gamma/N$ has property (T) by the assumption that $G$ has no real rank one factors. Since it is also discrete, it suffices to show it is amenable. So let $X$ be a compact metric space with a continuous $\Gamma/N$-action. Let $P$ be a minimal parabolic subgroup. Then $P$ is amenable, so $\Gamma$ acts amenably on $G/P$; therefore, we have a measurable essentially $\Gamma$-equivariant map $G/P \to \text{Prob}(X)$. We note that $\text{Prob}(X)$ is also a compact metric space. By the projective factors theorem, there exists a parabolic subgroup $Q$ containing $P$, such that $\text{Prob}(X)$ is measurably and $\Gamma$-equivariantly isomorphic to $G/Q$. Since $N$ acts trivially on $G/Q$, the $G$-action on $G/Q$ has infinite kernel, hence this kernel must be a factor. It follows it must be all of $G$. Therefore, $\Gamma$ acts essentially trivially on $\text{Prob}(X)$, so there exists a $\Gamma$-invariant probability measure on $X$.

Actually this is also true if $G$ has rank one factors \[13\]. However, the result often fails in rank one (see \[19\] building on an earlier argument by Gromov \[6\]). Also note that if $\Gamma$ is as in the theorem, any finite normal subgroup of $\Gamma$ is normal in $G$ by the Borel density theorem, and it it has dimension zero since $\Gamma$ does; therefore it is central. On the other hand, if a normal subgroup has finite index, it must be a lattice in $\Gamma$. This will be extremely useful in the next section.

5 Rigidity of Actions

The superrigidity theorem above generalizes by considering cocycles over an irreducible finite measure base, instead of homomorphisms. This leads to the idea one may be able to classify irreducible actions on compact manifolds of such lattices, much like linear actions as pointed out in the previous section. For more details and some of the progress that has been made, see the theorems by Ghys, Farb and Shalen below. We first define a cocycle:

Definitions. Let $G \curvearrowright X$, $\mu$ a measure on $X$ and $H$ a second countable group. A Borel function $\alpha : G \times X \to H$ is a cocycle if for every $g, h \in G$, $\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$ for a.e. $x \in X$. Two cocycles $\alpha, \beta : G \times X \to H$ are cohomologous if there exists Borel $\phi : X \to H$ such that for all $g \in G$, $\beta(g, x) = \phi(gx)^{-1}\alpha(g, x)\phi(x)$ for a.e. $x \in X$.

Example. Let $G \curvearrowright X$ and $\pi : G \to H$ a homomorphism. Then $\alpha_{\pi}(g, x) := \pi(g)$ is a cocycle.

Superrigidity for cocycles addresses the question when a cocycle is cohomologous to a cocycle of the type from the example.

Theorem 7 (Zimmer’s Superrigidity for Cocycles). Let $G$ be an algebraic group over $\mathbb{R}$ of higher rank such that $G^0_{\mathbb{R}}$ has no compact factors. Let $G \curvearrowright S$ ergodically and irreducibly with finite invariant measure, $H$ be a connected algebraic group over a local characteristic zero field $k$, almost simple over $k$, and $\alpha : G^0_{\mathbb{R}} \times S \to H(k)$ a cocycle that is not cohomologous to any cocycle mapping into the $k$-locus of a proper algebraic subgroup over $k$. Then

(i) If $k = \mathbb{R}$, $H(\mathbb{R})$ has trivial center and is non-compact, then $\alpha$ is cohomologous to the restriction to $G(\mathbb{R})^0$ of an $\mathbb{R}$-rational morphism $G \to H$.

(ii) If $k = \mathbb{C}$ and $H$ has trivial center, then either $\alpha$ is cohomologous to a cocycle mapping into a compact subgroup of $H$ or $\alpha$ is cohomologous to a rational morphism $G \to H$.
(iii) If \( k \) is totally disconnected, then \( \alpha \) is cohomologous to a cocycle mapping into a compact subgroup of \( H \).

The proof is very similar to the proof for Margulis Superrigidity given above, and will not be included here (see [20]). Cocycles arise naturally when considering group actions: A \( G \)-action on a fiber bundle over some base \( S \) by bundle maps, induces a cocycle over \( S \) with values in the automorphism group of the fiber. Zimmer used the superrigidity of cocycles and normal subgroups theorems to make progress on some of his conjectures involving a classification of lattices on compact manifolds. The idea is that a lattice that is in some sense ‘large’ cannot act non-trivially on a low-dimensional manifold. For example, if the action has a periodic orbit, a finite index subgroup (that is also a lattice) fixes a point. Then one obtains a representation of the lattice by considering the action on the tangent space; this extends to a representation of \( G \) by superrigidity, and \( G \) usually does not have low-dimensional non-trivial representations. Here one should understand ‘trivial’, ‘large’ and ‘low-dimensional’ in the following ways:

**Definition.** A group action is finite if the kernel has finite index.

**Definition.** Let \( G \) be a semisimple Lie group of higher real rank. Set \( c_1 \) to be the lowest number \( n \) such that \( G \) admits an infinite \( n \)-dimensional real representation, and let \( c_2 \) be such that \( \frac{c_2(c_2+1)}{2} \) is greater than or equal to the minimal dimension of a simple factor of \( G \). Set \( c(G) := \min\{c_1, c_2\} \).

**Conjecture 1** (Zimmer). Let \( \Gamma \) be an irreducible lattice in a higher real rank semisimple Lie group \( G \). Then any smooth action of \( \Gamma \) on a compact \( k \)-manifold is finite whenever \( k < c(G)-1 \). If the action is assumed to be volume-preserving, the action is finite whenever \( k < c(G) \).

The bounds are necessary: \( SL(n, \mathbb{Z}) \) acts on \( \mathbb{R}P^{n-1} \) without preserving volume, and it acts on \( \mathbb{T}^n \) preserving volume. In the case that \( M \) is one-dimensional, this conjecture has mostly been solved:

**Theorem 8** (Ghys). Let \( \Gamma \) be an irreducible lattice in \( G \), and assume \( G \) has no rank one factors. Then any \( C^1 \) \( \Gamma \)-action on the circle is finite.

**Sketch of proof due to Witte-Morris, Zimmer [18].** Without loss of generality the action preserves orientation. Any action induces a cocycle \( \alpha \) over \( G/\Gamma \). Since the group of orientation-preserving isometries of \( \mathbb{S}^1 \) is abelian and \( \Gamma \) has property (T), it suffices to show \( \Gamma \) acts by isometries, i.e. \( \alpha \) maps into the group of isometries of \( \mathbb{S}^1 \). Straightforward measure theory shows this is equivalent to the existence of a finite \( G \)-invariant measure on \( G/\Gamma \times_\alpha \mathbb{S}^1 \) in the same measure class as \( \mu \times \lambda \) that projects to \( \mu \) (where \( \mu \) is Haar measure and \( \lambda \) is Lebesgue measure). A generalization of the Reeb-Thurston stability theorem (for \( \mathbb{S}^1 \)) to cocycles shows such a measure exists if and only if there exists an \( \alpha \)-equivariant map \( G/\Gamma \to \mathbb{S}^1 \).

By using structure theory of \( G \), one can obtain an \( \alpha \)-equivariant map \( G/\Gamma \to \text{Prob}(\mathbb{S}^1) \), i.e. a finite \( G \)-invariant measure on \( G/\Gamma \times_\alpha \mathbb{S}^1 \) that projects to \( \mu \).

\( \alpha \) induces a natural cocycle \( \beta \) over \( G/\Gamma \times_\alpha \mathbb{S}^1 \). The projection to \( \mathbb{S}^1 \) is a \( \beta \)-equivariant map. Applying the previous comments to this space instead of \( G/\Gamma \) gives a finite \( G \)-invariant measure on \( (G/\Gamma \times_\alpha \mathbb{S}^1) \times_\beta \mathbb{S}^1 \) in the same measure class as \( (\mu \times \lambda) \times \lambda \) that projects to \( \mu \times \lambda \). Then projecting this measure by forgetting the middle factor is a finite \( G \)-invariant measure on \( G/\Gamma \times_\alpha \mathbb{S}^1 \) in the same measure class as \( \mu \times \lambda \) with projection \( \mu \), as desired.

The same proof generalizes to the situation of an action on a circle bundle over a compact base (see [18]). Progress on Zimmer’s conjecture if \( M \) has dimension two has been made if one assumes the action is, for example, real-analytic:
**Theorem 9** (Farb, Shalen). Let $\Gamma$ be an irreducible lattice of rational rank at least 4. Let $M$ be a closed oriented surface such that $\Gamma$ acts by real-analytic maps on $M$. Then the action is finite.

For example, the theorem applies if $\Gamma$ is a finite-index subgroup of $SL(n, \mathbb{Z})$ where $n \geq 5$.

**Proof.** Because $\Gamma$ has rational rank at least 4, there is a torsion-free nilpotent subgroup $N \subseteq \Gamma$ that is not metabelian. On the other hand, by a result of Ghys [5] and Rebelo [16], any nilpotent subgroup of $Diff^\omega(M)$ is metabelian. Hence some elements of $N$ act trivially. Since $N$ is torsion-free, this implies the kernel of the action is infinite. Again by the Margulis Normal Subgroups theorem, the action is finite.

Actually, Farb and Shalen originally employed a method that reduces the problem from surfaces to one- or zero-dimensional submanifolds but only works for $\chi(M) \neq 0$. Such an inductive procedure still works for 4-manifolds:

**Theorem 10** (Farb, Shalen). Let $\Gamma$ be an irreducible lattice of rational rank at least 7. Let $M$ be a closed 4-manifold with $\chi(M) \neq 0$ and suppose $\Gamma$ acts on $M$ by real-analytic and volume-preserving maps. Then the action is finite.

**Proof.** Suppose the action is infinite. Since $\Gamma$ has rational rank at least seven, it contains commuting subgroups $A, B$ with rational ranks at least 2 and 4 (respectively). Then $A$ contains a non-central, torsion-free element $\gamma$. Since $\chi(M) \neq 0$, the Lefschetz fixed point theorem shows some power of $\gamma$ has a fixed point. Let $F$ be this fixed set. $B$ acts on $F$. Using facts about analytic sets, one obtains a codimension $\geq 1$ real-analytic submanifold $W \subseteq M$, contained in $F$, such that some finite index subgroup $B' \subseteq B$ acts infinitely on $W$.

The claim is that $W$ cannot be codimension one. Suppose it is. If $W$ is separating, one can compare the volumes of the different components of $M \setminus W$, and use that $B'$ preserves volume to reach a contradiction. If $W$ does not separate $M$, it gives an infinite cyclic covering. This induces an order on $B'$. However, since $B'$ has higher rational rank, this is impossible.

So $W$ is either a finite set of points, circles, or surfaces. $W$ cannot be a finite set of points since $B'$ acts infinitely; the remaining cases produce a contradiction with either Ghys’ theorem or Farb and Shalen’s theorem above.

There is hope a similar argument produces results about actions on higher-dimensional manifolds.

**References**


